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POINTS OF CONTINUITY OF QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. We study points of joint continuity of multi-variable separately quasi-continuous functions which are continuous with respect to one variable. We will show that the assumption of complete regularity in the paper of V. Maslyuchenko et al in "Tatra Mt. Math. Pub. 68, (2017), 47–58" can be removed in some situations. Moreover, we use a topological game argument to prove that the set of points of continuity of functions with closed graph into $w\Delta$-spaces is a G_δ subset of its domain provided that its domain is a W-space.

1. Introduction

Let X and Z be topological spaces. Following [12], a function f from X to Z is called quasi-continuous at x_0 if for neighborhoods U and W of x_0 and $f(x_0)$, respectively, there exists a nonempty open set $V \subseteq U$ such that $f(V) \subseteq W$. The function f is called quasi-continuous if it is quasi-continuous at each point of X. The notion of quasi-continuity plays an important role in the proof that some semi-topological groups are actually topological groups (see, e.g., [4, 5, 21]) and in the proof of some generalizations of Michael's selection theorem [7]. The notion of quasi-continuity is frequently used for establishing the existence of points of joint continuity of two variables functions. Kempisty [12] used these functions to improve some results of Hahn [9] and Baire [2] on joint continuity of separately continuous functions. The notion of quasi-continuity suggests the following weaker form of separate continuity.

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A function $f : X \times Y \to Z$ is called a KC-function if it is quasi-continuous with respect to the first variable and continuous with respect to the second variable. A few mathematicians studied points of joint continuity of KC-functions see, e.g., [3, 6, 10, 16, 17, 20, 22, 23] and the survey paper of Neubrunn [25]). In particular, Piotrowski [26] proved the following result.

Theorem 1. *Let X be a Baire space, Y a first countable, and let Z be a metric space. If $f : X \times Y \to Z$ is a KC-function, the points of joint continuity of f are dense in $X \times \{y\}$ for all $y \in Y$.*

Let $\{\mathcal{G}_n\}$ be a sequence of open covers of a space X. This sequence is called a weak development [1] for X if for each $x \in X$, $x \in G_n \in \mathcal{G}_n$ for each $n \in \mathbb{N}$ implies that the sequence $\{\bigcap_{i \le n} G_i\}_n$ is a base at x. A space with a weak development is called a weakly developable space.

Holá and Piotrowski used this notion to improve the above result as follows.

Theorem 2. [11, Theorem 4.2] *Let X be a Baire space, Y a first countable space, and Z a regular weakly developable space. If $f : X \times Y \to Z$ is a KC--function, for each $y \in Y$, the points of joint continuity of f are dense in $X \times \{y\}$ for all $y \in Y$.*

In 1976, Gruenhage [8] introduced W-spaces, a class of topological spaces which strictly contains first countable spaces. This motivated the second author [21] to improve Holá-Piotrowski's theorem as follows.

Theorem 3. *Let X be a Baire space, Y a W-space, and Z a weakly developable regular space. Then, for each KC-function $f : X \times Y \to Z$ and each $y \in Y$, there is a dense G_δ set A_y of X such that f is jointly continuous at each point of $A_y \times \{y\}$.*

Following [18, 28], a function $f : X \times Y \to Z$ is said to have the *Weston*--property if

$$C_y(f) = \{x \in X : (x, y) \in C(f)\},$$

for each $y \in Y$, is a residual subset of X, where $C(f)$ denotes the set of points of continuity of the function f.

We say that (X, Y, Z) is a Weston-triple, if each KC-function $f : X \times Y \to Z$ has the Weston property. The space Y is a Weston-space with respect to Z if for each Baire space X, (X, Y, Z) is a Weston-triple.

It follows from Theorem 3 that every W-space is a Weston-space with respect to any weakly developable regular space.

V. Maslyuchenko et al. [18] investigated Weston property for two variable functions into a completely regular spaces. In this paper, we improve their results by considering KC-functions into regular spaces. More precisely, we will show that in order to prove strong quasi-continuity of KC-functions into completely regular spaces it suffices to study only real-valued KC-functions. We also investigate Weston property for this class of functions. We also give some applications of our results, which improve some old results in [13] and [18]. Finally, in Section 3, we study points of continuity of functions with closed graph into $w\Delta$-spaces, which generalize the corresponding result in [11].

2. Points of joint continuity of KC-functions

In this section, we will investigate points of joint continuity of KC-functions. In order to achieve this goal, we need the following weaker form of joint continuity of two variable functions.

Definition 1. Let $f : X \times Y \to Z$ be a function and $(x_0, y_0) \in X \times Y$. The function f is said to be *strongly quasi-continuous* at (x_0, y_0) with respect to the second variable [21, 24] if for each non-empty open subset $G \subseteq Z$ with $f(x_0, y_0) \in G$ and a neighborhood U of x_0, there are non-empty open subset U' of U and a neighborhood V of y_0 such that $f(U' \times V) \subseteq G$. If f is strongly quasi-continuous with respect to the second variable at points of its domain, it will be called strongly quasi-continuous with respect to the second variable.

Remark 1. The above notion was first introduced for metric spaces under the name "symmetrically quasi-continuous with respect to the second variable" in [12].

The following result shows that in order to prove strong quasi-continuity for the class of two variable functions into completely regular spaces, one can consider only real-valued functions.

Theorem 4. *Let X, Y be topological spaces and Z a completely regular space. Then, the following assertions are equivalent:*

(1) *Every KC-function $f : X \times Y \to \mathbb{R}$ is strongly quasi-continuous with respect to the second variable.*
(2) *Every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous with respect to the second variable.*

Proof. Clearly, (1) follows from (2). Suppose that (1) holds and $f : X \times Y \to Z$, where Z is completely regular space, is a KC-function. Suppose that f is not strongly quasi continuous with respect to the second variable at some point (x_0, y_0). By the definition, we can find neighborhoods U, V and G of x_0, y_0 and $f(x_0, y_0)$ such that for each nonempty open sets $U' \subseteq U$ and neighborhood $V' \subseteq V$ of y_0, $f(U' \times V') \nsubseteq G$. Take an open set $G_1 \subseteq G$ such that $f(x_0, y_0) \in G_1 \subseteq \overline{G_1} \subseteq G$. Since Z is completely regular, there is a continuous function $g : Z \to [0, 1]$ such that $g(f(x_0, y_0)) = 1$ and $g(z) = 0$ for each $z \in Z \setminus \overline{G_1}$.

Let $H = \{z \in G : g(z) > \frac{1}{2}\}$, then H is an open subset of G which contains $f(x_0, y_0)$. The function $h = gof : X \times Y \to \mathbb{R}$ is a KC-function. By (1), we can find nonempty open sets $U_1 \subseteq U$ and a neighborhood $V_1 \subseteq V$ of y_0 such that $h(U_1 \times V_1) \subseteq (\frac{1}{2}, 1]$. However, by our assumption, there is some $(x_1, y_1) \in U_1 \times V_1$ such that $f(x_1, y_1) \in Z \setminus G$. It follows that $\frac{1}{2} < h(x_1, y_1) = 0$. This contradiction proves our result. □

In 1996, Piotrowski [27] proved that if X is Baire, Y a topological space and Z is a developable space, then for every $f : X \times Y \to Z$ which is strongly quasi-continuous with respect to the second variable and $y_0 \in Y$, there is dense G_δ set $D \subseteq X$ such that f is jointly continuous at each point of $D \times Y$. The following result states that Piotrowski's theorem is true when Z is a weakly developable space.

Theorem 5. *Let X be a Baire space and Z a weakly developable space. If $y_0 \in Y$ and $f : X \times Y \to Z$ is strongly quasi-continuous with respect to the second variable at each point of $X \times \{y_0\}$, there is a dense G_δ subset D of X such that f is jointly continuous at each point of $D \times \{y_0\}$.*

Proof. Let $\{\mathcal{G}_n\}$ be a weak development for Z. For each $n \in \mathbb{N}$, we define D_n to be the union of all open sets $U \subseteq X$ such that $f(U \times V) \subseteq G_n$ for some neighborhood V of y_0 and $G_n \in \mathcal{G}_n$. Then, each D_n is open in X. We will show that each D_n is dense in X. Take some $n_0 \in \mathbb{N}$ and let U_0 be an arbitrary nonempty open subset of X. Let $x_0 \in U_0$ and choose some $G_{n_0} \in \mathcal{G}_{n_0}$ with $f(x_0, y_0) \in G_{n_0}$. By strong quasi-continuity of f at (x_0, y_0) with respect to the second variable, we can find a nonempty open subset U of U_0 and a neighborhood V of y_0 such that $f(U \times V) \subseteq G_{n_0}$. Therefore, $U \subset D_n \cap U_0$. Hence, D_n is dense in X for each $n \in \mathbb{N}$. Since X is Baire, $D = \bigcap_{n \geq 1} D_n$ is dense in X. We will show that f is continuous at each point $D \times \{y_0\}$. Let $(x, y_0) \in D \times \{y_0\}$ and G be an open subset of Z with $f(x, y_0) \in G$. For each $n \in \mathbb{N}$, there are open sets U_n, V_n and G_n with $x \in U_n$, $y_0 \in V_n$ and $f(x, y_0) \in G_n$. By the definition, the sequence $\{\bigcap_{i \leq n} G_i\}_n$ is a base for $f(x, y_0)$. Therefore, $\bigcap_{i \leq n_0} G_i \subseteq G$ for some $n_0 \in \mathbb{N}$. Let $U = \bigcap_{i=1}^{n_0} U_i$ and $V = \bigcap_{i=1}^{n_0} V_i$. Then, $f(U, V) \subseteq G$. This proves joint continuity of f at each point of $D \times \{y_0\}$. □

The above results enable us to state the following.

Corollary 1. *A topological space Y is a Weston-space with respect to every weakly developable completely regular spaces if and only if it is a Weston-space with respect to $\mathbb{R}$.*

In [18, Theorem 5.1], the authors have shown that if a topological space Y is a Weston-space with respect to $\mathbb{R}$, then for each Baire space X, every KC-function $f : X \times Y \to \mathbb{R}$ is quasi-continuous. The following result shows that one can replace $\mathbb{R}$ with any topological space.

Theorem 6. *Let X be a Baire space, Y a Weston-space with respect to a topological space Z, and let $f : X \times Y \to Z$ be a KC-function. Then, f is strongly quasi-continuous with respect to second variable at each point of $X \times Y$.*

Proof. Let $(x_0, y_0) \in X \times Y$ be an arbitrary point. We show that f is strongly quasi-continuous at (x_0, y_0) with respect to the second variable. Let G be a neighborhood of $f(x_0, y_0)$ and U a neighborhood of x_0. By the definition,

$$C_{y_0}(f) \cap U \neq \varnothing.$$

Let $x_1 \in C_{y_0}(f) \cap U$. The continuity of f at (x_1, y_0) implies that there exist a neighborhood $U_1 \subseteq U$ of x_1 and V of y_0 such that $f(U_1 \times V) \subseteq G$. This proves our claim. □

Definition 2. Let $\mathcal{B}$ be the class of all Baire spaces and let $\mathcal{B}^*$ denote the class of all topological spaces Y such that the product $X \times Y$ is a Baire space for every $X \in \mathcal{B}$.

The following result improves [18, Theorem 6.1].

Theorem 7. *Let $X_1, \dots, X_{n+1}, Z$ be topological spaces such that $X_1 \in \mathcal{B}$ and $X_2, \dots, X_n \in \mathcal{B}^*$, $X_2, \dots, X_{n+1}$ be Weston-spaces with respect to Z. Suppose that $f : X_1 \times \dots \times X_{n+1} \to Z$ is separately continuous. Then, f is strongly quasi-continuous with respect to last variable at each point of $X_1 \times \dots \times X_{n+1}$.*

Proof. We prove the result by induction. If $n = 1$, X_1 is a Baire space and $f : X_1 \times X_2 \to Z$ is a KC-function, since X_2 is a Weston-space with respect to Z, by Theorem 6, f is strongly quasi-continuous with respect to second variable. Suppose that the assertion is true when the number of spaces equals to n. Let $X = X_1 \times \dots \times X_n$ and $Y = X_{n+1}$. By the induction assumption for any fixed $y \in Y$, the function $g_y : X \to Z$ defined by $g_y(x_1, \dots, x_n) = f(x_1, \dots, x_n, y)$ is quasi-continuous. Since $f_x : Y \to Z$ is continuous at each point of $x \in X$, $f : X \times Y \to Z$ is a KC-function. Since X is a Baire space and Y is a Weston--space with respect to Z, by Theorem 6, $f : X \times Y \to Z$ is strongly quasi--continuous with respect to the second variable on $X \times Y$. □

Corollary 2. *Under conditions of Theorem 7, for each $y \in X_{n+1}$, there is a dense G_δ subset D of $X_1 \times \dots \times X_n$ such that f is continuous at each point of $D \times \{y\}$.*

Proof. Let $X = X_1 \times \dots \times X_n$, then X is Baire. Thanks to Theorem 7, $f : X \times X_{n+1} \to Z$ is a KC-function.

Since X_{n+1} is a Weston-space with respect to Z, the result follows. □

The above result suggests to study joint quasi-continuity of separately quasi-continuous functions. In 1961, Martin [15] proved the following result.

Theorem 8. *Let X be a Baire space, Y a second countable, and Z a metric space. If $f : X \times Y \to Z$ is separately quasi-continuous, then it is jointly quasi-continuous.*

In [19], the authors introduced the following extension of separate quasi-continuity for two variable functions.

Definition 3. Let X, Y and Z be topological spaces, a function $f : X \times Y \to Z$ is called horizontally quasi-continuous with respect to the second variable at (x_0, y_0) if for each neighborhood W of $f(x_0, y_0)$ in Z and for each product of open sets $U \times V \subseteq X \times Y$ containing (x_0, y_0), there are a nonempty open set $U_1 \subseteq U$ and a point $y_1 \in V$ such that $f(U_1 \times \{y_1\}) \subseteq W$. The function f is called horizontally quasi-continuous if it is horizontally quasi-continuous at each point of $X \times Y$.

They also established the following extension of Martin's theorem.

Theorem 9. [19, Theorem 3] *Suppose that X is a Baire space, Y satisfies the second axiom of countability, and Z is a completely regular space. If a horizontally quasi-continuous function $f : X \times Y \to Z$ is quasi-continuous with respect to the second variable, then f is jointly quasi-continuous.*

Definition 4. Let $\mathcal{G}$ be a collection of nonempty open sets in a topological space. Then, $\mathcal{G}$ is called a pseudo-base (or π-base) for this space if any nonempty open set contains some member of $\mathcal{G}$.

Note that the Stone-Čech compactification of the natural numbers has a countable dense set of isolated points, and thus has a countable pseudobase. Clearly, this space is not second countable. So, the class of spaces which have a countable pseudo-base is larger than the class of second countable spaces.

The second author in [22] obtained the following extension of Theorem 9.

Theorem 10. [22, Theorems 6 and 11] *Let X be a Baire space, let Y either be a W-space or have a countable π-base, and let Z be a regular space. If $f : X \times Y \to Z$ is a horizontally quasi-continuous function which is quasi-continuous with respect to the second variable, then f is jointly quasi-continuous.*

The above result enables us to state the following, which improves the results of Lee-Piotrowski [13, Theorem 1].

Theorem 11. *Let X_1 be a Baire space, $X_2, \ldots, X_n$ the elements of $\mathcal{B}^*$ with countable π-bases, and let Z be regular. Suppose that $f : X_1 \times \cdots \times X_n \times Y \to Z$ is quasi-continuous with respect to $X_1, \ldots, X_n$ and continuous with respect to Y.*

Put $X = X_1 \times \cdots \times X_n$. Then, under one of the following conditions, $f : X \times Y \to Z$ has Weston property.

(1) *Y is a W-space and Z is weakly developable.*

(2) *Y is a Weston-space with respect to Z.*

Proof. Let $y_0 \in Y$. By applying Theorem 10 $(n-1)$-times, we see that the function $f_{y_0} : X_1 \times \cdots \times X_n \to Z$, defined by

$$f_{y_0}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, y_0) \quad (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n,$$

is quasi-continuous. Thus, $f : X \times Y \to Z$ is a KC-function. If Y is a weakly developable W-space, the result follows from Theorem 3, while if Y has Weston property, the result follows from the definition. □

Definition 5. Let X be a topological space and $\mathcal{F}$ be a family of nonempty closed and separable subspaces of X. Then, $\mathcal{F}$ is called rich [14] if the following conditions are satisfied:
(i) For every separable subspace Y of X, there exists an $F \in \mathcal{F}$ such that $Y \subseteq F$.
(ii) For every increasing sequence $\{F_n\}$ in $\mathcal{F}$, $\overline{\bigcup_{n \geq 1} F_n} \in \mathcal{F}$.

Corollary 3. *Let X_1 be a Baire space, $X_2, \ldots, X_n$ be W-spaces which have a rich family of Baire spaces, and let Z be regular. Suppose that $f : X_1 \times \cdots \times X_n \times Y \to Z$ is quasi-continuous with respect to $X_1, \ldots, X_n$ and continuous with respect to Y. Put $X = X_1 \times \cdots \times X_n$. The under one of the following conditions, $f : X \times Y \to Z$ has Weston property.*

(1) *Y is a weakly developable W-space.*

(2) *Y is a Weston-space with respect to Z.*

Proof. By [14, Theorem 4], if a W-space has a rich family of Baire spaces, its product with every Baire space is again Baire. So, by induction, one can show that $X_1 \times \cdots \times X_k$ for each $k \leq n$ is Baire. Therefore, $X_2, \ldots, X_n \in \mathcal{B}^*$. The result of the proof is similar to the proof of Theorem 11. □

3. Continuity of functions with closed graph

We start this section by recalling the following definition.

Definition 6. A topological space Y is called $w\Delta$-space [8] if there exists a sequence $\{\mathcal{G}_n\}$ of open covers of Y such that for each $y \in Y$ if $y_n \in st(y, \mathcal{G}_n)$ for each $n \in \omega$, the sequence $\{y_n\}$ has a cluster point.

Holá and Piotrowsi [11, Theorem 5.3] obtained the following result.

Theorem 12. *Let X be a first countable topological space and Y be a $w\Delta$-space. Let $f : X \to Y$ be a function with a closed graph. Then, the set $C(f)$ of continuity points of f is a G_δ set in X.*

In order to improve the above result, we need the following topological game, which was introduced by Gruenhage in [8].

The topological game $G(X, x_0)$

Let X be a topological space and $x_0 \in X$. The topological game $G(X, x_0)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows. In step $n \geq 1$, $\mathcal{O}$ selects a neighborhood H_n of x_0 and $\mathcal{P}$ responds by choosing a point $x_n \in H_n$. We say, $\mathcal{O}$ wins the game $g = (H_n, x_n)_{n\geq 1}$ if $x_n \to x_0$. If

$$g_1 = (H_1, x_1), \ldots, g_n = (H_1, x_1, \ldots, H_n, x_n)$$

are the first n moves of some play (of the game), we call g_n the nth partial play of the game. A strategy s for the player $\mathcal{O}$ is a rule which assignes a neighborhood H_{n+1} of x_0 to g_n. This strategy is called winning for $\mathcal{O}$ if he/she wins all plays provided that his/her moves are according to the strategy s. Similarly, one can define a strategy for the player $\mathcal{P}$. We call $x_0 \in X$ a W-point in X if $\mathcal{O}$ has a winning strategy in the game $G(X, x_0)$. A space X in which each point of X is a W-point is called a W-space.

Clearly, every first countable space is a W-space, however, the converse is not true in general [14, Example 2.7]. Therefore, the following result improves Theorem 12.

Theorem 13. *Let X be a W-space and Y a $w\Delta$-space. If $f : X \to Y$ has a closed graph, the set $C(f)$ of continuity points of f is a G_δ-set in X.*

Proof. Let $\{\mathcal{G}_n : n \in \mathbb{N}\}$ satisfy the conditions of previous definition. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in X : f(U) \subseteq V \text{ for some neighborhood } U \text{ of } x \text{ and } V \in \mathcal{G}_n\}.$$

Clearly, each A_n is open and $C(f) \subseteq \bigcap_{n=1}^{\infty} A_n$. Suppose that $x \in \bigcap_{n=1}^{\infty} A_n \setminus C(f)$. Then, there is a neighborhood V of $f(x)$ such that for each neighborhood U of x, $f(U) \nsubseteq V$. We inductively define a strategy t for the player $\mathcal{P}$ in $\mathcal{G}(X, x)$ as follows. Let H_1 be the first move of the player $\mathcal{O}$. By our assumption, there is a neighborhood U_1 of x and $V_1 \in \mathcal{G}_1$ such that $f(U_1) \subseteq V_1$ but $f(U_1 \cap H_1) \setminus V \neq \varnothing$. The player $\mathcal{P}$ chooses a point $x_1 \in H_1 \setminus f^{-1}(V)$. In general, in step n, when $H_1, x_1, \ldots, H_n$ are selected, we find some neighborhood U_n of x and $V_n \in \mathcal{G}_n$ such that $f(U_n) \subseteq V_n$. By our assumption, $f(H_n \cap U_n) \nsubseteq V$. Let $t(H_1, \ldots, H_n)$ be a point $x_n \in H_n \cap U_n$ such that $f(x_n) \notin V$. Of course, $f(x_n) \in V_n \subseteq st(f(x), \mathcal{G}_n)$. Since X is a W-space, the strategy t is not a winning one for the player $\mathcal{P}$. Hence, there is a t-play $g = (H_n, x_n)$ which is won by the player $\mathcal{O}$. It follows that $x_n \to x$ as $n \to \infty$. Since Y is $w\Delta$-space, the corresponding sequence $\{f(x_n)\}$ has a cluster point $y \in Y$. Therefore, (x, y) is a cluster point of the sequence $\{(x_n, f(x_n)\}$ in graph of f. Since f has a closed graph, $y = f(x)$. However, $\{f(x_n)\}$ is a sequence in the closed set $Y \setminus V$. So, $y = f(x) \notin V$. This contradiction proves that $C(f) = \bigcap_{n=1}^{\infty} A_n$ is a G_δ-set. □

Corollary 4. *Under conditions of* Theorem 13*, if f is also quasi-continuous on a dense subset of X and X is Baire, then the set of points of continuity of f is a dense G_δ subset of X.*

Proof. For each $n \in \mathbb{N}$, let A_n be the same as in the proof of Theorem 13. Since X is Baire, it suffices to show that each A_n is dense in X. Let U be an arbitrary non-empty open subset of X and choose a point x_0 of quasi-continuity of f in U. For each $n \in \mathbb{N}$, there is some $V_n \in \mathcal{G}_n$ such that $f(x_0) \in V_n$. By quasi-continuity of f at x_0, we can find a non-empty open subset $U_n \subseteq U$ such that $f(U_n) \subseteq V_n$. Therefore, $U_n \subseteq A_n \cap U$. This proves that each A_n is dense in X. □

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