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COMPACTNESS OF MULTIPLICATION OPERATORS ON RIESZ BOUNDED VARIATION SPACES

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ABSTRACT. We prove compactness of the operator M_hC_g on a subspace of the space of 2π-periodic functions of Riesz bounded variation on $[-\pi, \pi]$, for appropriate functions g and h. Here M_h denotes multiplication by h and C_g convolution by g.

1. Introduction

The spaces of functions of Riesz bounded variation were originally defined by F. Riesz in [4] and they constitute one of the possible generalizations of the classical spaces of functions of bounded variation introduced by C. Jordan in [3]. Much work has been done since then on those spaces and their generalizations. An excelent source to consult many of these develoments is the recent book by Appel, Banaś and Merentes [1].

Our goal in this work is to study boundedness and compactness of convolution and multiplication operators acting on appropriate subspaces of the space of functions of Riesz bounded variation. In order to state such results, the first step will be to obtain a sufficient condition to determine if a family in these spaces is totally bounded. Later, with this tool, we will formulate and prove our main result. All of this will be done in the following sections.

We will use standard notation, and, as usual, we will employ the letter C to denote a constant that could be changing line by line.

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2. Preliminaries

In this section, we will present some notation and definitions, as well as some known results necessary for our exposition.

We will consider 2π-periodic versions of Riesz bounded variation spaces. To this end denote by I the closed interval $[-\pi, \pi]$.

Definition 2.1. Let $1 < p < \infty$. The set of 2π-periodic functions of Riesz bounded variation is defined as follows

$$\mathrm{RBV}_P^p(I) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is } 2\pi\text{-periodic and } V_R^p(f, I) < \infty\},$$

where

$$V_R^p(f, I) := \sup \frac{1}{2\pi} \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{(x_i - x_{i-1})^{p-1}},$$

and the supremum is taken over all posible finite partitions $x_0 = -\pi < x_1 < \cdots \cdots < x_n = \pi$ of the interval I.

In what follows, we will simply write RBV_P^p to denote the above described space.

Notice that if we consider $p = 1$ in the above definition, then RBV_P^1 reduces to the classical space of (2π-periodic) bounded variation functions.

The space RBV_P^p enjoys several properties. Here is a list of some of them (see [1], for more information).

Proposition 2.2 ([1]). *For $1 < p < \infty$ we have the following assertions.*

1. RBV_P^p *is a Banach space with the norm*

$$\|f\|_{\mathrm{RBV}_P^p} := |f(-\pi)| + V_R^p(f, I)^{1/p}.$$

2. $\mathrm{Lip}_P(I) \subset \mathrm{RBV}_P^p(I) \subset \mathrm{AC}_P(I) \subset \mathrm{BV}_P(I)$, *where* $\mathrm{Lip}_P(I)$ *denotes the space of 2π-periodic* Lip*schitz functions on I,* $\mathrm{AC}_P(I)$ *denotes the space of 2π-periodic absolutely continuous functions on I, and* $\mathrm{BV}_P(I)$ *denotes the space of 2π-periodic bounded variation functions on I.*

3. *A 2π-periodic function f belongs to* RBV_P^p *if and only if $f \in \mathrm{AC}_P(I)$ and $f' \in L^p(I)$. In such case*

$$V_R^p(f, I)^{1/p} = \|f'\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^p \, \mathrm{d}t\right)^{1/p}.$$

4. *For $1 < p < q < \infty$ we have*

$$\mathrm{RBV}_P^q(I) \subset \mathrm{RBV}_P^p(I).$$

Remark 1. Notice that we also consider in RBV_P^p the norm

$$\|f\|^*_{\mathrm{RBV}_P^p} := \|f\|_\infty + V_R^p\,(f, I)^{1/p}\,.$$

Since RBV_P^p is also complete with this norm and $\|f\|_{\mathrm{RBV}_P^p} \leq \|f\|^*_{\mathrm{RBV}_P^p}$ for every f, from Open Mapping Theorem it follows that both norms are equivalent.

In view of Remark 1, we will sometimes use the norm $\|\cdot\|^*_{\mathrm{RBV}_P^p}$ when convenient.

We will denote by $\mathrm{RBV}_{P,0}^p$ the following subspace of RBV_P^p

$$\mathrm{RBV}_{P,0}^p = \left\{f \in \mathrm{RBV}_P^p : f\,(-\pi) = 0\right\}.$$

Since it is a closed subspace of RBV_P^p, it is also complete.

Next, we will state a criterion for total boundedness in the subspace $\mathrm{RBV}_{P,0}^p$. To do this, we will make use of the following result proved by Hanche-Olsen and Holden in [2].

Lemma 2.3 ([2]). *Let X be a metric space. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$, a metric space W, and a mapping $\Phi : X \to W$ so that $\Phi\,(X)$ is totally bounded, and whenever $x, y \in X$ are such that $d_W\,(\Phi\,(x)\,, \Phi\,(y)) < \delta$, then $d_X\,(x, y) < \varepsilon$. Then, X is totally bounded.*

Now, we can set the following criterion.

Proposition 2.4. *Let $\mathcal{F}$ be a family of functions in $\mathrm{RBV}_{P,0}^p$ satisfying the following conditions:*

i): *There exists a constant $M > 0$ such that*

$$\|f\|_{\mathrm{RBV}_P^p} \leq M \quad \text{for every } f \in \mathcal{F}.$$

ii): *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in \mathcal{F}$*

$$\left(\frac{1}{2\pi}\int\limits_{\delta\leq|x|\leq\pi} |f'\,(x)|^p\,\mathrm{d}x\right)^{1/p} < \varepsilon.$$

iii): *Given $\varepsilon > 0$, there exists $\eta > 0$ such that for every $f \in \mathcal{F}$*

$$\left(\frac{1}{2\pi}\int\limits_I |\tau_{-y}f'\,(x) - f'\,(x)|^p\,\mathrm{d}x\right)^{1/p} < \varepsilon$$

as long as $|y| < \eta$. Here, τ_{-y} denotes the operator $\tau_{-y}\varphi\,(\cdot) = \varphi\,(\cdot + y)$.

Then, $\mathcal{F}$ is totally bounded.

Proof. Let $J = \left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$. We can find $N \in \mathbb{N}$ large enough so that the closure of the union of the intervals

$$\left(\frac{-N\eta}{2}, \frac{-(N-2)\eta}{2}\right), \dots,$$
$$\left(\frac{-3\eta}{2}, \frac{-\eta}{2}\right), \left(\frac{-\eta}{2}, \frac{\eta}{2}\right), \left(\frac{\eta}{2}, \frac{3\eta}{2}\right), \dots, \left(\frac{(N-2)\eta}{2}, \frac{N\eta}{2}\right)$$

covers the interval $(-\delta, \delta)$. As in [2], Theorem 5, we can consider the map P of RBV_P^p into V, the linear span of the characteristic functions of the intervals I_k given above, defined as

$$Pf(x) = \begin{cases} \frac{1}{\eta} \int\limits_{I_k} f'(t)\,\mathrm{d}t & x \in I_k, \quad \text{for every } k, \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding in the same way that [2], we can prove that given $\varepsilon > 0$ and any $f \in \mathcal{F}$, there is a constant C such that

$$\|f' - Pf\|_p \le C\varepsilon.$$

From here, we get that

$$\|f'\|_p \le \|f' - Pf\|_p + \|Pf\|_p \le C\varepsilon + \|Pf\|_p,$$

hence, for any f, $g \in \mathcal{F}$ such that $\|Pf - Pg\|_p < \varepsilon$ we have

$$\begin{aligned} \|f-g\|_{\mathrm{RBV}_P^p} &= |(f-g)(-\pi)| + \|f' - g'\|_p \\ &= \|f' - g'\|_p \\ &\le \|f' - Pf\|_p + \|Pf - Pg\|_p + \|Pg - g'\|_p \\ &\le 2C\varepsilon + \varepsilon. \end{aligned}$$

Now, notice that $P : \left(\mathrm{RBV}_{P,0}^p, \|\cdot\|_{\mathrm{RBV}_P^p}\right) \to \left(V, \|\cdot\|_p\right)$ is a bounded operator since

$$\|Pf\|_p \le \|f'\|_p.$$

This implies that $\|Pf\|_p \le \|f\|_{\mathrm{RBV}_P^p} \le M$ for every $f \in \mathcal{F}$, that is, $P(\mathcal{F})$ is bounded. Being $P\left(\mathrm{RBV}_{P,0}^p\right)$ finite dimensional, it follows that $P(\mathcal{F})$ is totally bounded. Now, we can invoke Lemma 2.3, and conclude that $\mathcal{F}$ is totally bounded. □

3. Main result

Before stating our main result, we will prove a couple of auxiliary propositions.

Proposition 3.1. *If g is a 2π-periodic function in $L^1(I)$ and $f \in \mathrm{RBV}_P^p$, $1 < p < \infty$, then $f * g \in \mathrm{RBV}_P^p$ and*

$$\|f * g\|_{\mathrm{RBV}_P^p} \leq \|f\|_{\mathrm{RBV}_P^p} \|g\|_1 .$$

Proof. Since f is absolutely continuous on I, given $\varepsilon > 0$, we can find $\delta > 0$ such that for any finite collection of pairwise nonoverlapping intervals $[a_k, b_k] \subset [-\pi, \pi]$, $k = 1, \ldots, n$ satisfying $\sum_{k=1}^{n} (b_k - a_k) < \delta$, we have that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\varepsilon}{\|g\|_1}.$$

Notice that being f absolutely continuous on I, the convolution $f * g$ is well defined.

Thus,

$$\begin{aligned}\sum_{k=1}^{n} |(f * g)(b_k) - (f * g)(a_k)| &\leq \sum_{k=1}^{n} \frac{1}{2\pi} \int_I |\tau_y f(b_k) - \tau_y f(a_k)| \, |g(y)| \, \mathrm{d}y \\ &= \frac{1}{2\pi} \int_I \left(\sum_{k=1}^{n} |\tau_y f(b_k) - \tau_y f(a_k)| \right) |g(y)| \, \mathrm{d}y \\ &< \varepsilon\end{aligned}$$

since $\sum_{k=1}^{n} \big(b_k - y - (a_k - y)\big) < \delta$. Therefore, $f * g$ is absolutely continuous on I.

Moreover, $(f * g)' = f' * g \in L^p(I)$ and so, $f * g \in \mathrm{RBV}_P^p$.

Finally,

$$\begin{aligned}\|f * g\|_{\mathrm{RBV}_P^p} &= |f * g(-\pi)| + V_R^p(f * g, I)^{1/p} \\ &= |f * g(-\pi)| + \|f' * g\|_p \\ &\leq |f * g(-\pi)| + \|f'\|_p \|g\|_1 \\ &\leq \left(\|f\|_\infty + V_R^p(f, I)^{1/p} \right) \|g\|_1 \\ &\leq C \|f\|_{\mathrm{RBV}_P^p} \|g\|_1 .\end{aligned}$$

This concludes the proof. □

We will denote by C_g the operator of convolution with $g \in L^1(I)$ defined on the space RBV_P^p, for $1 < p < \infty$. Also, for a fixed function φ, we shall denote by M_φ the operator of multiplication by φ.

The following proposition is very simple, but it will be very useful for us.

Proposition 3.2. *Let $1 < p < \infty$ and $\varphi \in \mathrm{RBV}_P^p$. Then, the operator M_φ is bounded from RBV_P^p into RBV_P^p.*

Proof. It is known that RBV_P^p is an algebra of functions ([1]). Thus, $f\varphi \in \mathrm{RBV}_P^p$ for every $f \in \mathrm{RBV}_P^p$. Moreover,

$$\begin{aligned}\|M_\varphi(f)\|_{\mathrm{RBV}_P^p} &= |f\varphi(-\pi)| + \left\|(f\varphi)'\right\|_p \\ &\leq \|f\|_\infty \|\varphi\|_\infty + \|f'\|_p \|\varphi\|_\infty + \|f\|_\infty \|\varphi'\|_p \\ &\leq C\|\varphi\|_\infty \|f\|_{\mathrm{RBV}_P^p} + \|f\|_\infty \|\varphi\|_{\mathrm{RBV}_P^p} \\ &\leq C' \|\varphi\|_{\mathrm{RBV}_P^p} \|f\|_{\mathrm{RBV}_P^p}\end{aligned}$$

and this implies that

$$\|M_\varphi\|_{\mathrm{RBV}_P^p \to \mathrm{RBV}_P^p} \leq C' \|\varphi\|_{\mathrm{RBV}_P^p},$$

proving our claim. □

We will denote by $C_0^1(I)$ the set of 2π-periodic functions f with continuous derivative on I and such that $f(x) \to 0$ as $|x| \to \pi$.

Now, we state and prove our main result.

Theorem 3.3. *Given $g \in L^1$ and $h \in \mathrm{RBV}_P^p \cap C_0^1(I)$, the operator $M_h C_g$ is a compact operator from $\mathrm{RBV}_{P,0}^p$ into $\mathrm{RBV}_{P,0}^p$, $1 < p < \infty$.*

Proof. First of all, notice that in view of Propositions 3.1 and 3.2, for every $f \in \mathrm{RBV}_{P,0}^p$, $M_h C_g(f) \in \mathrm{RBV}_{P,0}^p$.

par Let $\mathcal{F}$ be a bounded subset of $\mathrm{RBV}_{P,0}^p$. Then, there exists a positive constant $M > 0$ such that for every $f \in \mathcal{F}$

$$\|f\|_{\mathrm{RBV}_P^p} \leq M.$$

Since $\mathrm{RBV}_{P,0}^p$ is complete, it is suffices to prove that $\{M_h C_g(f) : f \in \mathcal{F}\}$ is totally bounded. In order to do this, we only need to show that the three conditions of Proposition 2.4 are satisfied.

Let $f \in \mathcal{F}$, then by Propositions 3.1 and 3.2 we have

$$\begin{aligned}\|M_h C_g(f)\|_{\mathrm{RBV}_P^p} &\leq C\|h\|_{\mathrm{RBV}_P^p} \|f\|_{\mathrm{RBV}_P^p} \|g\|_1 \\ &\leq CM\|h\|_{\mathrm{RBV}_P^p} \|g\|_1,\end{aligned}$$

uniformly on $\mathcal{F}$, and so, condition (i) of Proposition 2.4 is satisfied.

Now, let $y \in I$ and f an arbitrary element of $\mathcal{F}$. Then, we get

$$\begin{aligned}
&\left[\frac{1}{2\pi}\int\limits_I \left|\tau_{-y}\big(M_h C_g(f)\big)'(x) - \big(M_h C_g(f)\big)'(x)\right|^p \mathrm{d}x\right]^{1/p} \\
&= \left[\frac{1}{2\pi}\int_I \left|\big(M_h C_g(f)\big)'(x+y) - \big(M_h C_g(f)\big)'(x)\right|^p \mathrm{d}x\right]^{1/p} \\
&= \left[\frac{1}{2\pi}\int_I |h'(f*g)(x+y) + h(f'*g)(x+y)\right. \\
&\qquad \left. - h'(f*g)(x) - h(f'*g)(x)\right|^p \mathrm{d}x\Bigg]^{1/p} \\
&\le \left[\frac{1}{2\pi}\int_I \left|\big(\tau_{-y}h'(x) - h'(x)\big)\right|^p |\tau_{-y}(f*g)(x)|^p\right]^{1/p} \\
&\quad + \left[\frac{1}{2\pi}\int_I |h'(x)|^p |\tau_{-y}(f*g)(x) - (f*g)(x)|^p \mathrm{d}x\right]^{1/p} \\
&\quad + \left[\frac{1}{2\pi}\int_I |\tau_{-y}h(x) - h(x)|^p |\tau_{-y}(f'*g)(x)|^p \mathrm{d}x\right]^{1/p} \\
&\quad + \left[\frac{1}{2\pi}\int_I |h(x)|^p |\tau_{-y}(f'*g)(x) - (f'*g)(x)|^p \mathrm{d}x\right]^{1/p} \\
&\le \|\tau_{-y}h' - h'\|_\infty \|\tau_{-y}(f*g)\|_p + \|h'\|_\infty \|\tau_{-y}(f*g) - f*g\|_p \\
&\quad + \|\tau_{-y}h - h\|_\infty \|\tau_{-y}(f'*g)\|_p + \|h\|_\infty \|\tau_{-y}(f'*g) - f'*g\|_p \\
&= \|\tau_{-y}h' - h'\|_\infty \|f*g\|_p + \|h'\|_\infty \|\tau_{-y}(f*g) - f*g\|_p \\
&\quad + \|\tau_{-y}h - h\|_\infty \|f'*g\|_p + \|h\|_\infty \|\tau_{-y}(f'*g) - f'*g\|_p .
\end{aligned}$$

Thus, given $\varepsilon > 0$, we can find $\eta > 0$ such that for $|y| < \eta$

$$\begin{aligned}
&\left\|\tau_{-y}\big(M_h C_g(f)\big)' - \big(M_h C_g(f)\big)'\right\|_p \\
&\le \varepsilon \|f\|_p \|g\|_1 + \varepsilon \|h'\|_\infty + \varepsilon \|f'\|_p \|g\|_1 + \varepsilon \|h\|_\infty \\
&\le C\varepsilon \left(\|f\|_{\mathrm{RBV}_p^p} \|g\|_1\right) + \varepsilon \left(\|h\|_\infty + \|h'\|_\infty\right)
\end{aligned}$$

since the operator τ_{-y} is continuous on L^p and h, h' are continuous on I.

So, condition (iii) of Proposition 2.4 is satisfied.

Let us finally verify condition (ii).

Let $\varepsilon > 0$ be any positive number. For $\delta > 0$ whose size will be conveniently chosen later, we have

$$\begin{aligned}
&\left(\frac{1}{2\pi} \int\limits_{\delta \le |x| \le \pi} \left| \left(M_h C_g(f) \right)' (x) \right|^p \mathrm{d}x \right)^{1/p} \\
&= \left(\frac{1}{2\pi} \int\limits_{\delta \le |x| \le \pi} \left| h' \left(f * g \right)(x) + h \left(f' * g \right)(x) \right|^p \mathrm{d}x \right)^{1/p} \\
&\le \left\| h' \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \left\| f * g \right\|_p + \left\| h \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \left\| f' * g \right\|_p \\
&\le \left(\left\| h' \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \left\| f \right\|_p + \left\| h \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \left\| f' \right\|_p \right) \left\| g \right\|_1 \\
&\le C \left(\left\| h' \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty + \left\| h \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \right) \left\| f \right\|_{\mathrm{RBV}_P^p} \left\| g \right\|_1 \\
&\le CM \left(\left\| h' \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty + \left\| h \chi_{\{\delta \le |x| \le \pi\}} \right\|_\infty \right) \left\| g \right\|_1 \\
&< CM\varepsilon
\end{aligned}$$

if we choose $\delta > 0$ sufficiently small, since h belongs to C_0^1.

Therefore, $M_h C_g$ is a compact operator. □

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