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REAL FUNCTIONS, COVERS AND BORNOLOGIES

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ABSTRACT. The paper tries to survey the recent results about relationships between covering properties of a topological space X and the space $\mathrm{USC}(X)$ of upper semicontinuous functions on X with the topology of pointwise convergence. Dealing with properties of continuous functions $\mathrm{C}(X)$, we need shrinkable covers. The results are extended for $\mathcal{A}$-measurable and upper $\mathcal{A}$-semimeasurable functions where $\mathcal{A}$ is a family of subsets of X. Similar results for covers respecting a bornology and spaces $\mathrm{USC}(X)$ or $\mathrm{C}(X)$ endowed by a topology defined by using the bornology are presented. Some of them seem to be new.

1. Introdution

The paper is a part of the lecture I intended to read at 34th International Summer Conference on Real Functions Theory. The paper tries to give a survey of recent results about the relationships of the properties of some covers of a topological space X and the properties of the families of real functions on X. We tried to extend the known results also for covers respecting a bornology on X and families of real functions with the corresponding topology defined by the bornology.

We indicate which results are already known and give credits to authors. We suppose that the results with no indication of an author are new.

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2. Covers and bornology

A topological space $\langle X, \vartheta \rangle$ is always an infinite Hausdorff topological space, ϑ is the family of open subsets of X. Unexplained notions and terminology are those of R. Engelking [6].

A family $\mathcal{U}$ of subsets of X is a *cover* of X if $\bigcup \mathcal{U} = X$. For some technical reasons, a cover will be called also *o-cover*. A cover $\mathcal{V} \subseteq \mathcal{U}$ is said to be a *subcover* of $\mathcal{U}$. If we deal with a countable cover of X, we can consider it a sequence of subsets. A cover is *open* if every element of the cover is an open set.

We say that a family $\mathcal{V} \subseteq \mathcal{P}(X)$ is a *refinement* of the family $\mathcal{U} \subseteq \mathcal{P}(X)$ if

$$(\forall V \in \mathcal{V})(\exists U \in \mathcal{U})\, V \subseteq U.$$

A *bornology* $\mathcal{B}$ on a topological space X is a proper ideal[1] of subsets of X such that $\bigcup \mathcal{B} = X$. This notion should be a generalization of the notion of a bounded subset. A subset $\mathcal{B}_0 \subseteq \mathcal{B}$ is a *base* of the bornology $\mathcal{B}$ if for every $B \in \mathcal{B}$ there exists a $B_0 \in \mathcal{B}_0$ such that $B \subseteq B_0$. Note that a bornology $\mathcal{B}$ has a closed base if and only if for every $B \in \mathcal{B}$ also $\overline{B} \in \mathcal{B}$. The smallest bornology on X is the ideal $\mathrm{Fin} = \mathrm{Fin}(X)$ of all finite subsets of X.

We shall use the following convention. If the lower case letters φ or ψ denote one of the symbols o, λ, ω or γ, then the capital letters Φ or Ψ denote the corresponding symbol $\mathcal{O}$, Λ, Ω or Γ, respectively, and vice versa.

Let $\mathcal{B}$ be a bornology on a topological space X. We shall consider covers respecting this bornology. We assume that a *bornological cover*, briefly $\mathcal{B}$-o-cover, is identical with an o-cover. Similarly, a *large bornological cover* $\mathcal{U}$, briefly $\mathcal{B}$-λ-cover, is simply a large cover, i.e., for every $x \in X$ the set $\{U \in \mathcal{U} : x \in U\}$ is infinite. A cover $\mathcal{U}$ is a *bornological* ω*-cover*, briefly $\mathcal{B}$-ω*-cover*[2], if $X \notin \mathcal{U}$ and for every $B \in \mathcal{B}$ there exists $U \in \mathcal{U}$ such that $B \subseteq U$. A cover $\mathcal{U}$ is a *bornological* γ*-cover*, briefly $\mathcal{B}$-γ*-cover*, if $\mathcal{U}$ is infinite and for every $B \in \mathcal{B}$ the set $\{U \in \mathcal{U} : B \nsubseteq U\}$ is finite. If $\mathcal{U}$ is a $\mathcal{B}$-γ-cover, then $\mathcal{U} \setminus \{X\}$ is a $\mathcal{B}$-γ-cover as well. So, we can assume that X does not belong to a $\mathcal{B}$-γ-cover. We denote by $\Phi_{\mathcal{B}}(X)$ the family of all open $\mathcal{B}$-φ-covers of X for $\varphi = o, \lambda, \omega, \gamma$.

If $\mathcal{B} = \mathrm{Fin}(X)$, then a $\mathcal{B}$-φ-cover is the classical φ-cover and

$$\Gamma_{\mathrm{Fin}}(X) = \Gamma(X), \quad \Omega_{\mathrm{Fin}}(X) = \Omega(X).$$

Evidently,

$$\Gamma_{\mathcal{B}}(X) \subseteq \Omega_{\mathcal{B}}(X) \subseteq \Lambda_{\mathcal{B}}(X) = \Lambda(X) \subseteq \mathcal{O}_{\mathcal{B}}(X) = \mathcal{O}(X).$$

[1] Sometimes the authors ask that the empty set does not belong to a bornology.

[2] In [9], the authors call such a cover simply $\mathcal{B}$-cover. We respect the case $\mathcal{B} = \mathrm{Fin}(X)$. Similarly for a $\mathcal{B}^s$-cover, see also [5]

Let the family $\mathcal{V} \subseteq \mathcal{P}(X)$ be a refinement of the family $\mathcal{U} \subseteq \mathcal{P}(X)$. If $\mathcal{V}$ is an $\mathcal{B}$-o- or an $\mathcal{B}$-ω-cover, then $\mathcal{U}$ is such a cover as well. This is not true for $\mathcal{B}$-λ- and $\mathcal{B}$-γ-covers. If we add finitely many subsets of X to a $\mathcal{B}$-γ-cover, we obtain a $\mathcal{B}$-γ-cover. Moreover, each infinite subset of a $\mathcal{B}$-γ-cover is a $\mathcal{B}$-γ-cover as well. Omitting finitely many elements of an $\mathcal{B}$-λ- or an $\mathcal{B}$-ω-cover, we obtain a cover of same type. This is not true for $\mathcal{B}$-o-cover.

A $\mathcal{B}$-φ-cover $\mathcal{U}$ is *shrinkable* if there exists an open $\mathcal{B}$-φ-cover $\mathcal{V}$ such that

$$(\forall V \in \mathcal{V})(\exists U_V \in \mathcal{U} \setminus \{X\})\, \overline{V} \subseteq U_V. \tag{1}$$

The family $\{U_V : V \in \mathcal{V}\} \subseteq \mathcal{U}$ is a $\mathcal{B}$-φ-cover as well. The family of all open shrinkable $\mathcal{B}$-φ-covers of X will be denoted by $\Phi^{sh}_{\mathcal{B}}(X)$, or simply $\Phi^{sh}_{\mathcal{B}}$.

Similarly to F. Gerlits and Z. Nagy [7], we define: X *has the property* $(\varepsilon_{\mathcal{B}})$ if every open $\mathcal{B}$-ω-cover contains a countable $\mathcal{B}$-ω-subcover[3].

G. Beer and S. Levy in [1] introduced the notion of a strong $\mathcal{B}$-cover of a metric space. It is easy to define that notion for a uniform space. So, let $\langle X, \upsilon\rangle$ be a uniform space. We recall just some notions. The ball about $B \subseteq X$ and radius $V \in \upsilon$ is the set

$$\mathbb{B}(B, V) = \{x \in X : (\exists y \in B)\, \langle x, y\rangle \in V\}.$$

If $B = \{x\}$, we write simply $\mathbb{B}(x, V)$.

The topology ϑ_υ generated on X by the uniformity υ is defined by

$$U \in \vartheta_\upsilon \equiv (\forall x \in U)(\exists V \in \upsilon)\, \mathbb{B}(x, V) \subseteq U.$$

Let $\mathcal{B}$ be a bornology on X. An open cover $\mathcal{U}$ is a *strong* $\mathcal{B}$-ω-*cover*, briefly a $\mathcal{B}$-ω^s-*cover*, if $X \notin \mathcal{U}$ and for every $B \in \mathcal{B}$ there exist a $U \in \mathcal{U}$ and a $V \in \upsilon$ such that $\mathbb{B}(B, V) \subseteq U$. An open cover $\mathcal{U}$ is a *strong* $\mathcal{B}$-γ-*cover*, briefly a $\mathcal{B}$-γ^s-*cover*, if $\mathcal{U}$ is infinite and for every $B \in \mathcal{B}$ the set $\{U \in \mathcal{U} : \neg(\exists V \in \upsilon)\, \mathbb{B}(B, V) \subseteq U\}$ is finite. As above, we can assume that X does not belong to a $\mathcal{B}$-γ^s-cover. We denote by $\Omega^s_{\mathcal{B}}(X)$ and $\Gamma^s_{\mathcal{B}}(X)$ the family of all open $\mathcal{B}$-ω^s-covers and open $\mathcal{B}$-γ^s-covers of X, respectively. Then, we have

$$\Gamma^s_{\mathcal{B}}(X) \subseteq \Omega^s_{\mathcal{B}}(X) \subseteq \mathcal{O}(X).$$

Similarly as above, we have

$$\Gamma^s_{\mathrm{Fin}}(X) = \Gamma(X), \quad \Omega^s_{\mathrm{Fin}}(X) = \Omega(X).$$

One can easily see that for $\Phi = \Omega, \Gamma$ we have

$$\Phi^s_{\mathcal{B}}(X) \subseteq \Phi_{\mathcal{B}}(X) \subseteq \Phi(X).$$

[3] X has the property $(\varepsilon) = (\varepsilon_{\mathrm{Fin}})$ if and only if X^n is Lindelöf for every $n > 0$.

Both types of covers suggest to introduce corresponding topology on ${}^X\mathbb{R}$. The topology $\tau_{\mathcal{B}}$ is defined by typical neighborhoods of a function $h \in {}^X\mathbb{R}$ of the form

$$\mathcal{N}_{B,\varepsilon}(h) = \big\{f \in {}^X\mathbb{R} : (\forall x \in B)\, |h(x) - f(x)| < \varepsilon\big\} \tag{2}$$

for a set $B \in \mathcal{B}$ and $\varepsilon > 0$.

The product topology τ_p on ${}^X\mathbb{R}$ is actually the topology τ_{Fin}.

For a uniform space $\langle X, \upsilon\rangle$, the topology[4] related to $\mathcal{B}$-φ^s-covers $\tau^s_{\mathcal{B}}$ defined by typical neighborhoods of a function $h \in {}^X\mathbb{R}$ of the form

$$\mathcal{N}^s_{B,\varepsilon}(h) = \big\{f \in {}^X\mathbb{R} : (\exists V \in \upsilon)\big(\forall x \in \mathbb{B}(B, V)\big)\, |h(x) - f(x)| < \varepsilon\big\} \tag{3}$$

for $B \in \mathcal{B}$ and $\varepsilon > 0$.

One can easily see that

$$\tau_p \subseteq \tau_{\mathcal{B}} \subseteq \tau^s_{\mathcal{B}}.$$

3. Families of real functions

If $F \subseteq {}^X\mathbb{R}$, then we denote

$$F^+ = \{f \in F : (\forall x \in X)\, f(x) \geq 0\}, \quad F^* = \{f \in F : f \text{ is bounded}\}.$$

Instead of $\mathrm{C}_p(X)^*$ and $\mathrm{USC}_p(X)^*$, we shall write $\mathrm{C}^*_p(X)$ and $\mathrm{USC}^*_p(X)$, respectively. If $c \in \mathbb{R}$ is a real, then $\mathbf{c}$ denotes the constant function on X with the value c.

Let $\mathcal{B}$ be a bornology on X. We shall consider three properties of an infinite family $F \subseteq {}^X\mathbb{R}$ of real functions and a function $h \in {}^X\mathbb{R}$.

$(\mathcal{O}_h)_{\mathcal{B}}$ $h(x) \in \overline{\{f(x) : f \in F\}}$ *for every* $x \in X$.

$(\Omega_h)_{\mathcal{B}}$ $h \notin F$ *and* $h \in \overline{F}$ *in the topology* $\tau_{\mathcal{B}}$.

$(\Gamma_h)_{\mathcal{B}}$ F *is infinite, for every* $\varepsilon > 0$ *and for every* $B \in \mathcal{B}$ *the set* $\{f \in F : (\exists x \in B)\, |f(x) - h(x)| \geq \varepsilon\}$ *is finite.*

Omitting h from a set F with $(\Gamma_h)_{\mathcal{B}}$, we obtain

$$(\Gamma_h)_{\mathcal{B}} \to (\Omega_h)_{\mathcal{B}} \to (\mathcal{O}_h)_{\mathcal{B}}.$$

One can easily see that for $\Phi = \mathcal{O}, \Omega, \Gamma$ we have

If $\langle F,+\rangle$, $F \subseteq {}^X\mathbb{R}$ is a group, then F has the property $(\Phi_h)_{\mathcal{B}}$ if and only if F has the property $(\Phi_{h+f})_{\mathcal{B}}$ for every $f \in F$.

[4]The referee of the paper suggested the notation $\tau^s_{\mathcal{B}}$ instead of $\tau_{\mathcal{B}^s}$ used in [1] and [5]. I agree since the notation $\tau_{\mathcal{B}^s}$ suggests that $\mathcal{B}^s$ is a bornology. The same for the notion of a $\mathcal{B}$-φ^s-cover.

Let Φ be one of the symbols $\mathcal{O}, \Omega, \Gamma$. If $F \subseteq {}^X\mathbb{R}$ is a set of real functions, we set

$$\Phi_{h,\mathcal{B}}(F) = \{H \subseteq F : H \text{ has the property } (\Phi_h)_{\mathcal{B}}\}.$$

If $f \in {}^X\mathbb{R}$, $\varepsilon > 0$, we denote

$$U_f^\varepsilon = \{x \in X : |f(x)| < \varepsilon\}.$$

If $F \subseteq {}^X\mathbb{R}$, we set

$$\mathcal{U}^\varepsilon(F) = \{U_f^\varepsilon : f \in F\}.$$

In our consideration we must be careful, since it may happen that

$$U_{f_1}^{\varepsilon_1} = U_{f_2}^{\varepsilon_2} \quad \text{for } f_1 \neq f_2 \quad \text{or } \varepsilon_1 \neq \varepsilon_2.$$

If f is a continuous function or a non-negative upper semicontinuous function, then U_f^ε is an open set.

The following rather simple generalization of Theorem 4.1 of the author [2] will play a crucial role. Note that for $\mathcal{B} = \mathrm{Fin}$ we obtain Theorem 4.1 of [2].

Theorem 1. *Let $\mathcal{B}$ be a bornology on a topological space X. We assume that $F \subseteq {}^X\mathbb{R}$ is an infinite family of real functions.*

a) *The family F has the property $(\mathcal{O}_\mathbf{0})$ if and only if for every $\varepsilon > 0$ the family $\mathcal{U}^\varepsilon(F)$ is an o-cover of X.*

b) *The family F has the property $(\Omega_\mathbf{0})_\mathcal{B}$ if and only if $\mathbf{0} \notin F$ and either there exists a subsequence of F uniformly converging to $\mathbf{0}$ or there exists $\delta > 0$ such that for every $\varepsilon < \delta$ the family $\mathcal{U}^\varepsilon(F)$ is a $\mathcal{B}$-ω-cover of X.*

c) *The family F has the property $(\Gamma_\mathbf{0})_\mathcal{B}$ if and only if either F is countable and $F \rightrightarrows \mathbf{0}$ or there exists $\delta > 0$ such that for every $\varepsilon < \delta$ the family $\mathcal{U}^\varepsilon(F)$ is a $\mathcal{B}$-γ-cover of X and for every $f \in F$, such that $U_f^\varepsilon \neq X$, the set $\{g \in F : U_g^\varepsilon = U_f^\varepsilon\}$ is finite.*

Proof. The part a) is the part 1) of [2, Theorem 4.1].

Let F be an infinite family of real functions on X.

We set

$$\eta = \inf\{\varepsilon : X \in \mathcal{U}^\varepsilon(F) \vee \varepsilon = 1\}. \tag{4}$$

Let F possess the property $(\Omega_\mathbf{0})_\mathcal{B}$.

Assume that $\eta = 0$. Then, for every $\varepsilon > 0$ we have $X \in \mathcal{U}^\varepsilon(F)$. Let $\varepsilon_n = 2^{-n}$. Then, for every n there exists a function $h_n \in F$ such that $X = U_{h_n}^{\varepsilon_n}$. Since $(\forall x \in X)\, |h_n(x)| < 2^{-n}$, we obtain $h_n \rightrightarrows \mathbf{0}$.

Let η be positive. We set $\delta = \eta$. Let $\varepsilon < \delta$. Since $\mathbf{0} \in \overline{F}$ in the topology $\tau_\mathcal{B}$, for any set $B \in \mathcal{B}$ there exists an $f \in F$ such that $f \in \mathcal{N}_{B,\varepsilon}(\mathbf{0})$. Then, $B \subseteq U_f^\varepsilon$. Since $X \notin \mathcal{U}^\varepsilon(F)$, the family $\mathcal{U}^\varepsilon(F)$ is a $\mathcal{B}$-ω-cover.

The proof of opposite implication is similar. Either there exists a sequence $\langle f_n : n \in \omega\rangle$ of members of F such that $f_n \rightrightarrows \mathbf{0}$. Then F trivially has the property $(\Omega_\mathbf{0})_\mathcal{B}$. Or for any $\varepsilon < \delta$ and $B \in \mathcal{B}$ there exists $f \in F$ such that $B \subseteq U_f^\varepsilon$. Then, $f \in \mathcal{N}_{B,\varepsilon}(\mathbf{0})$. Consequently, $\mathbf{0} \in \overline{F}$ in the topology $\tau_\mathcal{B}$.

Assume now that the family F has the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$.

Assume $\eta > 0$. Then, we set $\delta = \eta$. Let $\varepsilon < \delta$. Then, $X \notin \mathcal{U}^{\varepsilon}(F)$. Given $h \in F$ only for finitely many $g \in F$, it may happen that $U_g^{\varepsilon} = U_h^{\varepsilon}$. Indeed, since $U_h^{\varepsilon} \neq X$, there exists an $x \notin U_h^{\varepsilon}$. If $U_g^{\varepsilon} = U_h^{\varepsilon}$, then g is in the finite set $\{f \in F : |f(x)| \geq \varepsilon\}$. So, $\mathcal{U}^{\varepsilon}(F)$ is infinite. By $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$, for any $B \in \mathcal{B}$ the set $\{f \in F : B \nsubseteq U_f^{\varepsilon}\}$ is finite, and therefore, $\mathcal{U}^{\varepsilon}(F)$ is a $\mathcal{B}$-γ-cover.

Assume now $\eta = 0$. For a positive real β we denote $F_{\beta} = \{f \in F : U_f^{\beta} \neq X\}$. Evidently, $F_{\beta_1} \subseteq F_{\beta_2}$ for any $\beta_2 < \beta_1$. If there exists a $\beta > 0$ such that F_{β} is infinite, set $\delta = \beta$. By the same argument as above, we obtain that for every $\varepsilon < \delta$ and for every $f \in F_{\varepsilon}$ the set $\{g \in F_{\varepsilon} : U_g^{\varepsilon} = U_f^{\varepsilon}\}$ is finite. Therefore, the family $\mathcal{U}^{\varepsilon}(F_{\varepsilon})$ is infinite. As above, it is easy to see that for any $\varepsilon < \delta$ the family $\mathcal{U}^{\varepsilon}(F)$ is a $\mathcal{B}$-γ-cover.

Assume now that for every $\beta > 0$ the set F_{β} is finite. One can easily show that $F \setminus \{\mathbf{0}\} = \bigcup_n F_{2^{-n}}$. Hence, F is countable. Let $F = \{f_n : n \in \omega\}$. We show that $f_n \rightrightarrows 0$. Indeed, if $\varepsilon > 0$, then there exists a k such that $2^{-k} < \varepsilon$. Since $F_{2^{-k}}$ is finite, there exists an n_0 such that for any $n > n_0$ we have $f_n \notin F_{2^{-k}}$, i.e., $|f_n(x)| < \varepsilon$ for every $x \in X$.

We prove the opposite implication for Γ.

If F is countable and $F \rightrightarrows \mathbf{0}$, then F satisfies $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$.

If for every $\varepsilon < \delta$ the family $\mathcal{U}^{\varepsilon}(F)$ is a $\mathcal{B}$-γ-cover and for every $f \in F$, such that $U_f^{\varepsilon} \neq X$, the set $\{g \in F : U_g^{\varepsilon} = U_f^{\varepsilon}\}$ is finite, then for every $\varepsilon < \delta$ and every $B \in \mathcal{B}$ the set $\{f \in F : B \nsubseteq U_f^{\varepsilon}\}$ is finite. Hence, F satisfies the condition $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$. □

Lemma 6.1 of [2] may be easily generalized as follows.

Lemma 2. *Assume that $\langle \varepsilon_n : n \in \omega \rangle$ is a sequence of positive reals converging to* 0*, and $f_n \in {}^{X}\mathbb{R}$ for $n \in \omega$. If $\{U_{f_n}^{\varepsilon_n} : n \in \omega\}$ is a $\mathcal{B}$-λ-, a $\mathcal{B}$-ω- or a $\mathcal{B}$-γ-cover, then there exists either an increasing sequence $\langle n_k : k \in \omega \rangle$ of integers such that $f_{n_k} \rightrightarrows \mathbf{0}$ or there exists a $\delta > 0$ such that for every positive $\varepsilon < \delta$, the family $\mathcal{U}^{\varepsilon}(\{f_n : n \in \omega\})$ is a $\mathcal{B}$-o-, a $\mathcal{B}$-ω- or a $\mathcal{B}$-γ-cover, respectively.*

4. Selections from sequence of covers

We recall the definition and some basic properties of selections from seuqences of covers important for our work, compare M. Scheepers [15] and W. Just, A. W. Miller, M. Scheepers, and P. J. Szeptycki [10].

If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(Y)$ are sets of subsets of a set Y, then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ means the following: for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ and for every $n \in \omega$ there exists a $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\} \in \mathcal{B}$. $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ means that for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ and for every $n \in \omega$ there exists a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n \mathcal{V}_n \in \mathcal{B}$.

If $Y = \mathcal{P}(X)$ for some set X, $\mathsf{U}_{\rm fin}(\mathcal{A}, \mathcal{B})$ means that for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ and for every $n \in \omega$ there exists a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \omega\} \in \mathcal{B}$. Evidently,

$$\mathsf{S}_1(\mathcal{A}, \mathcal{B}) \to \mathsf{S}_{\rm fin}(\mathcal{A}, \mathcal{B}) \to \mathsf{U}_{\rm fin}(\mathcal{A}, \mathcal{B}),$$

where the latter implication supposes that $\bigcup_n \mathcal{V}_n \in \mathcal{B} \to \{\bigcup \mathcal{V}_n : n \in \omega\} \in \mathcal{B}$ for finite $\mathcal{V}_n$.

If $Y \subseteq {}^X\mathbb{R}$ for some X, then $\mathsf{U}^*_{\rm fin}(\mathcal{A}, \mathcal{B})$ means that for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of elements of $\mathcal{A}$ and for every $n \in \omega$ there exists a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that[5] $\{\min \mathcal{V}_n : n \in \omega\} \in \mathcal{B}$.

If $Y = \mathcal{P}(X)$, where X is a topological space, then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is called a *covering property* and the topological space possessing the property $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is called $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$-space. If $Y \subseteq {}^X\mathbb{R}$, then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is called a *sequence selection property*. Similarly, for $\mathsf{S}_{\rm fin}(\mathcal{A}, \mathcal{B})$, $\mathsf{U}_{\rm fin}(\mathcal{A}, \mathcal{B})$ and $\mathsf{U}^*_{\rm fin}(\mathcal{A}, \mathcal{B})$.

Let Φ, Ψ be one of the symbols $\mathcal{O}, \Lambda, \Omega, \Gamma$, $F \subseteq {}^X\mathbb{R}$. If $\mathsf{S}_1\big(\Phi_{h,\mathcal{B}}(F), \Psi_{h,\mathcal{B}}(F)\big)$ holds true, we say that the set F *satisfies the selection principle* $\mathsf{S}_1(\Phi_{h,\mathcal{B}}, \Psi_{h,\mathcal{B}})$. Similarly for $\mathsf{S}_{\rm fin}$ and $\mathsf{U}^*_{\rm fin}$.

No topological space is a $\mathsf{S}_1(\mathcal{O}_\mathcal{B}, \Psi_\mathcal{B})$-space, neither $\mathsf{U}_{\rm fin}(\mathcal{O}^{sh}_\mathcal{B}, \Psi_\mathcal{B})$-space for $\Psi = \Lambda, \Omega, \Gamma$, compare [15] or [2]. By Theorem 17, (3)$\to$(1) of [15], we have

$$\mathsf{S}_1(\mathcal{O}, \mathcal{O}) \equiv \mathsf{S}_1(\Omega, \mathcal{O}), \quad \mathsf{S}_1(\mathcal{O}^{sh}, \mathcal{O}) \equiv \mathsf{S}_1(\Omega^{sh}, \mathcal{O}). \tag{5}$$

Thus, we do not need to investigate S_1 with the first argument $\mathcal{O}$ or $\mathcal{O}^{sh}$. We do not know similar results for bornological covers.

The proof of the equivalence (6) in [2] can be easily modified to a proof of[6]

$$\mathsf{S}_1\big((\Omega_\mathcal{B})_{ctbl}, \mathcal{O}\big) \equiv \mathsf{S}_1\big((\Omega_\mathcal{B})_{ctbl}, \Lambda\big), \quad \mathsf{S}_1(\Gamma_\mathcal{B}, \mathcal{O}) \equiv \mathsf{S}_1(\Gamma_\mathcal{B}, \Lambda). \tag{6}$$

As a consequence, we obtain

$$\text{If } X \text{ has the property } (\varepsilon_\mathcal{B}), \text{ then } \mathsf{S}_1(\Omega_\mathcal{B}, \mathcal{O}) \equiv \mathsf{S}_1(\Omega_\mathcal{B}, \Lambda). \tag{7}$$

Similarly for $\mathsf{S}_{\rm fin}$ and $\mathsf{U}_{\rm fin}$, see, e.g., [3].

We shall need the following result, see, e.g., [2] and [3].

Lemma 3 (Folklore). *If $\Phi = \Omega, \Gamma$, then $\mathsf{S}_1(\Phi, \Gamma)$ is equivalent to the following: for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of $\mathcal{B}$-φ-covers for every $n \in \omega$ there exists a set $U_n \in \mathcal{U}_n$ such that*

$$(\forall B \in \mathcal{B})(\exists n_0)(\forall n \geq n_0)\, B \subseteq U_n.$$

Similarly for $\mathsf{S}_{\rm fin}$ and $\mathsf{U}_{\rm fin}$.

[5] By definition, $\min \emptyset = \mathbf{1}$.

[6] If $\mathcal{A}$ is a family of sets, then $\mathcal{A}_{ctbl}$ is the family of all countable sets from $\mathcal{A}$.

5. Measurable and Semimeasurable Functions

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a family of subsets of X. For simplicity, we assume that $\emptyset, X \in \mathcal{A}$. A real function $f \in {}^X\mathbb{R}$ is $\mathcal{A}$-*measurable* if for every open interval (a, b), including $a = -\infty$ and $b = +\infty$, we have $f^{-1}((a, b)) \in \mathcal{A}$. A real function $f \in {}^X\mathbb{R}$ is *upper* $\mathcal{A}$-*semimeasurable* if $f^{-1}((-\infty, b)) \in \mathcal{A}$ for every $b \in \mathbb{R}$. One can easily see that if $\mathcal{A}$ is a σ-algebra, then every upper $\mathcal{A}$-semimeasurable function is $\mathcal{A}$-measurable. We denote by $\mathrm{M}(X, \mathcal{A})$ the set of all $\mathcal{A}$-measurable real functions defined on X and by $\mathrm{USM}(X, \mathcal{A})$ the set of all upper $\mathcal{A}$-semimeasurable functions defined on X. If the sets $\mathrm{M}(X, \mathcal{A})$ and $\mathrm{USM}(X, \mathcal{A})$ are endowed with the subspace topology τ_p of ${}^X\mathbb{R}$, we write $\mathrm{M}_p(X, \mathcal{A})$ and $\mathrm{USM}_p(X, \mathcal{A})$, respectively. Note that

$$\mathrm{C}_p(X) = \mathrm{M}_p(X, \vartheta), \quad \mathrm{USC}_p(X) = \mathrm{USM}_p(X, \vartheta).$$

As above, we say that the pair $\langle X, \mathcal{A}\rangle$ *satisfies the covering property* $\mathsf{S}_1(\Phi, \Psi)$ if $\mathsf{S}_1\big(\Phi(\mathcal{A}), \Psi(\mathcal{A})\big)$ holds true. Similarly for $\mathsf{S}_{\mathrm{fin}}$ and $\mathsf{U}_{\mathrm{fin}}$.

In [3], we have proved

Theorem 4 (L.B. [3]). *Assume that Φ is one of the symbols Ω, Γ and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Let $\mathcal{A}$ be a family of subsets of a set X. If $\Psi = \Gamma$, we assume that $\mathcal{A}$ is closed under finite intersections. In part* c), *we assume that $\mathcal{A}$ is closed under finite unions. Assume that $\langle \Phi, \Psi\rangle \neq \langle \Omega, \mathcal{O}\rangle$.*

a) *Then, the pair $\langle X, \mathcal{A}\rangle$ possesses the covering property $\mathsf{S}_1(\Phi, \Psi)$ if and only if the set $\mathrm{USM}_p(X, \mathcal{A})^+$ satisfies the selection principle $\mathsf{S}_1(\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}})$.*

b) *Similarly for $\mathsf{S}_{\mathrm{fin}}$.*

c) *The pair $\langle X, \mathcal{A}\rangle$ possesses the covering property $\mathsf{U}_{\mathrm{fin}}(\Phi, \Psi)$ if and only if the set $\mathrm{USM}_p(X, \mathcal{A})^+$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}})$.*

If $\mathcal{A}$ is a σ-algebra of sets, then $\mathrm{USM}(X, \mathcal{A})^+$ may be replaced by $\mathrm{M}(X, \mathcal{A})$.

If $\mathcal{B}$ is a bornology on X, for $\Phi = \Omega, \Gamma$, we set

$$\Phi_{\mathcal{B}}(\mathcal{A}) = \{\mathcal{U} : \mathcal{U} \subseteq \mathcal{A} \wedge \mathcal{U} \text{ is } \mathcal{B}\text{-}\varphi\text{-cover}\}.$$

As above, we say that the pair $\langle X, \mathcal{A}\rangle$ *possesses the covering property* $\mathsf{S}_1(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$ if $\mathsf{S}_1\big(\Phi_{\mathcal{B}}(\mathcal{A}), \Psi_{\mathcal{B}}(\mathcal{A})\big)$ holds true. Then

Theorem 5. *Let $\mathcal{B}$ be a bornology on a topological space X. Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, and Γ. Let $\mathcal{A}$ be a family of subsets of a set X. If $\Psi = \Gamma$, we assume that $\mathcal{A}$ is also closed under finite intersections. In part* c), *we assume that $\mathcal{A}$ is closed under finite unions. Assume that $\langle \Phi, \Psi\rangle \neq \langle \Omega, \mathcal{O}\rangle$.*

a) *Then, the pair $\langle X, \mathcal{A}\rangle$ possesses the covering property $\mathsf{S}_1(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$ if and only if $\langle \mathrm{USM}(X, \mathcal{A})^+, \tau_{\mathcal{B}}\rangle$ satisfies the selection principle $\mathsf{S}_1(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

b) *Similarly for* $\mathsf{S}_{\mathrm{fin}}$.

c) *Let* $\mathcal{A}$ *be closed under finite unions. Then, the pair* $\langle X, \mathcal{A}\rangle$ *possesses the covering property* $\mathsf{U}_{\mathrm{fin}}(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$ *if and only if the set* $\langle \mathrm{USM}(X, \mathcal{A})^+, \tau_{\mathcal{B}}\rangle$ *satisfies the selection principle* $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.

If $\mathcal{A}$ *is a* σ*-algebra of sets, then* $\mathrm{USM}(X, \mathcal{A})^+$ *may be replaced by* $\mathrm{M}(X, \mathcal{A})$.

Proof. Let $\langle X, \mathcal{A}\rangle$ satisfy $\mathrm{S}_1(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$. Assume that $\langle F_n : n \in \omega\rangle$ is a sequence of sets of non-negative real upper $\mathcal{A}$-semimeasurable functions and each set F_n has the property $(\Phi_{\mathbf{0}})_{\mathcal{B}}$. Each family $\mathcal{U}^{\varepsilon}(F_n)$ is a subset of $\mathcal{A}$.

If $\Phi = \Omega$, then, by Theorem 1, for every n there exists either a subsequence of F_n uniformly converging to $\mathbf{0}$ or a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^{\varepsilon}(F_n)$ is a $\mathcal{B}$-φ-cover.

If $\Phi = \Gamma$, then, by Theorem 1, for every n either the family F_n is countable and $F_n \rightrightarrows \mathbf{0}$ or there exists a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^{\varepsilon}(F_n)$ is a $\mathcal{B}$-γ-cover and for every $f \in F_n$ such that $U^{\varepsilon}_f \neq X$, the set $\{g \in F_n : U^{\varepsilon}_g = U^{\varepsilon}_f\}$ is finite.

In both cases, $\Phi = \Omega$ or $\Phi = \Gamma$, let A be the set of those $n \in \omega$ for which there exists a $\delta_n > 0$ such that for every $\varepsilon < \delta_n$ the family $\mathcal{U}^{\varepsilon}(F_n)$ is a $\mathcal{B}$-φ-cover.

If A is finite, one can find a sequence $\langle f_n \in F_n : n \in \omega \setminus A\rangle$ such that $f_n \rightrightarrows \mathbf{0}$. The family $\{f_n : n \in \omega \setminus A\}$ has the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$. For $n \in A$ take $f_n \in F_n$, $f_n \neq \mathbf{0}$ arbitrary. Then, the family $\{f_n : n \in \omega\}$ has the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$ as well. If $\Psi = \Omega$, then $f_n \neq \mathbf{0}$ for every n. Hence, the family $\{f_n : n \in \omega\}$ has the property $(\Omega_{\mathbf{0}})_{\mathcal{B}}$.

So, let A be infinite. We set $\varepsilon_n = \min\{\delta_n/2, 2^{-n}\}$ for $n \in A$.

If $\Psi = \Omega$, we apply $\mathrm{S}_1(\Phi_{\mathcal{B}}, \Omega_{\mathcal{B}})$ to the sequence $\{\mathcal{U}^{\varepsilon_n}(F_n) : n \in A\}$. We obtain sets $U_n \in \mathcal{U}^{\varepsilon_n}(F_n)$, $n \in A$ such that $\{U_n : n \in A\}$ is a $\mathcal{B}$-ω-cover. For every $n \in A$ there exists a function $f_n \in F_n$ such that $U_n = U^{\varepsilon_n}_{f_n}$. By Lemma 2, there exists either a subsequence uniformly converging to $\mathbf{0}$ or there exists a $\delta > 0$ such that the family $\{U^{\varepsilon}_{f_n} : n \in A\}$ is a $\mathcal{B}$-ω-cover for each positive $\varepsilon < \delta$. Therefore, the family $\{f_n : n \in A\}$ has the property $(\Omega_{\mathbf{0}})_{\mathcal{B}}$.

Now, let $\Psi = \Gamma$. Since for every $n \in A$ the family $\{\mathcal{U}^{\varepsilon_n}(F_n)$ is a $\mathcal{B}$-γ-cover, by Lemma 3, for every $n \in A$ there exists $U_n \in \mathcal{U}^{\varepsilon_n}(F_n)$ such that

$$(\forall B \in \mathcal{B})(\exists n_0)(\forall n \geq n_0, n \in A)\, B \subseteq U_n.$$

For every $n \in A$ there exists a function $f_n \in F_n$ such that $U_n = U^{\varepsilon_n}_{f_n}$. Let $B \in \mathcal{B}$. Since $\varepsilon_n \to 0$, we obtain that there exists an integer n_1 such that $\varepsilon_n \leq \varepsilon$ for each $n \geq n_1, n \in A$. If $n \geq \max\{n_0, n_1\}, n \in A$, then $f_n(x) < \varepsilon$ for each $x \in B$. Thus, the family $\{f_n : n \in A\}$ possesses the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$.

If $\Psi = \mathcal{O}$, then $\Phi = \Gamma$, and by (6), we have $\mathrm{S}_1(\Gamma_{\mathcal{B}}, \mathcal{O}) \equiv \mathrm{S}_1(\Gamma_{\mathcal{B}}, \Lambda)$. Applying $\mathrm{S}_1(\Gamma_{\mathcal{B}}, \Lambda)$ to the sequence $\{\mathcal{U}^{\varepsilon_n}(F_n) : n \in A\}$, we obtain sets $U_n \in \mathcal{U}^{\varepsilon_n}(F_n)$, $n \in A$ such that the family $\{U_n : n \in A\}$ is a λ-cover. We apply Lemma 2 to the family $\{U_n : n \in A\}$ and continue as above.

If Ψ is $\mathcal{O}$ or Ω, for $n \notin A$ take $f_n \in F_n$, $f_n \neq \mathbf{0}$ arbitrary. Then, the family $\{f_n : n \in \omega\}$ has the property $(\Psi_{\mathbf{0}})_{\mathcal{B}}$ as well.

Assume that $\Psi = \Gamma$. If $\omega \setminus A$ is finite, similarly as above, take $f_n \in F_n$ for $n \in \omega \setminus A$ arbitrary. Then, $\{f_n : n \in \omega\}$ has the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$. So, let $\omega \setminus A$ be infinite. Then, one can easily construct a sequence $\langle f_n \in F_n : n \in \omega \setminus A\rangle$ such that $f_n \rightrightarrows 0$, $n \in \omega \setminus A$. Then, the family $\{f_n : n \in \omega\}$ has the property $(\Gamma_{\mathbf{0}})_{\mathcal{B}}$.

For $\mathsf{S}_{\mathrm{fin}}$ and $\mathsf{U}_{\mathrm{fin}}$, the proof works equally.

Now, we show the opposite implication for $\mathsf{U}_{\mathrm{fin}}$. So, assume that the family $\mathrm{USM}(X, \mathcal{A})^+$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$. Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of $\mathcal{B}$-φ-covers, each a subset of $\mathcal{A}$. If $\varphi = \gamma$, we may assume that each $\mathcal{U}_n$ is countable and $X \notin \mathcal{U}_n$.

For any $U \in \mathcal{U}_n$, $n \in \omega$ set

$$f_U(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

If we set

$$F_n = \{f_U : U \in \mathcal{U}_n\},$$

then F_n is a family of non-negative upper $\mathcal{A}$-semimeasirable functions.

For any $\varepsilon \leq 1$, any $n \in \omega$ and any $U \in \mathcal{U}_n$ we have $U^{\varepsilon}_{f_U} = U$. Hence, for any n and any $\varepsilon \leq 1$ the cover $\mathcal{U}^{\varepsilon}(F_n)$ equals to the cover $\mathcal{U}_n$. Thus, every $\mathcal{U}^{\varepsilon}(F_n)$ is a $\mathcal{B}$-φ-cover.

By Theorem 1, every set F_n has the property $(\Phi_{\mathbf{0}})_{\mathcal{B}}$. By the selection principle $\mathsf{U}^*_{\mathrm{fin}}\big(\Phi_{\mathbf{0},\mathcal{B}}(\mathrm{USM}^+), \Psi_{\mathbf{0},\mathcal{B}}(\mathrm{USM}^+)\big)$ for every $n \in \omega$ there exists a finite set $H_n \subseteq F_n$ such that $\{\min H_n : n \in \omega\}$ has the property $(\Psi_{\mathbf{0}})_{\mathcal{B}}$. For every $n \in \omega$ and for every $f \in H_n$ there exists a set $U_{n,f} \in \mathcal{U}_n$ such that $f = f_{U_{n,f}}$. Since $X \notin \mathcal{U}_n$ for each n, neither $U_n = X$ for each n. So, by Theorem 1, for any $\varepsilon \leq 1$ the family $\mathcal{U}^{\varepsilon}(\{\min H_n : n \in \omega\})$ is a $\mathcal{B}$-ψ-cover. However,

$$\mathcal{U}^{\varepsilon}(\{\min H_n : n \in \omega\}) = \Big\{\bigcup\{U_{n,f} : f \in H_n\} : n \in \omega\Big\}. \qquad \square$$

If X is a topological space, then we denote by Borel the σ-algebra of Borel subsets of X.

Corollary 6. *Let $\mathcal{B}$ be a bornology on a topological space X. Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, and Γ. Assume that $\langle \Phi, \Psi\rangle \neq \langle \Omega, \mathcal{O}\rangle$.*

Then, the pair $\langle X, \mathrm{Borel}\rangle$ possesses the covering property $\mathsf{S}_1(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$ if and only if $\langle \mathrm{M}(X, \mathrm{Borel}), \tau_{\mathcal{B}}\rangle$ satisfies the selection principle $\mathsf{S}_1(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$. Similarly for $\mathsf{S}_{\mathrm{fin}}$.

The pair $\langle X, \mathrm{Borel}\rangle$ possesses the covering property $\mathsf{U}_{\mathrm{fin}}(\Phi_{\mathcal{B}}, \Psi_{\mathcal{B}})$ if and only if $\langle \mathrm{M}(X, \mathrm{Borel}), \tau_{\mathcal{B}}\rangle$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.

A special case $\mathcal{A} = \vartheta$ of Theorem 5 is a generalization of the main results of the author, Theorems 6.2, 6.3, 7.4 in [2], and Theorems 4.1 and 4.2 in [3].

Corollary 7. *Let $\mathcal{B}$ be a bornology on a topological space X. Assume that Φ is one of symbols Ω, Γ, and Ψ is one of symbols $\mathcal{O}, \Omega, \Gamma$, and $\langle \Phi, \Psi \rangle \neq \langle \Omega, \mathcal{O} \rangle$.*

a) *Then, X is an $\mathsf{S}_1(\Phi_\mathcal{B}, \Psi_\mathcal{B})$-space if and only if $\langle \mathrm{USC}(X)^+, \tau_\mathcal{B} \rangle$ satisfies the selection principle $\mathsf{S}_1(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

b) *Similarly for $\mathsf{S}_{\mathrm{fin}}$.*

c) *Then, X is a $\mathsf{U}_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}), \Psi_{\mathbf{0},\mathcal{B}})$-space if and only if $\langle \mathrm{USM}(X, \mathcal{A})^+, \tau_\mathcal{B} \rangle$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

If X has the property $(\varepsilon_\mathcal{B})$, then the equivalences hold true for the pair $\langle \Omega, \mathcal{O} \rangle$ as well.

6. Bornological covers and families of continuous real functions

Similarly as above, we generalize Theorems 6.2, 6.3 and 7.4 of [2], and Theorems 4.1 and 4.2 of [3] for bornological covers. We recommend to a reader to follow the proofs of corresponding results in [2] and [3].

Theorem 8. *Let $\mathcal{B}$ be a bornology on a normal topological space X. Assume that Φ is one of the symbols Ω, Γ, that Ψ is one of the symbols $\mathcal{O}$, Ω, Γ, and $\langle \Phi, \Psi \rangle \neq \langle \Omega, \mathcal{O} \rangle$.*

a) *The topological space X is an $\mathsf{S}_1(\Phi^{sh}_\mathcal{B}, \Psi_\mathcal{B})$-space if and only if $\langle \mathrm{C}(X), \tau_\mathcal{B} \rangle$ satisfies the selection principle $\mathsf{S}_1(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

b) *Similarly for $\mathsf{S}_{\mathrm{fin}}$.*

c) *The topological space X is a $\mathsf{U}_{\mathrm{fin}}(\Phi^{sh}_\mathcal{B}, \Psi_\mathcal{B})$-space if and only if $\langle \mathrm{C}(X), \tau_\mathcal{B} \rangle$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

If X has the property $(\varepsilon_\mathcal{B})$, then the equivalences hold true for the pair $\langle \Omega, \mathcal{O} \rangle$ as well.

P r o o f. The implications from left to right may be proved equally as in Theorem 5, just note that if $F \subseteq \mathrm{C}^*_p(X)$ has property $(\Phi_\mathbf{0})_\mathcal{B}$, then $\mathcal{U}^\epsilon(F)$ is a shrinkable $\mathcal{B}$-φ-cover.

We prove the implications from right to left for $\mathsf{U}_{\mathrm{fin}}$. Assume that $\langle \mathrm{C}(X), \tau_\mathcal{B} \rangle$ satisfies the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of covers from $\Phi^{sh}_\mathcal{B}$. Since $\mathcal{U}_n$ is an open shrinkable $\mathcal{B}$-φ-cover, then there exists an open $\mathcal{B}$-φ-cover $\mathcal{V}_n$ such that for each $V \in \mathcal{V}_n$ there exists a $U_V \in \mathcal{U}_n$ such that

$\overline{V} \subseteq U_V$. For $V \in \mathcal{V}_n$ take a continuous functions f_V such that $\mathrm{rng}(f_V) \subseteq [0,1]$, $f_V(x) = 1$ if $x \in X \setminus U_V$ and $f_V(x) = 0$ if $x \in \overline{V}$. Set

$$F_n = \{f_V : V \in \mathcal{V}_n\}.$$

It is easy to see that $F_n \in \Phi_{\mathbf{0},\mathcal{B}}$. E.g., let $\Phi = \Omega$. Then for a set $B \in \mathcal{B}$, there exists $V \in \mathcal{V}_n$ such that $B \subseteq V$. Then, $f_V(x) = 0$ for $x \in B$. Thus, we can apply the selection principle $\mathsf{U}^*_{\mathrm{fin}}(\Phi_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$ and we obtain finite sets $H_n \subseteq F_n$ such that $\{\min H_n : n \in \omega\} \in \Psi_{\mathbf{0},\mathcal{B}}$. Let $\mathcal{W}_n = \{V \in \mathcal{V}_n : f_V \in H_n\}$. Then, the family $\{\bigcup \mathcal{W}_n : n \in \omega\}$ is a $\mathcal{B}$-ψ-cover. □

Theorem 9. *Let $\mathcal{B}$ be a bornology with a closed base on a normal topological space X. Assume that Ψ is one of the symbols $\mathcal{O}, \Omega, \Gamma$. Then, the following are equivalent:*

a) *X is an $\mathsf{S}_1(\Omega_{\mathcal{B}}, \Psi_{\mathcal{B}})$-space.*

b) *$\langle \mathrm{USC}^*(X)^+, \tau_{\mathcal{B}}\rangle$ satisfies the selection principle $\mathsf{S}_1(\Omega_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

c) *$\langle \mathrm{C}^*(X), \tau_{\mathcal{B}}\rangle$ satisfies the selection principle $\mathsf{S}_1(\Omega_{\mathbf{0},\mathcal{B}}, \Psi_{\mathbf{0},\mathcal{B}})$.*

Proof. The implication a) $\to$ b) is a case of Corollary 7. The implication b) $\to$ c) is trivial. We show c) $\to$ a).

Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of $\mathcal{B}$-ω-covers. For every closed $B \in \mathcal{B}$ there exists an open set $U^n_B \in \mathcal{U}_n$ such that $B \subseteq U^n_B$. Since X is normal, for each closed $B \in \mathcal{B}$ there exists a continuous function $f^n_B : X \longrightarrow [0,1]$ such that $f^n_B(x) = 1$ for $x \in X \setminus U^n_B$ and $f^n_B(x) = 0$ for $x \in B$. Set

$$F_n = \{f^n_B : B \in \mathcal{B} \wedge B \text{ is closed}\}.$$

One can easily see that F_n has the property $(\Omega_{\mathbf{0}})_{\mathcal{B}}$. By $\mathrm{S}_1\big((\Omega_{\mathbf{0}})_{\mathcal{B}}, (\Psi_{\mathbf{0}})_{\mathcal{B}}\big)$, for every n there exists $f_n \in F_n$ such that the set $\{f_n : n \in \omega\}$ has the property $(\Psi_{\mathbf{0}})_{\mathcal{B}}$. Every f_n is $f^n_{B_n}$ for some $B_n \in \mathcal{B}$.

If $\Psi = \mathcal{O}$, then for every $x \in X$ there exists an f_n such that $f_n(x) < 1/2$. Then, $x \in U^n_{B_n}$. Thus, $\{U^n_{B_n} : n \in \omega\}$ is an open $\mathcal{B}$-o-cover.

If $\Psi = \Omega$, then for every $B \in \mathcal{B}$ there exists an $f_n = f^n_{B_n} \in \mathcal{N}_{B,1/2}$. Thus, $B \subseteq U^n_{B_n}$. Hence, $\{U^n_{B_n} : n \in \omega\}$ is an open $\mathcal{B}$-ω-cover.

Finally, if $\Psi = \Gamma$, then for every $B \in \mathcal{B}$ the set $\{f_n : (\exists x \in B)\, f_n(x) > 1/2\}$ is finite. Hence, the set $\{n : B \not\subseteq U^n_{B_n}\}$ is finite as well. Thus, $\{U^n_{B_n} : n \in \omega\}$ is an open $\mathcal{B}$-γ-cover. □

7. Topological properties of a family of real functions and selection principles

The following theorem is a generalization of a result by Arhangel'skiî and Pytkeeff, see, e.g., M. Sakai [14, p. 199].

Theorem 10 (A. Caserta, G. Di Maio and Ľ. Holá [5])**.** *Let $\mathcal{B}$ be a bornology with a closed base on a normal uniform space $\langle X, \upsilon\rangle$. Then, the following are equivalent:*

a) $\langle \mathrm{C}(X), \tau^s_{\mathcal{B}}\rangle$ *has countable tightness.*

b) *Every open $\mathcal{B}$-ω^s-cover has a countable $\mathcal{B}$-ω^s-subcover.*

The following theorem is a generalization of a result by F. Gerlits and Z. Nagy [7].

Theorem 11 (A. Caserta, G. Di Maio and Ľ. Holá [5])**.** *Let $\mathcal{B}$ be a bornology with a closed base on a normal uniform space $\langle X, \upsilon\rangle$. Then, the following are equivalent:*

a) $\langle \mathrm{C}(X), \tau^s_{\mathcal{B}}\rangle$ *is Fréchet.*

b) *Every open $\mathcal{B}$-ω^s-cover has a countable $\mathcal{B}$-γ^s-subcover.*

Both theorems remain true for any normal topological space X if we replace ω^s, γ^s and $\tau^s_{\mathcal{B}}$ by ω, γ and $\tau_{\mathcal{B}}$, respectively.

Following the proof of $\gamma \to \gamma'$ by J. Gerlits and Zs. Nagy in [7], we obtain

Theorem 12. *Let $\mathcal{B}$ be a bornology on a topological space X with a closed base, φ being one of the symbols ω, γ. Then, the following are equivalent:*

a) *X is an $\mathsf{S}_1(\Omega_{\mathcal{B}}, \Phi_{\mathcal{B}})$-space.*

b) *Every open $\mathcal{B}$-ω-cover has a countable $\mathcal{B}$-φ-subcover.*

If X is a uniform space, then similar equivalence holds true for $\mathcal{B}$-φ^s-covers.

P r o o f. Evidently a) $\to$ b). We show b) $\to$ a).

Assume that $[X]^{\aleph_0} \subseteq \mathcal{B}$. Since the bornology $\mathcal{B}$ is a proper ideal, X is uncountable. Then, there is no countable $\mathcal{B}$-ω-cover and both a) and b) are trivially false, hence equivalent.

So, assume that $[X]^{\aleph_0} \nsubseteq \mathcal{B}$. Let $\{x_n : n \in \omega\} \in [X]^{\aleph_0} \setminus \mathcal{B}$, $x_n \neq x_m$ for $n \neq m$. Assume that $\langle \mathcal{U}_n : n \in \omega\rangle$ is a sequence of open $\mathcal{B}$-ω-covers. We can assume that each $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$. Then

$$\mathcal{U} = \{U \setminus \{x_n\} : U \in \mathcal{U}_n \wedge n \in \omega\}$$

is an open $\mathcal{B}$-ω-cover. By b), there exists a $\mathcal{B}$-φ-subcover $\{V_k : k \in \omega\} \subseteq \mathcal{U}$. Let n_k be such that $V_k = U_{n_k} \setminus \{x_{n_k}\}$ for a $U_{n_k} \in \mathcal{U}_{n_k}$.

If $\{x_0, \dots, x_n\} \subseteq V_k$, then $n_k > n$. Thus, the set $\{n_k : k \in \omega\}$ is infinite. So, there exists an infinite set $A \subseteq \omega$ such that $\langle n_k : k \in A\rangle$ is increasing. For $k \in A$ we set $W_{n_k} = U_{n_k}$. For m smaller than the minimal n_k, $k \in A$ (if any) take $W_m \in \mathcal{U}_m$ arbitrary. If $n_k < m < n_l$, $k, l \in A$ and there exists no $p \in A$, $k < p < l$, take $W_m \in \mathcal{U}_m$ such that $U_{n_l} \subseteq W_m$. One can easily see that $\{W_m : m \in \omega\}$ is a $\mathcal{B}$-φ-cover. □

Corollary 13. *Let $\mathcal{B}$ be a bornology with a closed base on a normal topological space X. Then, the following are equivalent:*

a) $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ *has countable tightness.*

b) *Every open $\mathcal{B}$-ω-cover has a countable $\mathcal{B}$-ω-subcover.*

c) X *is an $\mathsf{S}_1(\Omega_{\mathcal{B}}, \Omega_{\mathcal{B}})$-space.*

If X is a uniform space, then similar equivalences hold true for the topology $\tau_{\mathcal{B}}^s$ and $\mathcal{B}$-ω^s-covers.

Proof. The proof of a) $\equiv$ b) is a slight modification of the proof of Theorem 10 in A. Caserta, G. Di Maio and Ľ. Holá [5]. The equivalence b) $\equiv$ c) is Theorem 12. □

Corollary 14. *Let $\mathcal{B}$ be a bornology with a closed base on a normal topological space X. Then, the following are equivalent:*

a) $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ *is Fréchet.*

b) *Every open $\mathcal{B}$-ω-cover has a countable $\mathcal{B}$-γ-subcover.*

c) X *is an $\mathsf{S}_1(\Omega_{\mathcal{B}}, \Gamma_{\mathcal{B}})$-space.*

d) $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ *is strictly Fréchet.*

If X is a uniform space, then similar equivalences hold true for the topology $\tau_{\mathcal{B}}^s$ and $\mathcal{B}$-ω^s- and $\mathcal{B}$-γ^s-covers, respectively.

Proof. As above, the proof of a) $\equiv$ b) is a slight modification of the proof of Theorem 11 in A. Caserta, G. Di Maio and Ľ. Holá [5]. The equivalence b) $\equiv$ c) is Theorem 12. By definition, $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ is strictly Fréchet when $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ satisfies the selection principle $\mathrm{S}_1(\Omega_{\mathbf{0},\mathcal{B}}, \Gamma_{\mathbf{0},\mathcal{B}})$. Thus, the equivalence c) $\equiv$ d) follows by Theorem 8. □

Theorem 15. *Let $\mathcal{B}$ be a bornology with a closed base on a topological space X. Then, the following are equivalent:*

a) $\langle \mathrm{USC}(X)^+, \tau_{\mathcal{B}} \rangle$ *is Fréchet.*

b) *every open $\mathcal{B}$-ω-cover has a countable $\mathcal{B}$-γ-subcover.*

c) X *is an $\mathsf{S}_1(\Omega_{\mathcal{B}}, \Gamma_{\mathcal{B}})$ space.*

d) $\langle \mathrm{USC}(X)^+, \tau_{\mathcal{B}} \rangle$ *is strictly Fréchet.*

Proof. In the proof of Theorem 9 we define the upper semicontinuous functions

$$f_B^n(x) = \begin{cases} 0 & \text{if } x \in U_B^n, \\ 1, & \text{otherwise.} \end{cases}$$

□

Corollary 16. *Let $\mathcal{B}$ be a bornology with a closed base on a normal topological space $\langle X, \vartheta \rangle$. Then*

a) $\langle \mathrm{USC}(X)^+, \tau_{\mathcal{B}} \rangle$ *has countable tightness if and only* $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ *is such.*

b) $\langle \mathrm{USC}(X)^+, \tau_{\mathcal{B}} \rangle$ *is Fréchet if and only if* $\langle \mathrm{C}(X), \tau_{\mathcal{B}} \rangle$ *is such.*

Remarks

Many special cases of presented results are known. I will mention all those of them that I know. I am sorry if I omit any other known result. The only reason for such an omitting is the fact that I do not know the result.

Some special cases of Theorems 7 and 8 were already known. Theorem 7 for $\Phi = \Psi = \Gamma$ was proved by the author [3] and for $\Phi = \Gamma$ and $\Psi = \Omega$ by M. Sakai [14]. Theorem 8 for $\Phi = \Psi = \Gamma$ was proved by the author and J. Haleš [4]. A theorem close to the case $\Phi = \Gamma$ and $\Psi = \Omega$ of Theorem 8 was proved by A.V. Osipov [12].

Two equivalences (i) $\equiv$ (iii) of Theorem 9 for $\mathcal{B} = \mathrm{Fin}$ are well-known.

One is the already mentioned classical result by F. Gerlits and Z. Nagy [7]:

$$X \textit{ has } \mathsf{S}_1(\Omega, \Gamma) \equiv \mathrm{C}_p(X) \textit{ is strictly Fréchet.}$$

For a Tychonoff topological space X we obtain a result of M. Sakai [13]:

$$X \textit{ has } \mathsf{S}_1(\Omega, \Omega) \equiv \mathrm{C}_p(X) \textit{ has countable strong fan tightness.}$$

The equivalence a) $\equiv$ b) of Corollary 13 is a special case of Theorem 3.5 by Ľ. Holá and B. Novotný [9].

H. Ohta and M. Sakai [11] investigated the property (USC) of a topological space: for every sequence $\langle f_n : n \in \omega \rangle$ of non-negative upper semicontinuous functions such that $f_n \to \mathbf{0}$, there exists a sequence $\langle g_n : n \in \omega \rangle$ of continuous functions such that $g_n \to \mathbf{0}$ and $f_n \leq g_n$ for each n. One can easily see that for a topological space X with the property (USC), $\mathsf{S}_1(\Gamma_{\mathbf{0}}, \Psi_{\mathbf{0}})$ for continuous functions implies $\mathsf{S}_1(\Gamma_{\mathbf{0}}, \Psi_{\mathbf{0}})$ for non-negative upper semicontinuous functions for any $\Psi = \Lambda, \Omega, \Gamma$. Therefore, if X is a normal topological space with the property (USC), then $\mathsf{S}_1(\Gamma^{sh}, \Psi) \equiv \mathsf{S}_1(\Gamma, \Psi)$. Moreover, they show that a subset X of the real line satisfies the property (USC) if and only if X is a σ-space, i.e., if $\mathrm{F}_\sigma(X) = \mathrm{G}_\delta(X)$. In this case, by J. Haleš [8], the property $\mathsf{S}_1(\Gamma, \Gamma)$ is hereditary, i.e., $\mathsf{S}_1\big(\Gamma(A), \Gamma(A)\big)$ holds true for any subset $A \subseteq X$.

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