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ŁUKASIEWICZ LOGIC AND THE DIVISIBLE EXTENSION OF PROBABILITY THEORY

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Professor Roman Frič has passed away unexpectedly during the last stages of the editorial process of this article. He was a respected editor and contributor of Tatra Mt. Math. Publ. We will keep in our memory the name of a distinguished scientist and an excellent colleague.

ABSTRACT. We show that measurable fuzzy sets carrying the multivalued Łukasiewicz logic lead to a natural generalization of the classical Kolmogorovian probability theory. The transition from Boolean logic to Łukasiewicz logic has a categorical background and the resulting divisible probability theory possesses both fuzzy and quantum qualities. Observables of the divisible probability theory play an analogous role as classical random variables: to convey stochastic information from one system to another one. Observables preserving the Łukasiewicz logic are called conservative and characterize the "classical core" of divisible probability theory. They send crisp random events to crisp random events and Dirac probability measures to Dirac probability measures. The nonconservative observables send some crisp random events to genuine fuzzy events and some Dirac probability measures to nondegenerated probability measures. They constitute the added value of transition from classical to divisible probability theory.

1. Introduction

At previous ISCRFT conferences (International Summer Conference on Real Functions Theory) we have presented our results related to the transition

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from classical probability space $(\Omega, \mathbf{A}, p)$, see [10, 13], to its fuzzification

$$\Big(\Omega, \mathcal{M}(\mathbf{A}), \int (\cdot)\, \mathrm{d}p\Big),$$

where $\mathcal{M}(\mathbf{A})$ is the set of all measurable functions into [0,1] and $\int(\cdot)\,\mathrm{d}p$ is the probability integral with respect to p.

In the present contribution, we clarify the role of Łukasiewicz logic in the transition. Real functions, measure, and integration play a vital role.

2. Why measurable fuzzy events

L. A. Zadeh [23] proposed to extend classical **random events** represented by a σ-field $\mathbf{A}$ of crisp subsets of the set Ω of outcomes to the set $\mathcal{M}(\mathbf{A})$ of all measurable functions into the unit interval [0,1], i.e., measurable fuzzy subsets of Ω. As usual, we identify a subset A of Ω and its indicator function

$$\chi_A \in [0,1]^\Omega, \quad \chi_A(\omega) = 1 \quad \text{for } \omega \in A \quad \text{and} \quad \chi_A(\omega) = 0, \quad \text{otherwise.}$$

Both crisp and fuzzy random events can be interpreted as propositional functions (of the argument ω), the former equipped with the Boolean logic and the latter equipped with some fuzzy logic. Our choice, supported by deep arguments [12, 16, 18–20, 22], is the Łukasiewicz logic. Observe that $\mathcal{M}(\mathbf{A})$ is the smallest subset $\mathcal{X}$ of $[0,1]^\Omega$ containing $\mathbf{A}$ which is divisible and closed with respect to pointwise sequential limits, where "divisible" means that if $u \in \mathcal{X}$, then $u/n \in \mathcal{X}$ for each natural number n. The closedness with respect to pointwise sequential limits is assumed to facilitate limit stochastic constructions. It is known that $\mathcal{M}(\mathbf{A})$ is the smallest subset $\mathcal{X}$ of $[0,1]^\Omega$ containing $\mathbf{A}$, all constant fuzzy subsets of Ω and closed with respect to pointwise limits of sequences. The transition from crisp random events (represented by $\mathbf{A}$) to divisible random events (represented by $\mathcal{M}(\mathbf{A})$) is, in some sense, as advantageous as the transition from integers to rational (and real) numbers (cf. [1, 6, 7]). Another advantage of the extension of crisp sets to measurable fuzzy sets is the fact that the values in the unit interval [0,1], the range of probability measures, can be viewed as measurable fuzzy sets. Indeed, if Ω is the singleton set $\{\omega\}$, then $\mathbf{A}$ reduces to the trivial σ-field $\mathbf{T} = \{\emptyset, \Omega\}$ and $[0,1]$ can be viewed as $\mathcal{M}(\mathbf{T})$; note that we identify $r \in [0,1]$ and the function $r\chi_{\{\omega\}} \in \mathcal{M}(\mathbf{T})$. Consequently, in a **categorical approach** to probability theory, $[0,1] \equiv \mathcal{M}(\mathbf{T})$ becomes an object and the probability integral becomes a morphism of $\mathcal{M}(\mathbf{A})$ to $\mathcal{M}(\mathbf{T})$.

More detailed information on the extension of classical probability theory [10, 13] to divisible probability theory can be found in [6].

3. Why Łukasiewicz logic

Further, L. A. Zadeh has proposed to replace the classical probability measure with the probability integral.

In [17], M. Navara observed that no justification to define the probability of a fuzzy event $u \in \mathcal{M}(\mathbf{A})$ by the formula $\int u \,\mathrm{d}p$ was given by Zadeh, and discussed two distinct approaches to generalized probability: probability on tribes, and probability on MV-algebras with products [21], confirming the original proposal by Zadeh.

In [1], D. Babicová provided another (categorical) argument and characterized $\int(\cdot)\,\mathrm{d}p$ (mapping $\mathcal{M}(\mathbf{A})$ into $[0,1] \equiv \mathcal{M}(\mathbf{T})$) as a sequentially continuous additive linearization of $\mathcal{M}(\mathbf{A})$. Here, "sequential continuity" is considered with respect to the pointwise convergence of sequences, "additivity" means that if

$$u, v \in \mathcal{M}(\mathbf{A}) \quad \text{and} \quad u \leq 1 - v,$$

then

$$\int (u+v)\,\mathrm{d}p = \int u \,\mathrm{d}p + \int v\,\mathrm{d}p,$$

and "linearization" refers to an order preserving map from the lattice $\mathcal{M}(\mathbf{A})$ to the linearly (totally) ordered set $[0,1]$. Further, she proved that the embedding of $\mathbf{A}$ into $\mathcal{M}(\mathbf{A})$ and the extension of p into $\int(\cdot)\,\mathrm{d}p$ can be viewed as an epireflection. Observe that the additivity is closely related to the **Łukasiewicz strong disjunction** $\oplus$ restricted to orthogonal elements. Indeed, for

$$u, v \in \mathcal{M}(\mathbf{A}), \quad u \oplus v = \min\{1, u+v\} \quad \text{and} \quad u + v = u \oplus v$$

whenever $u \leq 1 - v$, hence from $\int u\,\mathrm{d}p \leq 1 - \int v\,\mathrm{d}p$ we get

$$\int (u \oplus v)\,\mathrm{d}p = \int u\,\mathrm{d}p \oplus \int v\,\mathrm{d}p.$$

The lattice $\mathcal{M}(\mathbf{A})$ of all measurable fuzzy subsets of the set Ω of outcomes is an accepted model of generalized random events. In the corresponding fuzzified probability theory (cf. [1–4, 6–9, 14]), $\mathcal{M}(\mathbf{A})$ is equipped with one of the three isomorphic structures: effect algebra, D-poset, A-poset. The structures are defined via (dual) partial binary operations of sum and difference, respectively [5]. States (generalized probability measures) are suitable sum-preserving maps of $\mathcal{M}(\mathbf{A})$ into [0,1]. Note that the Boolean logic on $\mathbf{A}$ coincides with the Łukasiewicz logic. Our aim is to show how Łukasiewicz logic makes the fuzzified probability theory more "transparent". Recall that $\mathcal{M}(\mathbf{A})$, besides the lattice structure, carries the pointwise convergence of sequences, the Łukasiewicz strong disjunction $\oplus$, the **Łukasiewicz strong conjunction** $\odot$: for $u, v \in \mathcal{M}(\mathbf{A})$, $u \odot v = \max\{0, u+v-1\}$, and the **Łukasiewicz negation** $(\cdot)^c$: for $u \in \mathcal{M}(\mathbf{A}), u^c = 1 - u$. Let $(\Xi, \mathbf{B})$ be a measurable space and let $\mathcal{M}(\mathbf{B})$ be the corresponding lattice of all measurable fuzzy subsets of Ξ. According to [1], a map $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is said to be an A-**homomorphism** provided it preserves the partial order, the top and bottom elements, and it is additive: for $u, v \in \mathcal{M}(\mathbf{B}), u \leq 1 - v$, we have $h(u + v) = h(u) + h(v)$. Sequentially continuous A-homomorphisms are called **observables** and in the corresponding categorical probability theory observables serve as morphisms.

For convenience, the Łukasiewicz operations in the domain and the range of a mapping $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ will be denoted by the same symbols $\oplus, \odot, (\cdot)^c$.

Definition 3.1. Let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be a map such that for all $u, v \in \mathcal{M}(\mathbf{B})$ we have:

(i) $h(u^c) = \big(h(u)\big)^c$,

(ii) $h(u \oplus v) = h(u) \oplus h(v)$,

(iii) $h(u \odot v) = h(u) \odot h(v)$.

Then we say that h preserves the Łukasiewicz operations.

Lemma 3.2. *Let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be a map preserving the Łukasiewicz operations. Then, h is an A-homomorphism.*

Proof. We have to prove that h preserves the partial order, the top and the bottom elements, and it is additive.

1. h preserves the partial order. Let $u, v \in \mathcal{M}(\mathbf{B}), u \leq v$. Put $w = v - u$. Then, from $v = u + w \leq 1$, we get $u + w = u \oplus w$ and
$$h(v) = h(u + w) = h(u \oplus w) = h(u) \oplus h(w) = \min\{1, h(u) + h(w)\}.$$
Hence, for all $\omega \in \Omega$ we have $h(v)(\omega) = 1$ or $h(v)(\omega) = h(u)(\omega) + h(w)(\omega)$. Consequently, $h(u)(\omega) \leq h(v)(\omega)$ for each $\omega \in \Omega$, which yields $h(u) \leq h(v)$.

2. h preserves the top and the bottom elements. From
$$h(0) = h(0 \oplus 0) = h(0) \oplus h(0) = \min\{1, h(0) + h(0)\}$$
we obtain
$$h(0) = h(0) + h(0) = 0.$$
Indeed, from $h(0)(\omega) = 1$ for some $\omega \in \Omega$ we infer that
$$h(1)(\omega) = h(0^c)(\omega) = \big(h(0)(\omega)\big)^c = 1 - 1 = 0,$$
which is impossible since $h(0) \leq h(1)$. Hence, for all $\omega \in \Omega$ the equality $h(0)(\omega) = h(0)(\omega) + h(0)(\omega)$ holds. Consequently, $h(0) = 0$ and $h(1) = h(0^c) = 1$.

3. h is additive. Let $u, v \in \mathcal{M}(\mathbf{B}), u \leq 1 - v$. Then,
$$u + v = u \oplus v \quad \text{and} \quad h(u + v) = h(u \oplus v) = h(u) \oplus h(v).$$
From $u \leq 1 - v$ we get
$$h(u) \leq h(1 - v) = 1 - h(v) \quad \text{and} \quad h(u) + h(v) \leq 1.$$
Thus, $h(u) + h(v) = h(u) \oplus h(v) = h(u + v)$. □

Observation 3.3. Let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be a map. Observe that if h preserves the Łukasiewicz operations, then h preserves also the **Łukasiewicz implication**: $u \to v = \min\{1, 1-u+v\}$. Indeed, $u \to v = u^c \oplus v$ and hence, $h(u \to v) = h(u^c \oplus v) = h(u^c) \oplus h(v) = \big(h(u)\big)^c \oplus h(v) = h(u) \to h(v)$.

Let $(\Xi, \mathbf{B})$ and $(\Omega, \mathbf{A})$ be a measurable spaces and let $\mathcal{M}(\mathbf{B})$ and $\mathcal{M}(\mathbf{A})$ be the corresponding lattices of all measurable fuzzy subsets. We consider the elements of $\mathbf{A}$ and $\mathbf{B}$ as indicator functions, i.e., $\mathbf{A} \subset \mathcal{M}(\mathbf{A})$ and $\mathbf{B} \subset \mathcal{M}(\mathbf{B})$. Observe that $\mathbf{A}$ and $\mathbf{B}$ are exactly idempotent elements (with respect to the Łukasiewicz strong disjunction) of $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$, respectively. It is known that if h_0 is an A-homomorphism of $\mathbf{B}$ into $\mathbf{A}$, then h_0 preserves the Boolean set operations. Further, if h_0 is a sequentially continuous A-homomorphism of $\mathbf{B}$ into $\mathbf{A}$, resp. into $\mathcal{M}(\mathbf{A})$, then h_0 can be uniquely extended to a sequentially continuous A-homomorphism of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$, cf. [1,2].

Theorem 3.4. *Let $(\Xi, \mathbf{B})$ and $(\Omega, \mathbf{A})$ be measurable spaces, let $\mathcal{M}(\mathbf{B})$ and $\mathcal{M}(\mathbf{A})$ be the corresponding lattices of all measurable fuzzy subsets, and let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be a sequentially continuous map. Then the following are equivalent*:

(i) h preserves the Łukasiewicz operations $\oplus, \odot, (\cdot)^c$;

(ii) h is an A-homomorphism and for each $B \in \mathbf{B}$ we have $h(B) \in \mathbf{A}$.

Proof.

(i) implies (ii). By the preceding lemma, h is an A-homomorphism. Assume, for sake of contradiction, that there exists $B \in \mathbf{B}$ such that for some $\omega \in \Omega$ we have $0 < \big(h(B)\big)(\omega) < 1$. Define $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow [0,1] \equiv \mathcal{M}(\mathbf{T})$ by putting $g(u) = \big(h(u)\big)(\omega)$, $u \in \mathcal{M}(\mathbf{B})$. Then, g is a sequentially continuous A-homomorphism. Clearly, g preserves $\oplus$. But from $B = B \cup B = B \oplus B$, it follows that $g(B) = g(B \oplus B) = g(B) \oplus g(B)$, a contradiction with $0 < \big(h(B)\big)(\omega) < 1$.

(ii) implies (i). Since Łukasiewicz operations (on $\mathcal{M}(\mathbf{A})$) are defined coordinatewise, it suffices to assume that $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow [0,1] \equiv \mathcal{M}(\mathbf{T})$. It is known that each sequentially continuous A-homomorphism $h\colon \mathcal{M}(\mathbf{B}) \to [0,1]$ is a probability integral, i.e., $h(\cdot) = \int (\cdot)\,\mathrm{d}p$ for some probability measure p on $\mathbf{B}$. Since every $u \in \mathcal{M}(\mathbf{B})$ is a limit of an increasing sequence of simple functions (finite linear combinations of elements of $\mathbf{B}$), and since integral preserves monotone limits and linear combinations, it suffices to show that Łukasiewicz operations are preserved by the restriction of h to the lattice $\mathbf{B}$ of all crisp events.

It is clear that $(\cdot)^c$, being a linear mapping $u^c = 1 - u$, is preserved by h. We show that $h(u \odot v) = h(u) \odot h(v)$ for $u, v \in \mathbf{B}$. Indeed, if $h(u) = 0$ or $h(v) = 0$ then $h(u \odot v) = 0$, since $u \odot v = \max\{0, u+v-1\} = \min\{u, v\}$ and h is order-preserving. If $h(u) = h(v) = 1$, then $h(u \odot v) = h(u) - h(u \odot v^c) = 1$ since $h(u \odot v^c) \le h(v^c) = 0$. Thus, $\odot$ is preserved by h. Finally, h preserves $\oplus$ since $u \oplus v = (u^c \odot v^c)^c$. ☐

Definition 3.5. Let $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ be an observable such that for each $B \in \mathbf{B}$ we have $h(B) \in \mathbf{A}$. Then, h is said to be **conservative**.

Corollary 3.6. *An observable* $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ *preserves the Łukasiewicz operations iff* h *is conservative.*

Corollary 3.7. *An observable* $h\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{T})$ *preserves the Łukasiewicz operations iff* h *is a probability integral* $\int(\cdot)\,\mathrm{d}p$ *with respect to a* $\{0,1\}$*-valued probability measure* p.

Observation 3.8. Normalized and additive maps on MV-algebras have been introduced by F. Kôpka and F. Chovanec in [11], and then by D. Mundici under the name of MV-algebraic states (or simply states) in [15]. As it is shown in [15], every state s on an MV-algebra M is modular, i.e., for all $x, y \in M$ we have $s(x \vee y) = s(x) + s(y) - s(x \wedge y)$.

Observe the following interesting fact. Let $(\Omega, \mathbf{A})$ be a measurable space and let $\mathcal{M}(\mathbf{A})$ be the corresponding lattice of measurable fuzzy subsets of Ω. Even though $\mathcal{M}(\mathbf{A})$ fails to be a lattice with respect to the Łukasiewicz operations $\oplus$ and $\odot$ each probability integral $\int(\cdot)\,\mathrm{d}p$ is "modular", i.e., for each $u, v \in \mathcal{M}(\mathbf{A})$ we have

$$\int u \oplus v\,\mathrm{d}p = \int u\,\mathrm{d}p + \int v\,\mathrm{d}p - \int u \odot v\,\mathrm{d}p. \qquad (m)$$

Indeed, (m) follows directly from $u + v = \min\{1, u+v\} + \max\{0, u+v-1\}$.

In the next section, we recall some background stochastic information and summarize the role of Łukasiewicz logic in the transition from classical to divisible probability theory.

4. Stochastic background

Probability spaces $(\Omega, \mathbf{A}, p)$, $p \in \mathcal{P}(\mathbf{A})$, describe classical random experiments having the same fixed component $(\Omega, \mathbf{A})$, and $p \in \mathcal{P}(\mathbf{A})$ represents the "choice" of suitable probability measure, representing the "law of randomness", one of all possible probability measures related to the experiment in question.

Let $(\Omega, \mathbf{A})$ and $(\Xi, \mathbf{B})$ be measurable spaces and let $f\colon \Omega \longrightarrow \Xi$ be a measurable map. The preimage map $f^{\leftarrow}$, $f^{\leftarrow}(B) = \{\omega \in \Omega;\ f(\omega) \in B\}$, $B \in \mathbf{B}$, is a sequentially continuous (with respect to the pointwise sequential convergence of indicator functions) Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$. Further, $f^{\leftarrow}$ defines a map $T_{f^{\leftarrow}}$ on the set $\mathcal{P}(\mathbf{A})$ of all probability measures on $\mathbf{A}$ into the set $\mathcal{P}(\mathbf{B})$ of all probability measures on $\mathbf{B}$: $T_{f^{\leftarrow}}(p)$ is the composition $p \circ f^{\leftarrow}$, $p \in \mathcal{P}(\mathbf{A})$. Then, a choice of $p \in \mathcal{P}(\mathbf{A})$ determines the choice $p \circ f^{\leftarrow} \in \mathcal{P}(\mathbf{B})$. We say that $f^{\leftarrow}$ pushes forward p to $p \circ f^{\leftarrow}$ or that $f^{\leftarrow}$ "conveys the stochastic information p on $\mathbf{A}$ to $p \circ f^{\leftarrow}$ on $\mathbf{B}$".

In the upgraded probability theory, the notion of random experiment is modified as follows. Classical random events $\mathbf{A}$ are embedded and extended to $\mathcal{M}(\mathbf{A})$ (each $A \in \mathbf{A}$ is considered as the indicator function $\chi_A \in \mathcal{M}(\mathbf{A})$), the set Ω of outcomes of a classical random experiment is extended to $\mathcal{P}(\mathbf{A})$ (each $\omega \in \Omega$ is considered as the corresponding Dirac measure $\delta_\omega \in \mathcal{P}(\mathbf{A})$), each probability measure $p \in \mathcal{P}(\mathbf{A})$ is extended to the corresponding probability integral $\overline{p} = \int(\cdot)\,\mathrm{d}p$ on $\mathcal{M}(\mathbf{A})$ ($\overline{p}$ reduced to $\mathbf{A}$ can be considered as p). Here, $\mathcal{P}(\mathbf{A})$ and $\mathcal{M}(\mathbf{A})$ represent the "hardware" and $\int(\cdot)\,\mathrm{d}p$ represents the "stochastics" of experiment. Observe that, since $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$, resp. p and $\overline{p}$, are in one-to-one correspondence, this part of the upgrade is "for free". The "added value" of the upgrade comes via the fact that not all observables on $\mathcal{M}(\mathbf{B})$ to $\mathcal{M}(\mathbf{A})$ are conservative.

Definition 4.1. Let $(\Omega, \mathbf{A})$ be a measurable space and let $p \in \mathcal{P}(\mathbf{A})$. Then, $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p\big)$ is said to be a **random experiment**.

Let $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p\big)$ be a random experiment and let $(\Xi, \mathbf{B})$ be a measurable space. Then, instead of a measurable map $f\colon \Omega \longrightarrow \Xi$ and its preimage map $f^{\leftarrow}\colon \mathbf{B} \longrightarrow \mathbf{A}$, we start with a sequentially continuous A-homomorphism $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ (g is an A-homomorphism which preserves sequential limits with respect to pointwise convergence), called **observable**, and f is replaced with a map $T_g\colon \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$, called **statistical map**. For each probability integral $\overline{s} = \int(\cdot)\,\mathrm{d}s$ on $\mathcal{M}(\mathbf{A})$, the composition $\overline{s} \circ g$ of two observables is an observable to $\mathcal{M}(\mathbf{T})$, hence a probability integral $\overline{t} = \int(\cdot)\,\mathrm{d}t$ on $\mathcal{M}(\mathbf{B})$. This yields the statistical map $T_g\colon \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$ sending s to $T_g(s)$. If $q = T(p)$, then for $u \in \mathcal{M}(\mathbf{B})$ we have $\int u\,\mathrm{d}\big(T_g(p)\big) = \int g(u)\,\mathrm{d}p$ and, for $p = \delta_\omega$, $\omega \in \Omega$, we get

$$\big(g(u)\big)(\omega) = \int g(u)\,\mathrm{d}(\delta_\omega) = \int u\,\mathrm{d}\big(T_g(\delta_\omega)\big).$$

We say that g is an observable on $\big(\Xi, \mathcal{M}(\mathbf{B}), \int(\cdot)\,\mathrm{d}q\big)$ into $\big(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p\big)$.

Recall [3,8,9] that a big difference is that an observable $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ can map a crisp event $B \equiv \chi_B \in \mathbf{B}$ to a genuine fuzzy random event $g(\chi_B) \in \mathcal{M}(\mathbf{A}) \setminus \mathbf{A}$ and, dually, T_g can map a Dirac probability measure $\delta_\omega, \omega \in \Omega$, to a genuine probability measure $q = T_g(\delta_\omega), 0 < q(B) < 1$, for some $B \in \mathbf{B}$. Indeed, for $\mathbf{A} = \mathbf{T} = \big\{\emptyset, \{\omega\}\big\}$, every probability integral $\int(\cdot)\,\mathrm{d}q$, $q \in \mathcal{P}(\mathbf{B})$, is an observable to $\mathcal{M}(\mathbf{A})$ and, if $0 < q(B) < 1$ for some $B \in \mathbf{B}$, then the observable fails to be conservative and $T_g(\delta_\omega) = q$.

In the divisible probability theory, an important role is played by "degenerated" observables. They capture stochastic independence of one random experiment on another one [2,6].

Example 4.2. First, recall the notion of a classical degenerated random variable. Denote R the real line and denote $\mathbf{B}_R$ the σ-algebra of all Borel measurable subsets of R. Let $(\Omega, \mathbf{A}, p)$ be a classical probability space and let r be a real number. Define $f\colon \Omega \longrightarrow R$ as follows: $f(\omega) = r$ for all $\omega \in \Omega$. This defines a degenerated classical random variable for which the preimage map $f^{\leftarrow}$ maps $B \in \mathbf{B}_R$ to Ω whenever $r \in B$ and $f^{\leftarrow}(B) = \emptyset$ otherwise. Clearly, $T_{f^{\leftarrow}}(t) = \delta_r$ for all $t \in \mathcal{P}(\mathbf{A})$. All outcomes $\omega \in \Omega$ are mapped to r, hence f models a "deterministic" experiment.

Now, let $\bigl(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p\bigr)$ and $\bigl(\Xi, \mathcal{M}(\mathbf{B}), \int(\cdot)\,\mathrm{d}q\bigr)$ be random experiments. Define $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ as follows: for $u \in \mathcal{M}(\mathbf{B})$, $g(u) \in \mathcal{M}(\mathbf{A})$ is a constant function and $\bigl(g(u)\bigr)(\omega) = \int u\,\mathrm{d}q$, $\omega \in \Omega$. Clearly, g is an observable and the corresponding statistical map $T_g\colon \mathcal{P}(\mathbf{A}) \longrightarrow \mathcal{P}(\mathbf{B})$ sends all $t \in \mathcal{P}(\mathbf{A})$ to q. Such observables and statistical maps are called **degenerated**. Observe that if $0 < q(B) < 1$ for some $B \in \mathbf{B}$, then $g(\chi_B) \in \mathcal{M}(\mathbf{A}) \setminus \mathbf{A}$, i.e., g is not conservative.

Summary 1. There is a natural extension of the classical probability theory, CPT for short, called divisible probability theory, DPT for short. Basic notions and constructions in DPT and the transition from CPT to DPT can be expressed in terms of elementary category theory. Objects are measurable fuzzy subsets of classical outcomes. Objects consist of fuzzy random events and are viewed as fuzzy propositional functions equipped with Łukasiewicz logic. Classical random events are exactly the idempotent elements of the objects. Morphisms, called observables, serve as channels through which a stochastic information is transmitted from one object to another one. Relevant constructions in DPT are described in terms of observables and commutative diagrams. Classical probability measures are replaced by probability integrals and constitute special morphisms evaluating how probable individual fuzzy random events are. A classical (idempotent, crisp) random event can be mapped by an observable to a genuine fuzzy random event. An observable mapping each classical random event to a classical event, is called conservative. In fact, conservative observables are stochastic tools of CPT. This way CPT is embedded into DPT. The embedding (cf. [1,6]) can be characterized as an epireflection.

Summary 2. Łukasiewicz logic plays an important role. First, it extends the Boolean logic of classical random events in a canonical way and, secondly, it exactly identifies the CPT within DPT. An observable is conservative iff it preserves the Łukasiewicz logical operations on random events: conjunction, disjunction, negation, implication. Observe that a nonconservative observable serves only to "push forward" probability of individual random events. If $\bigl(\Omega, \mathcal{M}(\mathbf{A}), \int(\cdot)\,\mathrm{d}p\bigr)$, $\bigl(\Xi, \mathcal{M}(\mathbf{B}), \int(\cdot)\,\mathrm{d}q\bigr)$ are random experiments and $g\colon \mathcal{M}(\mathbf{B}) \longrightarrow \mathcal{M}(\mathbf{A})$ is a nonconservative observable, then $\int u\,\mathrm{d}q = \int g(u)\,\mathrm{d}g$ for each $u \in \mathcal{M}(\mathbf{B})$, but g does not necessarily push forward complex probabilities in terms of logical operations.

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