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ON IDEALS GENERATED BY PARTITIONS INTO MEAGER AND NULL SETS

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ABSTRACT. We examine the σ-ideal generated by complements of sums of sets from partitions into meager and null sets. We prove some characterizations of this σ-ideal.

1. Definitions

Throughout this paper, we consider sets of real numbers and subsets of the Cantor space 2^ω equipped with " + " — the usual modulo 2 coordinate-wise addition in 2^ω.

By $\mathcal{N}$, $\mathcal{MGR}$ we denote the σ-ideals of sets of Lebesgue measure zero (null sets) and sets of first category (meager) in $\mathbf{R}$ (or in the Cantor space 2^ω), respectively. By $\mathcal{E}$ we denote the σ-ideal generated by all F_σ sets of measure zero.

If $X, Y \subseteq G$ are subsets of a topological group (typically $G = 2^\omega$ or $G = \mathbf{R}$), then we define $X \pm Y = \{x \pm y \colon x \in X, y \in Y\}$. If $X \subseteq \mathbf{R}$ and $r \in \mathbf{R}$, then define $r \cdot X = \{r \cdot x \colon x \in X\}$.

Define: $\mathcal{MGR}^* = \{X \subseteq G \colon \forall_{M \in \mathcal{MGR}} X + M \in \mathcal{MGR}\}$.

If $\mathcal{I}$ and $\mathcal{J}$ are two σ-ideals of subsets of a fixed space G (typically $G = 2^\omega$ or $G = \mathbf{R}$), then we write $\mathcal{I} \perp \mathcal{J}$ if and only if there exist $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that $A \cup B = X$ and $A \cap B = \emptyset$.

For any family $\mathcal{F}$ of subsets of the real line (or the Cantor set), we say that $\mathcal{F}$ is transitive invariant if and only if $\forall_{x \in \mathbf{R}} \forall_{F \in \mathcal{F}} x + F \in \mathcal{F}$.

For $A \subseteq G$ (where G is a fixed topological group, typically $G = 2^\omega$ or $G = \mathbf{R}$), $\langle A \rangle$ will denote a subgroup of G generated by A.

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If $H \subseteq G$ is a subgroup of the topological group G (again, typically $G = 2^\omega$ or $G = \mathbf{R}$), then we write $H \triangleleft G$.

$\triangle$ denotes the symmetric difference of sets.

Quantifiers $\forall^\infty$ and $\exists^\infty$ denote "for all except finitely many" and "for infinitely many", respectively.

Suppose that $\theta \in [0, \pi)$. By $\rho_\theta \colon \mathbf{R}^2 \to \mathbf{R}$ we denote "the projection in the direction θ", i.e., $\rho_\theta(x, y) = x \cdot \cos(\theta) + y \cdot \sin(\theta)$.

If $Z \subset \mathbf{R}^2$, then for $y \in \mathbf{R}$ define the horizontal section:

$$Z^{(y)} = \{x \in \mathbf{R} \colon \langle x, y\rangle \in Z\}.$$

We will need the following characterization of meager sets in 2^ω:

Characterization ([BJS, Prop. 9])**.** $F \subseteq 2^\omega$ is meager if and only if there exist $z \in 2^\omega$ and a partition of ω into finite intervals (I_n) such that $F \subseteq \{x \colon \forall_n^\infty x \restriction I_n \neq z \restriction I_n\}$.

Also, we will use the following definition:

Definition 1.1 ([BJ, Def. 2.5.3])**.** A set $G \subseteq 2^\omega$ is called *small* if there exist a partition of ω into finite intervals (I_n) and a sequence $H_n \subseteq 2^{I_n}$ such that $\sum_n \frac{|H_n|}{2^{|I_n|}} < \infty$ and $G \subseteq \{x \colon \exists_n^\infty x \restriction I_n \in H_n\}$.

Recall that every small set is a null set (i.e., Lebesgue measure zero) and every null set is a union of two small sets (see in [BJ, Theorem 2.5.7]).

Also, we have:

Characterization ([BJ])**.** A set $E \subseteq 2^\omega$ is in $\mathcal{E}$ if and only if there exist a partition of ω into finite intervals (I_n) and a sequence $H_n \subseteq 2^{I_n}$ such that $\sum_n \frac{|H_n|}{2^{|I_n|}} < \infty$ and $E \subseteq \{x \colon \forall_n^\infty x \restriction I_n \in H_n\}$.

2. Introduction

It is a classical result that the real line may be partitioned into two sets M and N where N is a Lebesgue measure null set ($N \in \mathcal{N}$) and its complement $M = N^c$ is a meager set ($M \in \mathcal{MGR}$) (see, for example, [Oxt]). It is well-known that by the theorems of Steinhaus and Picard, respectively, we have $M + M = \mathbf{R}$ and $N + N = \mathbf{R}$. However, it is also well-known that we can construct a partition $\mathbf{R} = M \cup N$ into meager and measure null sets such that $M + N$ is a "small" set in a special sense (for example, $\operatorname{int}(M + N) = \emptyset$). So, it is natural to investigate properties of the family of such a kind of sets. To simplify the investigations, it is good to consider a σ-ideal generated by relevant sets. Therefore, in our article we introduce new σ-ideals $\mathcal{I}_{M,N}$ and $\mathcal{I}_{M,N}^{(A)}$. Another motivation of this article is given by the following (open) problem from [N] (see [N, Problem 3.8]):

Problem 2.1. Suppose that $\langle \theta_i, \mu_i \in [0,\pi), i \in \mathbb{N}\rangle$ are pairwise distinct and $M \in \mathcal{MGR}$, $N \in \mathcal{N}$. Is it true that $\bigcup_{i\in\mathbb{N}} \rho_{\theta_i}^{-1}[M] \cup \bigcup_{i\in\mathbb{N}} \rho_{\mu_i}^{-1}[N] \neq \mathbf{R}^2$?

Notice that if $\mathbf{R} = M \cup N$ is a partition into meager and measure null sets, then $\mathbf{R} \setminus (M+N) = \{x \in \mathbf{R} \colon (x-M) \cup N = \mathbf{R}\} = \{y \in \mathbf{R} \colon Z^{(y)} = \mathbf{R}\}$, where $Z = \rho_0^{-1}[N] \cup \rho_{\frac{3\pi}{4}}^{-1}[-\sqrt{2}\cdot M]$, which shows that these two topics are related to each other.

3. Results

Let us define:

Definition 3.1. $\mathcal{I}_{M,N}$ denotes the σ- generated by the family

$$\{\mathbf{R} \setminus (M+N) \colon M \cup N = \mathbf{R} \wedge M \cap N = \emptyset \wedge M \in \mathcal{MGR} \wedge N \in \mathcal{N}\},$$

and

Definition 3.2. $\mathcal{I}_{M,N}^{(A)}$ denotes the σ- generated by the family

$$\{\mathbf{R}\setminus(M+N) \colon M\cup N = \mathbf{R}\wedge M\cap N = \emptyset\wedge M \in \mathcal{MGR}\wedge N \in \mathcal{N}\wedge M \in F_\sigma\wedge N \in G_\delta\}.$$

Notice that we can define these σ ideals in the same way for the Cantor set. Obviously, $\mathcal{I}_{M,N}^{(A)} \subseteq \mathcal{I}_{M,N} \subseteq \mathcal{MGR} \cap \mathcal{N}$, and $\mathcal{I}_{M,N}^{(A)}$ is a σ-ideal generated by coanalytic sets. We have:

Remark. If $X \in \mathcal{I}_{M,N}$ ($\mathcal{I}_{M,N}^{(A)}$ respectively) and $r \in \mathbf{R}$, then

$$X + r, rX \in \mathcal{I}_{M,N} \quad \big(\mathcal{I}_{M,N}^{(A)},\ \text{respectively}\big).$$

P r o o f. Clear, since

$$(N+r) + (M+r) = N + M + 2r \quad \text{and} \quad (rN) + (rM) = r(N+M). \quad \square$$

Theorem 3.3. $\mathcal{MGR}^* \subseteq \mathcal{I}_{M,N}$.

P r o o f. Suppose that $X \in \mathcal{MGR}^*$. Let us fix a partition $\mathbf{R} = N \cup M$ into sets of measure zero and meager, respectively. Assume that $-M = M$, M is an F_σ set and N is a G_δ. Let us define: $M_1 = M + \langle X\rangle$, and $N_1 = M_1^c$. We check that $X \subseteq (M_1 + N_1)^c$. Suppose, on the contrary, that there is an $x \in X$ such that $x \in M_1 + N_1$. Then, we have $x = m_1 + n_1$, where $m_1 \in M_1$ and $n_1 \in N_1$, and then, $m_1 = m + y$ for some $m \in M$ and $y \in \langle X\rangle$. Hence,

$$n_1 = (m_1+n_1) - m_1 = (m_1+n_1) - (m+y) = (m_1+n_1) - y - m \in \langle X\rangle + M,$$

which is a contradiction, since $n_1 \notin M_1$. $\square$

3.1. The Ξ operator

Thoroughout this paper, we use the following abbreviation: $\Xi(X)=(X+X^c)^c$.

Let us define for $X \subseteq \mathbf{R}$: $\mathrm{Sym}(X) = \{x_0 \colon \forall_{x\in\mathbf{R}} x \in X \Leftrightarrow 2x_0 - x \in X\}$ (this symbol was used by K. C. Ciesielski). We have:

Observation. $2 \cdot \mathrm{Sym}(X) = \Xi(X)$.

Proof. Suppose that $x_0 \in 2 \cdot \mathrm{Sym}(X)$ and $x_0 \notin \Xi(X)$. Then, $x_0 \in (X + X^c)$, so $x_0 = x + y$ for some $x \in X$ and $y \in \mathbf{R} \setminus X$. We have $2\frac{x_0}{2} - x \in X$ but $x_0 - x = y$, a contradiction. On the other hand, suppose that $x_0 \in \Xi(X)$ and $x_0 \notin 2 \cdot \mathrm{Sym}(X)$. Let $x \in \mathbf{R}$ be such that $\neg(x \in X \Leftrightarrow 2 \cdot \frac{x_0}{2} - x \in X)$. Suppose that $x \notin X \wedge 2 \cdot \frac{x_0}{2} - x \in X$ (the second case is similar). Then,

$$x \notin X \wedge x_0 - x \in X, \quad \text{so} \quad x_0 = (x_0 - x) + x \in X + X^c, \quad \text{a contradiction.} \quad \square$$

We now give several elementary lemmas concerning the previously defined σ-ideals.

Lemma 3.4. *Suppose that $X, Y \subseteq \mathbf{R}$. Then*

$$z \in (X+Y)^c \iff \left[\forall_{x\in X} z - x \in Y^c\right].$$

Proof.

"$\Rightarrow$" Suppose that $z \in (X + Y)^c$ and assume the contrary. Then, $z - x \in Y$ for some $x \in X$, hence $z = x + (z - x) \in X + Y$, which is a contradiction.

"$\Leftarrow$" Assume that $\left[\forall_{x\in X} z - x \in Y^c\right]$ and assume that $z \in X + Y$. Then, $z = x + y, x \in X, y \in Y$, hence $z - x = y \in Y$, contrary to the assumption.

$\square$

Lemma 3.5. *Suppose that $X \subseteq \mathbf{R}$. Let x_1, x_2, $x_3 \in \Xi(X)$. Then, $x_1 + x_2 - x_3 \in \Xi(X)$.*

Proof. We use Lemma 3.4. Let $x \in X$. Then, $x_1 - x \in X$, so $x_3 - x_1 + x \in X$. Finally, $x_2 - x_3 + x_1 - x \in X$, which proves that $x_1 + x_2 - x_3 \in \Xi(X)$. $\square$

Lemma 3.6. *Suppose that $X \subseteq \mathbf{R}$ is such that $\Xi(X) \neq \emptyset$. Let b_0 be an arbitrary element of $\Xi(X)$. Then, the set $H = \{b - b_0 \colon b \in \Xi(X)\}$ is an additive subgroup of $\mathbf{R}$.*

Proof. Suppose that $b_1, b_2 \in H$, then $h_1 = b_1 - b_0$ and $h_2 = b_2 - b_0$ for some $b_1, b_2 \in \Xi(X)$. Then, by Lemma 3.5 we have $b_1 + b_2 - b_0 \in \Xi(X)$. Hence, $b_1 + b_2 - 2b_0 \in H$ and therefore $(b_1 - b_0) + (b_2 - b_0) \in H$, and this gives us that $h_1 + h_2 \in H$. Next, let $h \in H$, then there exists $b \in \Xi(X)$ such that $h = b - b_0$. Let us define $b' = 2b_0 - b$. Again, by Lemma 3.5 we obtain that $b' \in \Xi(X)$. Since we have $b' - b_0 = -(b - b_0)$, we conclude that $-h \in H$. $\square$

3.2. The Fix operator

Recall the following definition from [CJKS]:

Definition 3.7 ([CJKS])**.** Suppose that J is a transitive invariant ideal of subsets of $\mathbf{R}$. For $X \subseteq \mathbf{R}$ define $\mathrm{Fix}(X, J) = \{x \in \mathbf{R} \colon (X + x) \triangle X \in J\}$.

Notice that (see [CJKS]) $\mathrm{Fix}(X, J)$ is an additive subgroup of $\mathbf{R}$. Denote $\mathrm{Fix}(X) = \mathrm{Fix}(X, \{\emptyset\})$.

Suppose that $\mathcal{F} \subseteq P(\mathbf{R}) \setminus \{\emptyset\}$ is a family of nonempty subsets of the real line. By $\mathrm{IFix}(\mathcal{F})$ let us denote the σ-ideal generated by the family $\{\Xi(X) \colon X \in \mathcal{F}\}$. It is easy to see that we have $\mathcal{I}_{M,N} = \mathrm{IFix}(\mathcal{F})$ for the family

$$\mathcal{F} = \{X \in \mathcal{M} \colon X^c \in \mathcal{N}\}.$$

We obtain:

Theorem 3.8. *Suppose that $\mathcal{F} \subseteq P(\mathbf{R})$ is a transitive invariant family of nonempty subsets of the real line. Assume that $\forall_{F \in \mathcal{F}} F \cup -F \in \mathcal{F}$. Then, the ideal* $\mathrm{IFin}(\mathcal{F})$ *is the same as the ideal generated by* $\{\mathrm{Fix}(F) + x \colon F \in \mathcal{F}, x \in \mathbf{R}\}$.

Proof. First, let us notice that $X = -X \iff 0 \in \Xi(X)$ and if $X = -X$, then $\mathrm{Fix}(X) = \Xi(X)$. Indeed, the first is evident and if $g \in \mathrm{Fix}(X)$, then, if $g = x + y$, $x \in X, y \in X^c$, we would have $g - x = y$, which is impossible. On the other hand, if $g \in \Xi(X)$ and $x \in X$, then $g + x \in X$ (otherwise, $g = -x + (g + x) \in X + X^c$). The same argument shows that $x - g \in X$.

Let $F \in \mathcal{F}$ and assume that $(F + F^c)^c \neq \emptyset$. Pick any $r_0 \in (F + F^c)^c$. Then, $F - \frac{r_0}{2} \in \mathcal{F}$ and $0 \in \big((F - \frac{r_0}{2}) + (F - \frac{r_0}{2})^c\big)^c$, so $(F + F^c)^c = \mathrm{Fix}(F - \frac{r_0}{2}) + r_0$.

On the other hand, it suffices to show that for any $F \in \mathcal{F}$, $\mathrm{Fix}(F) \subseteq (H + H^c)^c$ for some $H \in \mathcal{F}$. Put $H = F \cup -F$. Let $x \in \mathrm{Fix}(F)$ and suppose that $x = h_1 + h_2$ for $h_1 \in H$ and $h_2 \in H^c$. If $h_1 \in F$, then, since $x + (-h_2) = h_1$, we have $-h_2 \in F$, which is impossible. If $h_1 \in -F$, then $h_2 = x + (-h_1) \in F$, which is again impossible. □

This is an easy corollary from Theorem 3.8:

Corollary 3.9. *If $\mathcal{I}$ is a transitive invariant σ-ideal of subsets of the real line such that $A \in \mathcal{I} \Rightarrow -A \in \mathcal{I}$, then the σ-ideal* $\mathrm{IFix}(\mathcal{I} \setminus \{\emptyset\})$ *is the σ-ideal generated by the family $\{G + x \colon G \lhd \mathbf{R} \wedge G \in \mathcal{I}\}$.*

Proof. This follows from the observation that if $A \in \mathcal{I}$ and $x_0 \in A$, then $x_0 + \mathrm{Fix}(A) \subseteq A$ and if $G \lhd \mathbf{R}$, $G \in \mathcal{I}$, then $\mathrm{Fix}(G) = G$. □

Let us recall the following result:

Theorem 3.10 (See [MZ, Lemma 2.6]). *There exist a comeager null set $R \subseteq \mathbf{R}$ and a perfect nowhere dense null set $P \subseteq \mathbf{R}$ such that $R + P \subseteq R$.*

Let us notice that from the proof of this theorem one can deduce something more, namely we may require that moreover, $R - P \subseteq R$. So, $P \subseteq \mathrm{Fix}(R)$ (indeed, if $x_0 \in P$ and $r \in R$, then $x_0 + r \in R$ and $r - x_0 \in R$, thus $x_0 + R = R$, so $x_0 \in \mathrm{Fix}(R)$), hence, by virtue of Theorem 3.8, we obtain that $P \in \mathcal{I}_{M,N}$. Therefore, we obtain the conclusion that the σ-ideal $\mathcal{I}_{M,N}$ contains a perfect sets.

Now, let us go to the case of the Cantor space. Recall (see Lemma 3.6) that in the case of the real line, the set $\Xi(X)$ (if it is nonempty) is a coset of some addivite subgroup. However, in the case of the Cantor space, the situation is different, namely we have:

Theorem 3.11. *If $X \subseteq 2^\omega$ is such that $X \neq \emptyset$ and $X \neq 2^\omega$, then the set $\Xi(X)$ is a subgroup of 2^ω.*

P r o o f. Obviously, $0 \in \Xi(X)$. Suppose that $x_1, x_2 \in \Xi(X)$ and aiming for a contradiction suppose that $x_1 + x_2 \in X + X^c$. Then, $x_1 + x_2 = x + x'$, where $x \in X$ and $x' \notin X$. Since $x_1 \in \Xi$, we obtain that $x_1 + x \in X$ (otherwise, $x_1 = x + (x_1 + x) \in X + X^c$), and in the same way, we obtain that $x' = x_2 + x_1 + x \in X$, which is a contradiction. □

We have the following result:

Theorem 3.12. *Suppose that (I_n) is a partition of ω into finite intervals and $H_n \subseteq 2^{I_n}$ a sequence such that*

$$\sum_n \frac{|H_n|}{2^{|I_n|}} < \infty, \tag{1}$$

and H_n is a subgroup of the group 2^{I_n}. Then, the group $E = \{x \colon \forall_n^\infty x \restriction I_n \in H_n\} \subseteq 2^\omega$ belongs to the ideal $\mathcal{I}_{M,N}^{(A)}$ (on the Cantor space 2^ω).

P r o o f. Define

$$N = \{x \in 2^\omega \colon \exists_n^\infty x \restriction I_n \in H_n\},$$

$$M = \{x \in 2^\omega \colon \forall_n^\infty x \restriction I_n \notin H_n\}$$

and

$$M_1 = \{x \in 2^\omega \colon \forall_n^\infty x \restriction I_n \neq \underline{0} \restriction I_n\},$$

where $\underline{0} \in 2^\omega$ denotes the sequence $(0, 0, \ldots)$. By Characterization 1 and Definition 1.1, $N \in \mathcal{N}$ and $M_1 \in \mathcal{MGR}$. Moreover, M is a F_σ set and since $M \subseteq M_1$, $M \in \mathcal{MGR}$. We have

$$\begin{aligned} M + N &= \{x \in 2^\omega \colon \forall_n^\infty x \restriction I_n \notin H_n\} \\ &\quad + \{y \in 2^\omega \colon \exists_n^\infty x \restriction I_n \in H_n\} \\ &\subseteq \{z \in 2^\omega \colon \exists_n^\infty x \restriction I_n \notin H_n\}. \end{aligned}$$

On the other hand, suppose that $z \in 2^\omega$ is such that $\exists_n^\infty z \upharpoonright I_n \notin H_n$. Denote $B = \{n \in \omega\colon z \upharpoonright I_n \notin H_n\}$ and define

$$x \upharpoonright I_n = \begin{cases} z \upharpoonright I_n & \text{if} \quad n \in B, \\ \text{any element of } 2^{I_n} \setminus H_n & \text{if} \quad n \notin B \wedge H_n \neq 2^{I_n}, \\ \underline{0} \upharpoonright I_n & \text{if} \quad H_n = 2^{I_n} \end{cases}$$

and

$$y \upharpoonright I_n = \begin{cases} \underline{0} \upharpoonright I_n & \text{if} \quad n \in B, \\ (x+z) \upharpoonright I_n & \text{if} \quad n \notin B \wedge H_n \neq 2^{I_n}, \\ z \upharpoonright I_n & \text{if} \quad H_n = 2^{I_n}. \end{cases}$$

Then, $x \in M$, $y \in N$ and $x + y = z$. Finally, $E \subseteq (M+N)^c \in \mathcal{I}^{(A)}_{M,N}$.

□

4. Open problems

Problem 4.1. We know that the σ-ideal $\mathcal{I}^{(A)}_{M,N}$ is generated by coanalytic sets. This follows from the fact that if $E \subseteq \mathbf{R}$ is an F_σ set, then $\Xi(E)$ is coanalytic. However, we do not know whether this set is always Borel.

Suppose that the answer is yes. Then, by the following theorem from [L]:

Theorem 4.2 ([L, Theorem 1]). *If G is an additive proper analytic subgroup of $\mathbf{R}$, then $G \in \mathcal{E}$, and by Lemma 3.6 we would obtain that $\mathcal{I}^{(A)}_{M,N} \in \mathcal{E}$.*

Compare this with a problem due to Taras Banakh whether every subgroup G of 2^ω such that $G \in \mathcal{MGR} \cap \mathcal{N}$ belongs to $\mathcal{E}$ (oral communication).

Problem 4.3. Is it true that $\mathcal{MGR}^* \subseteq \mathcal{I}^{(A)}_{M,N}$?

Notice that if $M \cap N = \emptyset$, $M \cup N = \mathbf{R}$, then $\mathbf{R} \setminus (M+N) = \{x \in \mathbf{R}\colon (x - N) \cap M = \emptyset\} = \{x \in \mathbf{R}\colon (x-M) \cup N = \mathbf{R}\}$. Since $\mathbf{R} \setminus (M+N) \in \mathcal{I}_{M,N}$ and $\{x \in \mathbf{R}\colon (x-M) \cup N = \mathbf{R}\} \subseteq \{x\colon (x-M) \cup N \cup (M+x) = \mathbf{R}\}$, it is natural to ask the following problem:

Problem 4.4. Suppose that M and N are meager and null set, respectively, such that $M \cup N = \mathbf{R}$ and $M \cap N = \emptyset$. Is it true that the set $\{x\colon (x-M) \cup N \cup (M+x) = \mathbf{R}\}$ belongs to the σ − ideal $\mathcal{I}_{M,N}$?

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