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HAHN-BANACH AND SANDWICH THEOREMS FOR EQUIVARIANT VECTOR LATTICE-VALUED OPERATORS AND APPLICATIONS

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Dedicated to the loving memory
of Professor Domenico Candeloro and Professor Beloslav Riečan

ABSTRACT. We prove Hahn-Banach, sandwich and extension theorems for vector lattice-valued operators, equivariant with respect to a given group G of homomorphisms. As applications and consequences, we present some Fenchel duality and separation theorems, a version of the Moreau-Rockafellar formula and some Farkas and Kuhn-Tucker-type optimization results. Finally, we prove that the obtained results are equivalent to the amenability of G.

1. Introduction

The Hahn-Banach theorem plays a fundamental role in Functional Analysis and its developments and applications. By means of this theorem, it is possible to prove extension and sandwich-type theorems for operators and measures, results on separation of convex sets by means of hyperplanes and several other related theorems in abstract spaces (for a survey, see for example [11]).

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Among the consequences of this theorem we recall, for instance, the Fenchel-type duality theorem which has several applications in minimax theorems, in particular in Economics and stochastic programming (see, e.g., [3, 13, 37–39, 43, 44]) and image and/or document reconstruction (see, e.g., [2, 4, 9, 15, 16, 19, 20]).

Some other consequences of Hahn-Banach-type theorems deal with Subdifferential Calculus (see for instance [32] and the references therein), Operations Research and optimization problems with constraints, for example, Farkas and Kuhn-Tucker-type theorems (see also [24, 27, 31, 34]).

Another related topic is the study of invariant and equivariant linear functionals, invariant measures and amenable groups. For example, the group $SO(2,\mathbb{R})$ of all orthogonal 2×2-matrices whose determinant is equal to one (that is, the group of all rotations of $\mathbb{R}^2$) is amenable, while $SO(n,\mathbb{R})$ is not amenable for all $n \geq 3$ (see, e.g., [35]). In Probability and Statistics, there are several types of translation invariant or equivariant functionals (for example, deviation risk and acceptability functionals, respectively), invariant statistic models, invariant testing problems and equivariant estimators on the mean vector in Multivariate Analysis of Variance (see for instance [21, 36] and the references therein). There are some other recent studies of invariance and equivariance, for example, in applications to Machine Learning (see, e.g., [17, 26, 29]).

Observe that Hahn-Banach, sandwich and extension theorems for operators or measures, invariant or equivariant with respect to a given group G of homomorphisms are always valid if and only if G is amenable.

In the literature, many studies on these topics have been extended to the context of partially ordered space-valued operators and measures. It is often advisable to consider ordered vector space-valued functionals in order to investigate operators or probability measures which can depend not only on the considered event, but also on the time and on the state of knowledge, as well as conditional expected values, which can be viewed as operators taking values in the space L^1 of all integrable functions. For a related literature, see also [5–8, 10–12, 14, 35, 36, 40–42, 45].

In this paper, we extend to linear and equivariant vector lattice-valued operators the earlier results proved in [23, 47–49] in the linear case and in [6] in the invariant setting on Hahn-Banach, sandwich and extension theorems, Fenchel-type duality theorems, Moreau-Rockafellar formula, subdifferential calculus, Farkas and Kuhn-Tucker theorems on convex optimization under constraints. Furthermore, we prove the equivalence of given theorems with amenability. Some theorems of this kind for invariant operators were given, for instance, in [5–7, 14, 41, 42].

2. Preliminaries

Let X be a real vector space. An *affine combination* of elements $x_1, x_2, \dots, x_n$ of X is any linear combination of the form

$$\sum_{i=1}^{n} \lambda_i x_i, \quad \text{with} \quad \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1.$$

An *affine manifold* of X is a nonempty subset of X closed under affine combinations.

Let $\emptyset \neq D \subset X$. We denote by $\mathrm{span}(D)$ (resp. $\mathrm{span}_{\mathrm{aff}}(D)$) the smallest linear subspace (resp. affine manifold) of X which contains D.

A point $x_0 \in D$ is an *algebraic relative interior point of* D if and only if for each $x \in \mathrm{span}_{\mathrm{aff}}(D)$ there is $\lambda_0 > 0$ with $(1-\lambda)x_0 + \lambda\, x \in D$ whenever $\lambda \in [-\lambda_0, \lambda_0]$. We denote by $\mathrm{int}(D)$ the set of all algebraic relative interior points of D. It is not difficult to check the following property, which will be useful later.

Proposition 2.1. *It is* $0 \in \mathrm{int}(D)$ *if and only if for every* $x \in$ *span*(D) *there exists a positive real number* λ_x *such that* $\lambda\, x \in D$ *for all* $\lambda \in [-\lambda_x, \lambda_x]$.

A nonempty subset D of a real vector space X is said to be *convex* if and only if $\lambda x_1 + (1-\lambda)x_2 \in D$ for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$.

Given two real vector spaces X, Y, we say that a function $T \in Y^X$ is *affine* if and only if there exist a linear function $L \in Y^X$ and an element $y_0 \in Y$ with $T(x) = L(x) + y_0$ for every $x \in X$.

If Y is partially ordered and $D \subset X$ is convex, we say that a function $U \in Y^D$ is *convex* on D if and only if $U\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda U(x_1) + (1-\lambda)U(x_2)$ for every x_1, $x_2 \in D$ and $\lambda \in [0,1]$.

Let R, S be two vector lattices, $R^+ = \{y \in R : y \geq 0\}$, $S^+ = \{y \in S : y \geq 0\}$. An *$S$-valued functional* (*on* R) is any map $\varphi : R \to S$. A *real functional* is an $\mathbb{R}$-valued functional.

An S-valued functional φ is said to be *positive* if and only if $\varphi(r) \in S^+$ whenever $r \in R^+$. A *positive order continuous S-valued functional* is a positive S-valued functional φ such that, for each upward directed increasing net $(r_\lambda)_{\lambda \in \Lambda}$ of elements of R with

$$R \ni r = \bigvee_{\lambda} r_\lambda, \quad \text{it is} \quad \varphi(r) = \bigvee_{\lambda} \varphi(r_\lambda).$$

Now, we recall some notions and properties on vector lattices and their duals, which will be useful later (for a related literature, see , e.g., [1, 18, 25, 33, 46]).

The *algebraic dual* of R is the ordered vector space R^* whose elements are the linear real functionals on R, where the addition and the scalar multiplication are those inherited by $\mathbb{R}^R$, and whose order relation $\le$ is defined by setting $\varphi_1 \le \varphi_2$ if and only if $\varphi_2 - \varphi_1$ is a positive linear real functional on R. The *order dual* (resp. *order continuous dual*) of R is the ordered vector space $R^\sim$ (resp. $R^\times$) whose elements are the linear real functionals on R which can be expressed as a difference between two positive (order continuous) linear real functionals on R, where the addition, the scalar multiplication and the order relation are those inherited by R^*. Note that a linear real functional y belongs to $R^\sim$ if and only if it maps order bounded subsets of R into order bounded subsets of $\mathbb{R}$. The functionals of $R^\sim$ (resp. $R^\times$) are said to be *order bounded* (resp.*order continuous*).

Now, let $Y = R^\sim$ (resp. $R^\times$), and $c : R \to Y^*$ be the *evaluation map* defined by $c(r)(y) = y(r)$, $r \in R$, $y \in Y$. Note that c is a lattice homomorphism from R into $Y^\times$. A vector lattice R is a $\sim$*-space* (resp. π*-space*) if and only if the evaluation map $c : R \to Y^\times$ is one-to-one.

A subset I of R is said to be *order dense* in R if and only if for each $r \in R^+$, $r \neq 0$, there is $u \in I$ with $0 \le u \le r$ and $u \neq 0$. A subspace I of R is called an *ideal* of R if and only if $r_1, r_2 \in I$, $r_3 \in R$ and $r_1 \le r_3 \le r_2$ imply $r_3 \in I$.

Let Υ be the class of all order dense ideals of R, and set $\Phi = \bigcup_{I \in \Upsilon} I^\times$. Then, a function $\varphi \in \mathbb{R}^R$ belongs to Φ if and only if there exists $I \in \Upsilon$ such that φ is an order continuous linear functional on I. If $\varphi \in \Phi$, we denote by I_φ its domain. Given $\varphi \in \Phi$ and $r \in R^+$, set $|\varphi|(r) = \sup\{|\varphi(s)| : 0 \le s \le r\}$, and for $r \in R$, put $|\varphi|(r) = |\varphi|(r^+) - |\varphi|(r^-)$.

Let $D_\varphi = \{r \in R : |\varphi|(r) < +\infty\}$, then D_φ is the largest order dense ideal of R on which $|\varphi|$ can be extended finitely.

Given two elements $\varphi_1, \varphi_2 \in \Phi$, we say that $\varphi_1 \approx \varphi_2$ if and only if the set $\{r \in R : \varphi_1(r) = \varphi_2(r)\}$ contains an order dense ideal of R. Since Υ is closed with respect to finite intersections, we get that $\approx$ is an equivalence relation. We denote by the symbol $[\varphi]$ a generic class of equivalence with respect to $\approx$.

Let $R^\rho = \Phi/\approx$. Given $[\varphi_i] \in R^\rho$, $i = 1, 2, 3$, we say that $[\varphi_1] + [\varphi_2] = [\varphi_3]$ if and only if there exist $\varphi_i' \in [\varphi_i]$, $i = 1, 2, 3$, such that the set $\{r \in R : \varphi_1'(r) + \varphi_2'(r) = \varphi_3'(r)\}$ contains an order dense ideal. It is not difficult to see that this definition is independent of the choice of φ_i'. Similarly, it is possible to define $\alpha \cdot [\varphi]$, for each $[\varphi] \in R^\rho$ and $\alpha \in \mathbb{R}$. Moreover, we say that $[\varphi] \ge 0$ if and only if there is a $\varphi' \in [\varphi]$ such that the set $\{r \in R : \varphi'(r) \ge 0\}$ contains an order dense ideal of R, and $[\varphi_1] \le [\varphi_2]$ if and only if $[\varphi_2] - [\varphi_1] \ge 0$. Observe that for each element $[\varphi] \in R^\rho$ there exists $\varphi_* \in [\varphi]$ such that for every $\varphi' \in [\varphi]$ it is $I_{\varphi'} \subset I_{\varphi_*}$ and $\varphi' = \varphi_*$ on $I_{\varphi'}$. So, given $\varphi \in \Phi$, we say that φ_* is the *maximal element* determined by φ.

We will identify φ with φ_*. If $y = [\varphi] \in R^\rho$, then we put $\mathcal{I}_y = D_{\varphi_*}$. So, by means of this identification, y can be viewed as a linear functional on $\mathcal{I}_y$. It is possible to see that R^ρ, endowed with these structures, is a vector lattice.

Note that, in the space R^ρ, the lattice supremum can be identified with the pointwise supremum on an order dense ideal of R. Indeed, we get the following

Proposition 2.2 (see also [1, Theorems 1.67 (b) and 1.73], [33, Theorem 1.5]). *The spaces $R^\sim$ and $R^\times$ are Dedekind complete vector lattices. In $R^\sim$ and $R^\times$, the lattice supremum of any upward directed net of positive elements coincides with its pointwise supremum along the set R^+.*

Moreover, if $([\varphi_\lambda])_{\lambda\in\Lambda}$ is an upward directed net of positive elements of R^ρ, bounded from above by an element $[\varphi_0] \in R^\rho$, and $\varphi'_\lambda \in [\varphi_\lambda]$, $\lambda \in \Lambda$, $\varphi'_0 \in [\varphi_0]$, then there exists $\varphi' \in (D_{\varphi'_0})^\times$ with $\bigvee_{\lambda\in\Lambda} \varphi'_\lambda = \varphi'$, that is $\sup_{\lambda\in\Lambda} \varphi'_\lambda(r) = \varphi'(r)$ for each $r \in D_{\varphi'_0} \cap R^+$. If φ_* is the maximal element determined by φ', then $[\varphi_*]$ is the lattice supremum of the net $([\varphi_\lambda])_{\lambda\in\Lambda}$. Therefore, the space R^ρ is Dedekind complete.

Now, we define the evaluation map $c : R \to R^{\rho\rho} = (R^\rho)^\rho$ as follows. For each $r \in R^+$, set $I_r = \{y \in R^\rho : r \in \mathcal{I}_y\}$. For every $r \geq 0$, $y \in I_r$ and $\varphi \in y$, let $c(r)(y) = \varphi(r)$, and for each $r \in R$ set $c(r) = c(r^+) - c(r^-)$. Observe that I_r is an order dense ideal of R^ρ, the definition of c does not depend on the choice of $\varphi \in y$, $c(r)$ is an order continuous linear functional on I_r for every $r \in R^+$ and c is a lattice homomorphism. A vector lattice R is a *ρ-space* if and only if the evaluation map c is one-to-one.

Now, we recall the following properties of ρ-spaces.

Proposition 2.3 (see also [33, Corollary 2.4]). *A ρ-space R is Dedekind complete if and only if R is an ideal of $R^{\rho\rho}$ and, in this case, R is order dense in $R^{\rho\rho}$.*

Proposition 2.4 (see also [33, Theorem 2.5]). *A vector lattice R is a ρ-space if and only if there exists an order dense ideal I of R such that $I^\times \neq \{0\}$.*

Example. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where μ is any positive countably additive measure. We say that μ is *non-atomic* or *diffused* if and only if for every $A \in \mathcal{A}$ with $\mu(A) > 0$ there exists $B \in \mathcal{A}$ with $B \neq A$, $B \subset A$ and $\mu(B) > 0$. The measure μ is said to be *semi-finite* if and only if $\mu(\Omega) < +\infty$ or every element of $\mathcal{A}$ of infinite μ-measure contains an element of $\mathcal{A}$ of finite strictly positive μ-measure. We say that μ is *localizable* if and only if μ is semi-finite and for each class $\mathcal{E} \subset \mathcal{A}$ there exists $H \in \mathcal{A}$ such that:

1) $\mu(E \setminus H) = 0$ for all $E \in \mathcal{E}$, and
2) if $A \in \mathcal{A}$ and $\mu(E \setminus A) = 0$ for each $E \in \mathcal{E}$, then $\mu(H \setminus A) = 0$.

Let $L^0(\Omega, \mathcal{A}, \mu)$ be the set of all real-valued μ-measurable functions on Ω, with the identification of μ-null sets, and $L^\infty(\Omega, \mathcal{A}, \mu)$ be the subspace of $L^0(\Omega, \mathcal{A}, \mu)$ consisting of all μ-essentially bounded functions (a function f is *μ-essentially bounded* if and only if there exists a real number $C > 0$ with $|f(\omega)| \leq C$ for all $\omega \in \Omega$, with the exception of a μ-null set). Observe that $L^\infty(\Omega, \mathcal{A}, \mu)$ is an order dense ideal of $L^0(\Omega, \mathcal{A}, \mu)$, and $\big(L^\infty(\Omega, \mathcal{A}, \mu)\big)^\times \neq \{0\}$ (see [18, Corollary 5.3.11]). Thus, by Proposition 2.4, $L^0(\Omega, \mathcal{A}, \mu)$ is a ρ-space. However, if μ is non-atomic, then $L^0(\Omega, \mathcal{A}, \mu)$ is not a $\sim$-space, and a fortiori neither a π-space (see also [18, 46]).

If μ is semi-finite, then $L^0(\Omega, \mathcal{A}, \mu)$ is Dedekind complete if and only if μ is localizable (see [25, 241G]). For example, the usual Lebesgue measure is non-atomic and localizable (see also [25, 211G, 211L, 211M]). Moreover, if $\mathcal{A}$ is the countable-cocountable σ-algebra of $\mathbb{R}$ and μ is the restriction of the counting measure to $\mathcal{A}$, then μ is semi-finite, but not localizable (see [25, 211Y]). Thus, the associated space $L^0(\mathbb{R}, \mathcal{A}, \mu)$ is a ρ-space, but not Dedekind complete.

Let G be a group, R be a vector lattice, and $\mathcal{P}(G)$ be the class of all subsets of G. A finitely additive measure $\nu : \mathcal{P}(G) \to R$ is a *G-invariant mean* if and only if $\nu(G) \in R^+ \setminus \{0\}$ and $\nu(\{gh : g \in E\}) = \nu(E)$ whenever $E \subset G$ and $h \in G$. We say that G is *amenable* if and only if there exists a G-invariant mean $\nu : \mathcal{P}(G) \to [0, 1]$, with $\nu(G) = 1$.

From now on, let X be a Dedekind complete vector lattice, $R \subset X$ be a Dedekind complete ρ-space, $G \subset X^X$ be a group of linear positive X-functionals on X. A set $\emptyset \neq D \subset X$ is said to be *G-invariant* if and only if $hx \in D$ whenever $x \in D$ and $h \in G$. Note that, in this case, we get $hx \in D$ if and only if $x \in D$, since G is a group. We will assume that R is G-invariant. A function $L \in R^X$ is said to be *G-invariant* (resp.*G-equivariant*) if and only if $L(hx) = L(x)$ (resp. $h^{-1}\big(L(hx)\big) = L(x)$ or equivalently $h\big(L(x)\big) = L(hx)$) for all $x \in X$ and $h \in G$.

The following property will be useful later.

Proposition 2.5. *For every $h \in G$ and for each family $(r_\xi)_{\xi \in \Xi}$ in X it is*

$$h\left(\bigvee_\xi r_\xi\right) = \bigvee_\xi h(r_\xi), \tag{1}$$

and

$$h\left(\bigwedge_\xi r_\xi\right) = \bigwedge_\xi h(r_\xi). \tag{2}$$

P r o o f. We prove only (1), since the proof of (2) is analogous. Pick arbitrarily $h \in G$. Since h is positive, it is $h(\bigvee_\xi r_\xi) \geq h(r_\xi)$ for each $\xi \in \Xi$. Taking the supremum, we obtain

$$h\left(\bigvee_\xi r_\xi\right) \geq \bigvee_\xi h(r_\xi). \tag{3}$$

Moreover, as h is chosen arbitrarily and G is a *group*, the inequality in (3) holds also when h is replaced by h^{-1}. Hence, we get

$$h^{-1}\left(\bigvee_\xi h(r_\xi)\right) \geq \bigvee_\xi h^{-1}\big(h(r_\xi)\big) = \bigvee_\xi r_\xi. \tag{4}$$

By applying h in (4), we obtain

$$\bigvee_\xi h(r_\xi) = h\left(h^{-1}\left(\bigvee_\xi h(r_\xi)\right)\right) \geq h\left(\bigvee_\xi r_\xi\right). \tag{5}$$

Thus, (1) follows from (3) and (5). □

From now on, we denote by $l_b(G, R)$ the space of all bounded R-valued functions defined on G, by $\mathcal{L}(X, R)$ the set of all linear R-functionals on X, by $\mathcal{L}_{\text{equiv}}(X, R)$ the set of all G-equivariant R-functionals of $\mathcal{L}(X, R)$, and by $\mathcal{L}_{+,\,\text{equiv}}(X, R)$ the set of all positive R-functionals belonging to $\mathcal{L}_{\text{equiv}}(X, R)$.

Given $0 \neq L \in \mathcal{L}(X, R)$, $0 \neq L' \in \mathcal{L}(X, R)$ and u_0 in R, set

$$H = \{(x, y) \in X \times R : L(x) + L'(y) = u_0\}. \tag{6}$$

Observe that the set H defined in (6) is empty or an affine manifold of $X \times R$ (see also [28]).

If A, B are two nonempty subsets of $X \times R$ and $H \neq \emptyset$ is as in (6), then we say that H *separates* A and B if and only if $A \subset H^+$ and $B \subset H^-$, where

$$\begin{aligned} H^+ &= \{(x, y) \in X \times R : L(x) + L'(y) \leq u_0\}, \\ H^- &= \{(x, y) \in X \times R : L(x) + L'(y) \geq u_0\}. \end{aligned}$$

Let $U : D(U) \to R$, $V : D(V) \to R$ be two convex and G-equivariant functions, where $D(U)$, $D(V)$ are convex and G-invariant subsets of X. The *G-equivariant conjugate* of U is the function U^c defined by

$$U^c(L) = \bigvee\{L(x) - U(x) : x \in D(U)\}, \quad L \in D(U^c), \quad \text{where}$$

$$\emptyset \neq D(U^c) = \{L \in \mathcal{L}_{\text{equiv}}(X, R) : \bigvee\{L(x) - U(x) : x \in D(U)\} \text{ exists in } R\}.$$

If $x_0 \in D(U)$, then we call *G-equivariant subdifferential* (or *subdifferential*) *at* x_0 the set $\partial_{\text{equiv}} U(x_0)$ defined as

$$\partial_{\text{equiv}} U(x_0) = \{L \in \mathcal{L}_{\text{equiv}}(X, R) : L(x) - L(x_0) \leq U(x) - U(x_0)\}.$$

An element $L \in \partial_{\mathrm{equiv}} U(x_0)$ is called *(G-equivariant) subgradient of U at x_0*.

In our version of the duality theorem, we will consider the following problems (see also [47]).

2.7): Find $r = \bigwedge\{U(x) + V(x) : x \in D(U) \cap D(V)\}$ in R.

2.8): Find $s = \bigvee\{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\}$ in R (where $D(U^c) \cap D(V^c) \neq \emptyset$).

3. The main results

We begin with recalling the construction of an integral for functions with values in ρ-spaces, with respect to finitely additive real-valued measures (see also [5]).

Let $f \in l_b(G, R)$. There are $r_1, r_2 \in R$ with $0 \le [f(g)]^+ \le r_1$, $0 \le [f(g)]^- \le r_2$ for all $g \in G$. By identifying r_i with $c(r_i)$, $i = 1, 2$, we find an order dense ideal I of R^ρ such that $r_1|_I$, $r_2|_I$ are two linear positive order continuous real-valued functionals.

Now, fix $g \in G$. There exists an order dense ideal K_g of R^ρ with $0 \le [f(g)]^+(y) \le r_1(y)$ for every $y \in K_g, y \ge 0$. Without loss of generality, we can suppose $K_g \subset I$. Let $\overline{r_1}(y) = \sup\{r_1(u)\colon 0 \le u \le y, u \in K_g\}$, $\overline{[f(g)]^+}(y) = \sup\{f(g)(u)\colon 0 \le u \le y, u \in K_g\}$, for every $y \in R^\rho$, $y \ge 0$. It is possible to see that $\overline{r_1}(y_1 + y_2) = \overline{r_1}(y_1) + \overline{r_1}(y_2)$, $\alpha\,\overline{r_1}(y_1) = \overline{r_1}\ \alpha(y_1)$, and $[y_1 \le y_2] \Rightarrow [\overline{r_1}(y_1) \le \overline{r_1}(y_2)]$ for every $y_1, y_2 \in R^\rho, y_1, y_2 \ge 0$, and for each $\alpha \in \mathbb{R}_0^+$; similar results hold for $\overline{[f(g)]^+}$. Moreover, it is $0 \le r_1(u) \le r_1(y)$ for all $y \in I$, $y \ge 0$, and for every $u \in K_g$, $0 \le u \le y$. Taking the supremum, we obtain $0 \le \overline{r_1}(y) \le r_1(y)$. As $0 \le [f(g)]^+(u) \le r_1(u)$ for each $u \in K_g$, $u \ge 0$, then $0 \le \overline{[f(g)]^+}(y) \le \overline{r_1}(y) \le r_1(y)$ for any $y \in R^\rho$, $y \ge 0$. Thus, $\overline{[f(g)]^+}$ is an additive, monotone, positively homogeneous real functional on the set of all positive elements of I, and then, it can be extended linearly on I. We denote this extension by $\overline{[f(g)]^+}$ again. Since r_1 is positive order continuous on I, then so is $\overline{[f(g)]^+}$. Analogously, we can construct $\overline{[f(g)]^-}$. Note that $f(g)$ is equivalent (with respect to the relation $\approx$) to $\overline{[f(g)]^+} - \overline{[f(g)]^-}$. With this identification, $f(g)$ is an order continuous linear real-valued functional on I for every $g \in G$, and $-r_2(y) \le f(g)(y) \le r_1(y)$ for every $y \in I$, $y \ge 0$ and $g \in G$. Now, let $\mu : \mathcal{P}(G) \to [0, 1]$ be any finitely additive measure. For every $y \in I$, let

$$J_f(y) = (L) \int_G f(g)(y)\,\mathrm{d}\mu(g) \tag{7}$$

be the Lebesgue integral of the function $g \mapsto f(g)(y)$ with respect to μ. Note that $J_f(y)$ is well-defined for all $f \in l_b(G, R)$, since μ is defined on *all* subsets of G. It is not difficult to check that J_f is a linear real-valued functional on I.

Since $-r_2$ and r_1 are order continuous on I, then so is J_f. By identifying J_f with its class of equivalence with respect to the relation $\approx$, J_f will be an element of $R^{\rho\rho}$. It is not difficult to see that the map $f \mapsto J_f$ is a positive linear $R^{\rho\rho}$-valued functional. As R is Dedekind complete, by Proposition 2.4, R can be identified with an order dense ideal of $R^{\rho\rho}$, and thus, $J_f \in c(R)$ for each $f \in l_b(G, R)$.

Set $\mathcal{J}_f = c^{-1}(J_f)$, then $\mathcal{J}_f$ will be called the *ρ-integral* of f (with respect to μ), and we write

$$(\rho) \int_G f(g) \,\mathrm{d}\mu(g) = \mathcal{J}_f. \tag{8}$$

It is possible to see that $\mathcal{J}_f$ is well-defined and the map $f \mapsto \mathcal{J}_f$ is a positive linear R-valued functional.

The next lemma will be useful in proving the existence of G-equivariant R--functionals.

Lemma 3.1. *Let $\mu : \mathcal{P}(G) \to [0, 1]$ be a finitely additive measure with $\mu(G) = 1$, $f \in l_b(G, R)$, $h \in R^R$ be an affine order bounded R-valued functional, and $\mathcal{J}_f$ be as in* (8)*. Set $(hf)(g) = h\big(f(g)\big)$, $g \in G$. Then, $h(\mathcal{J}_f) = \mathcal{J}_{hf}$.*

P r o o f. Since h maps order bounded sets into order bounded sets, the union of the ranges of f and hf is order bounded. By proceeding analogously as in the construction of the integral in (7), we find an order dense ideal I_h of R^ρ such that $f(g)_{|I_h}$ and $(hf)(g)_{|I_h}$ are linear order continuous real functionals for every $g \in G$.

For every $r \in R$ and $y \in I_h$, define $h_y \in \mathbb{R}^{\mathbb{R}}$ by $h_y\big(y(r)\big) = h(r)(y)$. Since h is affine, then h_y is affine for all $y \in I_h$, and hence, there are a_y and $b_y \in \mathbb{R}$ with $h_y(x) = a_y x + b_y$ for all $x \in \mathbb{R}$. If $J_f(y)$ and $\mathcal{J}_f$ are as in (7) and (8), respectively, since $\mu(G) = 1$ we obtain

$$\begin{aligned} h(\mathcal{J}_f)(y) &= h_y\big(J_f(y)\big) = a_y\big(J_f(y)\big) + b_y \\ &= J_{a_y f}(y) + \mu(G)\, b_y \\ &= J_{a_y f}(y) + J_{\mathbf{b}_y}(y) \\ &= J_{a_y f + \mathbf{b}_y}(y) = \mathcal{J}_{hf}(y), \end{aligned} \tag{9}$$

where $\mathbf{b_y}$ denotes the constant function defined on G which assumes the value b_y. From (9) we deduce that $h(\mathcal{J}_f) = \mathcal{J}_{hf}$, that is the assertion. □

Remark 1. Observe that, when h is linear and positive, the assertion of Lemma 3.1 holds for any finitely additive measure $\mu : \mathcal{P}(G) \to [0, +\infty[$. Moreover, observe that in Lemma 3.1, the amenability of G is not required.

Now, we extend to the setting of equivariance some extension, sandwich, Hahn-Banach, Fenchel duality, Moreau-Rockafellar, Farkas and Kuhn-Tucker-type theorems which were proved in [6] in the case of invariant functionals. We begin with the following Hahn-Banach-type theorem, extending [49, Theorem 2.1]. Our technique is based on the existence of linear functionals, not necessarily equivariant, guaranteed by Hahn-Banach classical-like theorems, and on the ρ-integral by means of which it is possible to construct G-equivariant linear functionals.

Theorem 3.2. *Let $U : D(U) \to R$ be convex and G-equivariant, $R \subset D(U) \subset X$ be convex and G-invariant, $0 \in \mathrm{int}\big(D(U)\big)$ and $U(0) \geq 0$. Then, there exists $L \in \mathcal{L}_{\mathrm{equiv}}(X, R)$ with $L(x) \leq U(x)$ for any $x \in D(U)$.*

Proof. Let $X_1 = \mathrm{span}\big(D(U)\big)$. Observe that X_1 is G-invariant, because the elements of G are linear and $D(U)$ is G-invariant.

Pick arbitrarily $x \in X_1$. As $0 \in \mathrm{int}\big(D(U)\big)$, by Proposition 2.1 there is a positive real number λ_x with $\lambda x \in D(U)$ whenever $\lambda \in [-\lambda_x, \lambda_x]$, and hence $\lambda(gx) = g(\lambda x) \in D(U)$ for every $\lambda \in [-\lambda_x, \lambda_x]$ and $g \in G$. Without loss of generality, we can assume that $\lambda_x < 1$. For each $x \in X_1$, set

$$\begin{aligned} U^+(x) &= \bigwedge\Big\{\tfrac{1}{\lambda}U(\lambda x) : 0 < \lambda < 1, \lambda x \in D(U)\Big\}, \\ U^-(x) &= \bigvee\Big\{\tfrac{1}{\kappa}\big(-U(-\kappa x)\big) : 0 < \kappa < 1, -\kappa x \in D(U)\Big\}. \end{aligned} \tag{10}$$

We claim that $U^-(x) \leq U^+(x)$ for every $x \in X_1$. Indeed, since U is convex, for any λ, $\kappa \in (0,1)$ with $\lambda x, -\kappa x \in D(U)$, it is

$$0 \leq U(0) = U\left(\frac{1}{2}\lambda\kappa x - \frac{1}{2}\lambda\kappa x\right) \leq \frac{1}{2}U(\lambda\kappa x) + \frac{1}{2}U\left(-\lambda\kappa x\right),$$

and hence $-U(-\lambda\kappa x) \leq U(\lambda\kappa x)$. From this it follows that

$$\frac{1}{\kappa}\big(-U(-\kappa x)\big) = \frac{1}{\lambda\kappa}\big(-U(-\lambda\kappa x)\big) \leq \frac{1}{\lambda\kappa}U(\lambda\kappa x) = \frac{1}{\lambda}U(\lambda x). \tag{11}$$

Taking in (11) the infimum with respect to λ and the supremum with respect to κ, we deduce $U^-(x) \leq U^+(x)$, that is the claim.

By [49, Theorem 2.1] there is a linear function $L^* \in R^X$ (not necessarily G-equivariant) such that

$$\begin{aligned} &U^-(x) \leq L^*(x) \leq U^+(x) \qquad \text{for every} \qquad x \in X_1 \\ \text{and} \quad &L^*(x) \leq U^+(x) \leq U(x) \qquad \text{for any} \qquad x \in D(U). \end{aligned} \tag{12}$$

Since G is amenable, there is a G-invariant mean $\nu : \mathcal{P}(G) \to [0,1]$. For each $x \in X_1$, set

$$L'(x) = (\rho)\int_G g^{-1}\big(L^*(gx)\big)\,\mathrm{d}\nu(g). \tag{13}$$

We show that

3.2.1) L' is well-defined on X_1.

First of all, observe that, for every $g \in G$ and $x \in X$, we get that

$$gx \in D(U) \quad \text{if and only if} \quad x \in D(U), \text{ since } D(U) \text{ is } \quad G\text{-invariant}.$$

Moreover, thanks to Proposition 2.5, for every $g \in G$ and $x \in X_1$ we have

$$\begin{aligned} g\big(U^+(x)\big) &= g\Big(\bigwedge\Big\{\tfrac{1}{\lambda}U(\lambda x) : 0 < \lambda < 1, \lambda x \in D(U)\Big\}\Big) \\ &= \bigwedge\Big\{\tfrac{1}{\lambda}g\big(U(\lambda x)\big) : 0 < \lambda < 1, \lambda gx \in D(U)\Big\} \qquad (14) \\ &= \bigwedge\Big\{\tfrac{1}{\lambda}U(\lambda gx) : 0 < \lambda < 1, \lambda gx \in D(U)\Big\} = U^+(gx). \end{aligned}$$

Similarly as in (14), it is possible to see that $g\big(U^-(x)\big) = U^-(gx)$ for any $g \in G$ and $x \in X_1$. Thus, U^+ and U^- are G-equivariant. From this and (12), for each fixed $g \in G$ and $x \in X_1$, since g^{-1} is linear and positive, we get

$$U^-(x) = g^{-1}\big(U^-(gx)\big) \le g^{-1}\big(L^*(gx)\big) \le g^{-1}\big(U^+(gx)\big) = U^+(x). \qquad (15)$$

Taking the (ρ)-integral, since $\nu(G) = 1$, from (15) we deduce

$$\begin{aligned} U^-(x) &= (\rho)\int_G U^-(x)\,\mathrm{d}\nu(g) = (\rho)\int_G g^{-1}\big(U^-(gx)\big)\,\mathrm{d}\nu(g) \\ &\le (\rho)\int_G g^{-1}\big(L^*(gx)\big)\,\mathrm{d}\nu(g) = L'(x) \\ &\le (\rho)\int_G g^{-1}\big(U^+(gx)\big)\,\mathrm{d}\nu(g) \qquad (16) \\ &= (\rho)\int_G U^+(x)\,\mathrm{d}\nu(g) = U^+(x). \end{aligned}$$

Thus, $L'(x) \in R$ for every $x \in X_1$, and hence, L' is well-defined.

We now claim that

3.2.2) $L'(x) \le U(x)$ for every $x \in D(U)$.

Indeed, this follows from (12) and (16).

Now, we prove that

3.2.3) L' is G-equivariant on X_1.

Fix arbitrarily $x \in X_1$ and $h \in G$. Thanks to Lemma 3.1 applied to h^{-1}, the linearity of h^{-1} and the G-invariance of ν, we obtain

$$\begin{aligned} h^{-1}\big(L'(hx)\big) &= h^{-1}\left((\rho)\int_G g^{-1}\Big(L^*\big(g(hx)\big)\Big)\,\mathrm{d}\nu(g)\right) \\ &= (\rho)\int_G h^{-1}\left(g^{-1}\Big(L^*\big(g(hx)\big)\Big)\right)\mathrm{d}\nu(g) \\ &= (\rho)\int_G (gh)^{-1}\Big(L^*\big(g(hx)\big)\Big)\,\mathrm{d}\nu(gh) \\ &= (\rho)\int_G g^{-1}\big(L^*(gx)\big)\,\mathrm{d}\nu(g) = L'(x), \end{aligned} \tag{17}$$

getting the G-equivariance of L' on X_1.

Now, we prove that

3.2.4) L' is linear on X_1.

Pick arbitrarily $\alpha_1,\ \alpha_2 \in \mathbb{R}$ and $x_1,\ x_2 \in X_1$. By the linearity of L^*, of the g's and of the (ρ)-integral, we obtain

$$\begin{aligned} L'(\alpha_1\,x_1+\alpha_2\,x_2) &= (\rho)\int_G g^{-1}\Big(L^*\big(g(\alpha_1\,x_1+\alpha_2\,x_2)\big)\Big)\,\mathrm{d}\nu(g) \\ &= \alpha_1\,(\rho)\int_G g^{-1}\big(L^*(gx_1)\big)\,\mathrm{d}\nu(g) \\ &\quad + \alpha_2\,(\rho)\int_G g^{-1}\big(L^*(gx_2)\big)\,\mathrm{d}\nu(g) \\ &= \alpha_1\,L'(x_1)+\alpha_2\,L'(x_2). \end{aligned} \tag{18}$$

Let X_2 be an *algebraic complement* of X_1 in X, that is a subspace of X such that every element $x \in X$ can be expressed in a unique way as $x = x_1 + x_2$, where $x_i \in X_i$, $i = 1, 2$. Such a space X_2 does exist (see also [30, Theorem 7.3.3]).

Let us define $L : X \to R$ by $L(x) = L'(x_1)$. It is not difficult to check that L is well-defined and linear on X, since L' is linear on X_1. Note that $U^-(x_1) \le L'(x_1) = L(x) \le U^+(x_1)$ for any $x \in X$, and $L(x) = L'(x) \le U(x)$ for each $x \in D(U)$. Furthermore, for every $x \in X$, $x = x_1 + x_2$ and $h \in G$, we get $h\big(L(x)\big) = h\big(L'(x_1)\big) = L'(hx_1) = L(hx)$, since h is linear, X_1 and X_2 are G-invariant and L' is G-equivariant on X_1. Therefore, L is G-equivariant on X. This completes the proof. □

We now prove the following sandwich-type theorem, extending [6, Theorem 3.2] and [49, Sandwich Theorem 3.1] to the equivariance setting.

Theorem 3.3. *Let $U : D(U) \to R$, $V : D(V) \to R$ be two convex and G-equivariant functions, where*

$$R \subset D(U), D(V) \subset X \text{ are convex and } G\text{-invariant, } 0 \in \operatorname{int}\big(D(U) - D(V)\big), \tag{19}$$

and assume that

$$U(x) + V(x) \geq 0 \quad \text{for all} \quad x \in D(U) \cap D(V). \tag{20}$$

Then, there exist

$$L \in \mathcal{L}_{\text{equiv}}(X, R) \quad \text{and} \quad u_0 \in R$$

with

$$L(x) - u_0 \leq U(x) \quad \text{for any} \quad x \in D(U) \quad \text{and} \quad L(x') - u_0 \geq -V(x')$$

whenever

$$x' \in D(V).$$

Proof. Let $\mathcal{D} = D(U) - D(V)$. Note that $D(U) \cap D(V) \neq \emptyset$, since $0 \in \mathcal{D}$. Set

$$\begin{aligned} \mathcal{E} \;\;=\;\; & \{(x, y) \in X \times R : \text{ there are } x_1 \in D(U),\ x_2 \in D(V),\ z \in R^+ \\ & \text{with } x = x_1 - x_2,\ \ y = U(x_1) + V(x_2) + z\}. \end{aligned} \tag{21}$$

It is not difficult to see that the sets $\mathcal{D}$ and $\mathcal{E}$ are convex. Now, we claim that

$$\text{for every} \quad h \in G, (hx, hy) \in \mathcal{E} \quad \text{if and only if} \quad (x, y) \in \mathcal{E}. \tag{22}$$

Fix arbitrarily $(x, y) \in \mathcal{E}$ and $h \in G$, and let x_1, x_2, z be as in (21). We get

$$hx_1 \in D(U) \quad \text{and} \quad hx_2 \in D(V),$$

since $D(U)$ and $D(V)$ are G-invariant; $hx = hx_1 - hx_2$, as h is linear; $hz \in R^+$, since h is positive. As U and V are G-equivariant, we obtain that

$$\begin{aligned} hy &= h\big(U(x_1) + V(x_2) + z\big) \\ &= h\big(U(x_1)\big) + h\big(V(x_2)\big) + hz \\ &= U(hx_1) + V(hx_2) + hz. \end{aligned}$$

Thus, hx_1, hx_2 and hz satisfy the conditions in definition of $\mathcal{E}$. So, $(hx, hy) \in \mathcal{E}$, and this proves the "if" part of (22). The "only if" part follows by exchanging h with h^{-1}.

For any $x \in \mathcal{D}$, put

$$\mathcal{E}_x = \{y \in R : (x, y) \in \mathcal{E}\}.$$

Note that $\mathcal{E}_x \neq \emptyset$ for all $x \in \mathcal{D}$. Let us define $p : \mathcal{D} \to R$ by setting $p(x) = \bigwedge \mathcal{E}_x$, $x \in \mathcal{D}$. Note that p is well-defined and convex (see also [49]), and from (20) it follows that $p(0) \geq 0$.

Now, we prove that

3.3.1) p is G-equivariant.

Pick arbitrarily $h \in G$. Taking into account (22) and Proposition 2.5, we deduce

$$\begin{aligned} h\big(p(x)\big) &= h\Big(\bigwedge \mathcal{E}_x\Big) = \bigwedge h(\mathcal{E}_x) = \bigwedge \{hy \in R : (x,y) \in \mathcal{E}\} \\ &= \bigwedge \{y \in R : (x, h^{-1}y) \in \mathcal{E}\} = \bigwedge \{y \in R : (hx, y) \in \mathcal{E}\} = p(hx) \end{aligned} \tag{23}$$

for all $x \in \mathcal{D}$, getting the G-equivariance of p.

By Theorem 3.2 applied to $\mathcal{D}$ and p, there is $L \in \mathcal{L}_{\text{equiv}}(X, R)$ with

$$L(x) - L(x') = L(x - x') \le p(x - x') = \bigwedge \mathcal{E}_{x-x'} \le U(x) + V(x'),$$

and hence

$$L(x) - U(x) \le L(x') + V(x') \quad \text{for all} \quad x \in D(U), \quad x' \in D(V). \tag{24}$$

Put

$$u_0 = \bigvee \{L(x) - U(x) : x \in D(U)\}. \tag{25}$$

Note that $u_0 \in R$, as R is Dedekind complete. By (24) we have

$$L(x') - u_0 \ge -V(x') \quad \text{for each} \quad x' \in D(V),$$

and from (25) we get that

$$L(x) - u_0 \le U(x) \quad \text{for all} \quad x \in D(U).$$

This finishes the proof. □

Now, we prove the following monotone extension theorem for linear equivariant functionals which extends [49, Corollary 3.11].

Theorem 3.4. *Let $Z \subset X$ be a G-invariant subspace endowed with the order inherited by X, such that $0 \in \operatorname{int}(Z - X^+)$, and $L_0 \in \mathcal{L}_{+,\,\text{equiv}}(Z, R)$. Then, there is $L \in \mathcal{L}_{+,\,\text{equiv}}(X, R)$, with $L(z) = L_0(z)$ for all $z \in Z$.*

Proof. Let $U = L_0$ with $D(U) = Z$, and $V \equiv 0$ with $D(V) = X^+$. Observe that U and V are convex and G-equivariant maps, $D(U)$ and $D(V)$ are convex and G-invariant sets, and $U(x) + V(x) \ge 0$ for all $x \in D(U) \cap D(V)$, since L_0 is positive on Z. So, the hypotheses of Theorem 3.3 are satisfied, and hence, there exist $L \in \mathcal{L}_{\text{equiv}}(X, R)$ and $u_0 \in R$ with

$$L(z) - u_0 \le L_0(z) \text{ for every } z \in Z \text{ and } L(x) - u_0 \ge 0 \text{ for all } x \in X^+. \tag{26}$$

Putting in (26) $z = 0$ and $x = 0$, we obtain $u_0 = 0$. Thus, $L_0(z) - L(z) \ge 0$ for each $z \in Z$ and $L(x) \ge 0$ for any $x \in X^+$. Since L and L_0 are linear, by changing z with $-z$ in (26) we obtain $L(z) = L_0(z)$ for any $z \in Z$. Thus, the assertion follows. □

4. Applications

We begin with proving the existence of affine manifolds, separating two nonempty sets of a product space, by extending [28, Theorem 4.1] to equivariance.

Theorem 4.1. *Let A, $B \neq \emptyset$ be two G-invariant subsets of $X \times R$, such that $A - B$ is convex. Set*

$$P_X(A-B) = \{x \in X : \quad \text{there is} \quad y \in R \quad \text{with} \quad (x,y) \in A - B\}. \tag{27}$$

Suppose that

$$0 \in \operatorname{int}\big(P_X(A-B)\big), \quad \text{and} \tag{28}$$

$$y_1 \geq y_2 \quad \text{whenever} \quad (x, y_1) \in A \quad \text{and} \quad (x, y_2) \in B. \tag{29}$$

Then there exist $L \in \mathcal{L}_{\rm equiv}(X,R)$ and $u_0 \in R$ such that the affine manifold

$$H = \{(x,y) \in X \times R : L(x) - y = u_0\} \qquad \text{separates} \quad A \text{ and } B.$$

P r o o f. First of all, observe that the set $P_X(A-B)$ is convex and G-invariant, since is $A - B$.

Now, let $C = \bigcup_{\lambda > 0} \lambda\,(A-B)$. It is not difficult to see that C is convex. Now we claim that for every

$$h \in G, (hx, hy) \in C \quad \text{if and only if} \quad (x,y) \in C. \tag{30}$$

Indeed, if $(x,y) \in C$, and $h \in G$, there exist $\lambda_0 > 0$, $x' \in X$, $y' \in R$, such that $x = \lambda_0\, x'$, $y = \lambda_0\, y'$, and $(x', y') \in A - B$. Since A and B are G-invariant and h is linear, it is $(hx', hy') \in A - B$, and hence $(hx, hy) = \lambda_0\,(hx', hy') \in C$. This proves the “if part” of (30). The “only if” part follows from the “if” part, by exchanging h with h^{-1}.

Now, we define a convex and G-equivariant function $U : P_X(A-B) \to R$, which will satisfy the hypotheses of Theorem 3.2.

For every $x \in P_X(A-B)$, set $E_x = \{y \in R : (x,y) \in C\}$. By proceeding similarly as in [28] and [6], it is possible to show that $E_x \neq \emptyset$ for all $x \in P_X(A-B)$. Now, define

$$U(x) = \bigwedge E_x, \quad x \in P_X(A-B). \tag{31}$$

It is not difficult to check that U is convex, and $U(0) = 0$. Moreover, by arguing analogously as in (23), it is possible to prove that U is G-equivariant.

By Theorem 3.2, there is $T \in \mathcal{L}_{\rm equiv}(X,R)$ with $T(x) \leq U(x)$ for any $x \in P_X(A-B)$. From this, we deduce that

$$T(x_1) - T(x_2) = T(x_1 - x_2) \leq U(x_1 - x_2) \leq y_1 - y_2 \tag{32}$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. From (32) we obtain $T(x_1) - y_1 \leq T(x_2) - y_2$. As R is Dedekind complete, there exists $u_0 \in R$ with

$$\bigvee\{T(x_1) - y_1 : (x_1,y_1) \in A\} \leq u_0 \leq \bigwedge\{T(x_2) - y_2 : (x_2,y_2) \in B\},$$

getting the assertion. $\square$

Now, as a consequence of Theorem 3.3, we give the next Fenchel-type duality theorem, extending to equivariance [6, Theorem 2] and [47, Theorem 2].

Theorem 4.2. *Let $U : D(U) \to R$, $V : D(V) \to R$ be convex and G-equivariant functions, where $D(U)$ and $D(V)$ satisfy (19). Let*

$$r = \bigwedge \{U(x) + V(x) : x \in D(U) \cap D(V)\}$$

exist in R, where r is as in Problem 2.7) , and let U^c, V^c be as in Problem 2.8). Then, Problem 2.8) has a solution L_0, such that $-U^c(L_0) - V^c(-L_0) = r$.

Proof. Set $\widetilde{U}(x) = U(x) - r$. Then, $D(\widetilde{U}) = D(U)$ and $\widetilde{U}$ is convex on its domain. We claim that

4.2.1) $\widetilde{U}$ is G-equivariant.

Choose arbitrarily $h \in G$. Taking into account the G-equivariance of U and V, the linearity of h and Proposition 2.5, we get

$$\begin{aligned} hr &= h\Big(\bigwedge \{U(x) + V(x) : x \in D(U) \cap D(V)\}\Big) \\ &= \bigwedge \{hU(x) + hV(x) : x \in D(U) \cap D(V)\} \\ &= \bigwedge \{U(hx) + V(hx) : x \in D(U) \cap D(V)\} \\ &= \bigwedge \{U(x) + V(x) : x \in D(U) \cap D(V)\} = r, \end{aligned} \tag{33}$$

because $x \in D(U) \cap D(V)$ if and only if $hx \in D(U) \cap D(V)$, thanks to the G-invariance of $D(U) \cap D(V)$ and the existence of h^{-1}. From (33) we obtain $h\big(\widetilde{U}(x)\big) = h\big(U(x) - r\big) = h\big(U(x)\big) - hr = U(hx) - r = \widetilde{U}(hx)$, getting the G-equivariance of $\widetilde{U}$.

Now, observe that $U(x) + V(x) \geq 0$ for every $x \in D(U) \cap D(V)$. Thus, $\widetilde{U}$ and V satisfy the hypotheses of Theorem 3.3. So, there are $L_0 \in \mathcal{L}_{\text{equiv}}(X, R)$ and $u_0 \in R$ such that $L_0(x) - u_0 \leq \widetilde{U}(x) = U(x) - r$ for any $x \in D(U)$ and $L_0(x') - u_0 \geq -V(x')$ whenever $x' \in D(V)$. The assertion follows by arguing analogously as in [6, Theorem 2] and [47, Theorem 2]. □

As consequences of the previous theorems, it is possible to prove the following results, whose proofs are analogous to those given in [47], [49] and [6].

Theorem 4.3 (Optimality condition)**.** *Let U, V, $D(U)$, $D(V)$ be as in Theorem 4.2, and let $x_0 \in D(U) \cap D(V)$ be a solution to Problem* I. Then, $\partial_{\text{equiv}} U(x_0) \cap \big(-\partial_{\text{equiv}} V(x_0)\big) \neq \emptyset$.

Theorem 4.4. *Let $U : D(U) \to R$ be convex and G-equivariant, $D(U)$ be G-invariant, $U(0) = 0$, and $Z \subset X$ be a G-invariant subspace such that $0 \in \operatorname{int}(D(U) - Z)$. Let $T_0 \in \mathcal{L}_{\text{equiv}}(Z, R)$ be with $T_0(z) \leq U(z)$ for all $z \in D(U) \cap Z$. Then, T_0 admits an extension*

$$T \in \mathcal{L}_{\text{equiv}}(X, R), \quad \textit{with} \quad T(x) \leq U(x) \quad \textit{for every} \quad x \in D(U).$$

Theorem 4.5 (Moreau-Rockafellar formula). *Let U, V, $D(U)$, $D(V)$ be as in Theorem 4.2, $x_0 \in D(U) \cap D(V)$, and suppose that $\partial_{\text{equiv}} U(x_0) \neq \emptyset$ and $\partial_{\text{equiv}} V(x_0) \neq \emptyset$. Then*

$$\partial_{\text{equiv}}(U+V)(x_0) = \partial_{\text{equiv}}(U)(x_0) + \partial_{\text{equiv}}(V)(x_0). \tag{34}$$

Now, we give a Farkas-type theorem related to some minimization problems for vector lattice-valued equivariant functionals, extending the earlier results proved in [6, 23, 48].

Theorem 4.6. *Let $U : D(U) \to X$, $V : D(V) \to R$ be convex G-equivariant functions, where $D(U)$, $D(V)$ are G-invariant convex subsets of X, containing R. Let $X_0 = D(U) \cap D(V)$, assume that $0 \in \operatorname{int}\big(U(X_0) + X^+\big)$ and suppose that, for any $x \in X_0$,*

$$V(x) \geq 0 \text{ whenever } U(x) \leq 0. \tag{35}$$

Then, there is $L \in \mathcal{L}_{+,\,\text{equiv}}(X, R)$ with

$$V(x) + L\big(U(x)\big) \geq 0 \text{ for any } x \in X_0. \tag{36}$$

P r o o f. Set $\mathcal{W} = U(X_0) + X^+$. Observe that $\mathcal{W}$ is convex and G-invariant (see also [6, Theorem 11] and [48, Theorem 3]). We will define a convex and G-equivariant map $p : \mathcal{W} \to R$ satisfying the hypotheses of Theorem 3.2. Set

$$\begin{aligned} A &= \{(x, y) \in X \times R : \text{ there is } x_0 \in X_0 \text{ with } x \geq U(x_0) \text{ and } y \geq V(x_0)\}; \\ B &= \bigcup_{\lambda > 0} \lambda A. \end{aligned} \tag{37}$$

It is not difficult to check that A and B are convex sets, since U and V are convex functions.

For each $w \in \mathcal{W}$, put $S_w = \{y \in R : (w, y) \in B\}$ and set $p(w) = \bigwedge S_w$. Similarly as in [6, Theorem 11] and [48, Theorem 3], it is possible to show that $S_w \neq \emptyset$ for all $w \in \mathcal{W}$, p is well-defined and convex, and $p(0) = 0$.

Now, we claim that

4.6.1) p is G-equivariant on $\mathcal{W}$.

Observe that by arguing analogously as in (23), in order to prove the G-equivariance of p, it suffices to show that

$$\text{for any } h \in G,\ (hx, hy) \in B \text{ if and only if } (x, y) \in B. \tag{38}$$

Note that taking into account (37), it suffices to prove (38) when $(x, y) \in A$.

Choose arbitrarily $(x, y) \in A$ and $h \in G$, and let x_0 satisfy (37) in correspondence with (x, y). Since U and V are G-equivariant and h is positive, we get $hx \geq h\big(U(x_0)\big) = U(hx_0)$ and $hy \geq h\big(V(x_0)\big) = V(hx_0)$. As $D(U)$ and $D(V)$ are G-invariant, it is $hx_0 \in X_0$. Thus, hx_0 satisfies the condition in (37) in correspondence with (hx, hy), and hence $(hx, hy) \in A$. The converse implication is obtained by exchanging h with h^{-1}.

Thus, by Theorem 3.2 applied with $U = p$, $D(U) = \mathcal{W}$, there exists $L \in \mathcal{L}_{\text{equiv}}(X, R)$ with $-L(w) \leq p(w)$ for all $w \in \mathcal{W}$. The proofs of (36) and the positivity of L are analogous to those in [6] and [48]. Thus, L is the required functional. □

Now, we prove a Kuhn-Tucker-type theorem for equivariant functionals.

Theorem 4.7. *Under the same assumptions and hypotheses as in Theorem* 4.6*, let us consider the following problems:*
4.7.1) *find* $x_0 \in Z_0 := \{x \in X_0\colon U(x) \leq 0\}$ *such that* $V(x_0) \leq V(x)$ *for all* $x \in Z_0$*;*
4.7.2) *find* $x_0 \in X_0$ *and* $L_0 \in \mathcal{L}_{+,\,\text{equiv}}(X, R)$ *such that*

$$L\big(U(x_0)\big) + V(x_0) \leq L_0\big(U(x_0)\big) + V(x_0) \leq L_0\big(U(x)\big) + V(x) \tag{39}$$

for every $x \in X_0$ *and* $L \in \mathcal{L}_{+,\,\text{equiv}}(X, R)$*.*

If x_0 *is a solution to Problem* 4.7.1) *,then there is* $L_0 \in \mathcal{L}_{+,\,\text{equiv}}(X, R)$ *such that the pair* (x_0, L_0) *is a solution to Problem* 4.7.2)*.*

P r o o f. Let x_0 be a solution to Problem 4.7.1), and set $V'(x) := V(x) - V(x_0)$, $x \in D(V)$. We claim that
4.7.3) V' is G-equivariant.

To prove 4.7.3), we first observe that

$$\text{for every } h \in G, \text{ it is } x \in Z_0 \text{ if and only if } hx \in Z_0. \tag{40}$$

Indeed, if $x \in Z_0$, since X_0 is G-invariant, h is linear and positive, and U is G--equivariant, we get that $hx \in X_0$ and $U(hx) = h\big(U(x)\big) \leq h(0) = 0$. The converse implication follows by replacing h with h^{-1}. Now, we show that

$$\begin{aligned} &\text{for every } h \in G,\ hx_0 \text{ is a solution to Problem 4.7.1)} \\ &\text{if and only if } x_0 \text{ is a solution to Problem 4.7.1).} \end{aligned} \tag{41}$$

We first prove the "if" part. Let x_0 be such that $V(x_0) \leq V(x)$ whenever $x \in Z_0$. Observe that $h(Z_0) = Z_0$, thanks to (40). Pick arbitrarily $x \in Z_0$. There exists $x' \in Z_0$ such that $x = hx'$. Since V is G-equivariant and h is positive, we get $V(hx_0) = h\big(V(x_0)\big) \leq h\big(V(x')\big) \leq V(hx') = V(x)$, which proves the "if" part. The "only if" part follows from the "if" part, by exchanging h with h^{-1} and taking into account (40).

Now, we prove that

$$V(hx_0) = V(x_0) \quad \text{for every } h \in G. \tag{42}$$

Indeed, choose arbitrarily $h \in G$. Taking into account that Z_0 is G-invariant, from (41) it follows that $V(hx_0) \leq V(x_0)$ and $V(x_0) \leq V(hx_0)$, that is (42).

Finally, we are in position to prove 4.7.3). Pick arbitrarily $h \in G$. From (42) and the G-equivariance of V it follows that

$$h\big(V'(x)\big) = h\big(V(x)\big) - h\big(V(x_0)\big) = V(hx) - V(x_0) = V'(hx). \tag{43}$$

It is not difficult to check that U and V' satisfy (35). By Theorem 4.6, there is $L_0 \in \mathcal{L}_{+,\,\mathrm{equiv}}(X, R)$ with $V'(x) + L_0\big(U(x)\big) \geq 0$ for every $x \in X_0$, namely,

$$L_0\big(U(x)\big) + V(x) \geq V(x_0) \quad \text{for all } x \in X_0. \tag{44}$$

From (44) we deduce that $L_0\big(U(x_0)\big) \geq 0$. Since $U(x_0) \leq 0$ and L_0 is positive, it is $L_0\big(U(x_0)\big) \leq 0$, and hence $L_0\big(U(x_0)\big) = 0$. From this and (44) we obtain

$$L_0\big(U(x_0)\big) + V(x_0) \leq L_0\big(U(x)\big) + V(x) \quad \text{for every } x \in X_0. \tag{45}$$

As $U(x_0) \leq 0$, we get $L\big(U(x_0)\big) \leq 0$ for any $L \in \mathcal{L}_{+,\,\mathrm{equiv}}(X, R)$, and hence

$$L\big(U(x_0)\big) + V(x_0) \leq V(x_0) = L_0\big(U(x_0)\big) + V(x_0) \tag{46}$$

for each $L \in \mathcal{L}_{+,\,\mathrm{equiv}}(X, R)$, taking into account that $L_0\big(U(x_0)\big) = 0$. From (45) and (46) it follows that (x_0, L_0) is a solution to 4.7.2). □

Now, we will prove that for each fixed Dedekind complete ρ-space R, amenability is a condition not only sufficient but also necessary in order that the given theorems are always valid. We begin with the following

Theorem 4.8. *Under the given notations and assumptions, let R be a fixed Dedekind complete ρ-space. If G satisfies Theorem 3.4 with respect to R, then G is amenable.*

Proof. Choose $X = \{f \in R^G,\ f \text{ is bounded}\}$, $Z = \{f \in R^G,\ f \text{ is constant}\}$, and let $G \subset X^X$ be a group of positive linear maps $h \in X^X$ defined by setting $(hf)(g) = h\big(f(h(g))\big)$, $f \in X$, $h \in G$. Pick any element $y \in (R^\rho)^+$, $y \neq 0$, and define $\mu : Z \to R$ by $\mu(\mathbf{r}) = ry$, where $\mathbf{r}$ is the constant function, which assumes the value r. It is not difficult to check that $0 \in \mathrm{int}(Z - X^+)$, and that μ is a linear, positive and G-equivariant R-functional. By Theorem 3.4, μ admits a linear, positive and G-equivariant extension $\widetilde{\mu}$ on the whole of X. For each $A \subset G$, set $\widehat{\mu}(A) = \widetilde{\mu}(\chi_A)$. We have

$$(h\chi_A)(\omega) = \begin{cases} h\big(\widehat{\mu}(A)\big), & \text{if } g \in A, \\ 0, & \text{if } g \notin A; \end{cases} \quad ; \quad h\Big(\chi_A\big(h(\omega)\big)\Big) = \begin{cases} h\big(\widehat{\mu}(A)\big), & \text{if } hg \in A, \\ 0, & \text{if } hg \notin A. \end{cases}$$

Since $\widetilde{\mu}$ is G-equivariant, we deduce that $\widehat{\mu}$ is G-invariant.

Now, we prove that G is amenable. Since R is a Dedekind complete ρ-space, R can be embedded as an ideal of $R^{\rho\rho}$. Since $\widehat{\mu}(G) = u \geq 0$, $\widehat{\mu}(G) = u \neq 0$, there is $\overline{y} \in R^\rho$ such that $\widehat{\mu}(G)(\overline{y}) = u(\overline{y})$ is a strictly positive real number.

For every $A \subset G$, set $\nu(A) = \frac{\widehat{\mu}(A)(\overline{y})}{\widehat{\mu}(G)(\overline{y})}$. It is readily seen that $\nu(\emptyset) = 0$, $\nu(G) = 1$ and ν takes values in $[0,1]$. Moreover, ν is G-invariant, because $\widehat{\mu}$ is G-invariant. Furthermore, if A and B are any two disjoint subsets of G, then

$$\nu(A \cup B) = \frac{\widehat{\mu}(A \cup B)(\overline{y})}{\widehat{\mu}(G)(\overline{y})} = \frac{\widehat{\mu}(A)(\overline{y}) + \widehat{\mu}(B)(\overline{y})}{\widehat{\mu}(G)(\overline{y})} = \nu(A) + \nu(B).$$

Therefore, G is amenable. This finishes the proof. □

Now, we are in position to give the following characterization of amenability of groups, extending [23, Theorems 1 and 2] and [6, Theorem 17] to equivariance.

Theorem 4.9. *Let $G \subset X^X$ be a group. Then, theorems* (3.2)–(3.4) *and* (4.1)-‑(4.7) *(with respect to a fixed Dedekind complete ρ-space) are equivalent to the amenability of G.*

Proof. Amenability $\Longrightarrow$ (3.2), see Theorem 3.2.

(3.2) $\Longrightarrow$ (3.3), see Theorem 3.3.

(3.3) $\Longrightarrow$ (3.4), see Theorem 3.4.

(3.4) $\Longrightarrow$ Amenability, see Theorem 4.8.

(3.2) $\Longrightarrow$ (4.1), see Theorem 4.1.

(3.3) $\Longrightarrow$ (4.2), see Theorem 4.2.

(4.1) $\Longrightarrow$ (4.2), let r be as in Problem 2.7), and set $\widetilde{U}(x) = U(x) - r$, $A = \{(x, y)\colon x \in D(U),\ y \in R,\ y \geq \widetilde{U}(x)\}$, $B = \{(x, y)\colon x \in D(V),\ y \in R,\ y \leq -V(x)\}$. It is not difficult to check that A and B are nonempty, and $A - B$ is convex. Now, we claim that

$$\text{for every } h \in G, (hx, hy) \in A \text{ (resp. } B) \text{ if and only if } (x, y) \in A \text{ (resp. } B). \quad (47)$$

Now, pick arbitrarily $(x, y) \in A$. Since h is positive and $\widetilde{U}$ is G-equivariant (see 4.2.1)), we obtain $hy \geq h\big(\widetilde{U}(x)\big) = \widetilde{U}(hx)$, and hence, $(hx, hy) \in A$. The converse implication follows by exchanging h with h^{-1}. The proof that $(hx, hy) \in B$ if and only if $(x, y) \in B$ is analogous.

Now, we claim that

$$0 \in \operatorname{int}\big(P_X(A - B)\big). \quad (48)$$

Indeed, we get $P_X(A) = D(U)$, $P_X(B) = D(V)$, $P_X(A - B) = P_X(A) - P_X(B) = D(U) - D(V)$. Since, by hypothesis, $0 \in \operatorname{int}\big(D(U) - D(V)\big)$, then we obtain (48) (see also [22, Satz 1 (6)]).

Now, we prove that

$$y_1 \geq y_2 \text{ whenever } (x, y_1) \in A \text{ and } (x, y_2) \in B. \tag{49}$$

If $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_2 \leq -V(x) \leq U(x) - r = \widetilde{U}(x) \leq y_1$, getting (49).

Therefore, A and B satisfy the hypotheses of Theorem 4.1. Thus, there are $L_0 \in \mathcal{L}_{\text{equiv}}(X, R)$ and $u_0 \in R$ with $L_0(x) - y \leq u_0$ whenever $(x, y) \in A$ and $L_0(x') - y' \geq u_0$ for all $(x', y') \in B$. In particular, $L_0(x) - u_0 \leq \widetilde{U}(x) = U(x) - r$ for any $x \in D(U)$ and $L_0(x') - u_0 \geq -V(x')$ for all $x' \in D(V)$. The assertion follows by proceeding analogously as in [6, Theorem 2] and [47, Theorem 2].

(4.2) $\Longrightarrow$ (4.3), see Theorem 4.3 and [6].

(4.3) $\Longrightarrow$ (4.4), see Theorem 4.4 and [6].

(4.4) $\Longrightarrow$ (4.1) The proof is analogous to that of implication (3.2) $\Longrightarrow$ (4.1) by applying Theorem 4.4 with $Z = \{0\}$ and U as in (31 (this is possible, since $U(0) = 0$).

(4.2) $\Longrightarrow$ (4.5), see Theorem 4.5 and [6].

(3.2) $\Longrightarrow$ (4.6), see Theorem 4.6.

(4.6) $\Longrightarrow$ (4.7), see Theorem 4.7.

(4.5) $\Longrightarrow$ (3.4), (4.7) $\Longrightarrow$ (3.4) it suffices to argue analogously as in [6]. □

Open problem Prove similar results when R is any Dedekind complete vector lattice.

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