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AROUND TAYLOR'S THEOREM ON THE CONVERGENCE OF SEQUENCES OF FUNCTIONS

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ABSTRACT. Egoroff's classical theorem shows that from a pointwise convergence we can get a uniform convergence outside the set of an arbitrary small measure. Taylor's theorem shows the possibility of controlling the convergence of the sequences of functions on the set of the full measure. Namely, for every sequence of real-valued measurable fnctions $\{f_n\}_{n\in\mathbb{N}}$ pointwise converging to a function f on a measurable set E, there exist a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of positive reals converging to 0 and a set $A \subseteq E$ such that $E \setminus A$ is a nullset and $\lim_{n\to+\infty} \frac{|f_n(x)-f(x)|}{\delta_n} = 0$ for all $x \in A$. Let $J(A, \{f_n\})$ denote the set of all such sequences $\{\delta_n\}_{n\in\mathbb{N}}$. The main results of the paper concern basic properties of sets of all such sequences for a given set A and a given sequence of functions. A relationship between pointwise convergence, uniform convergence and the Taylor's type of convergence is considered.

The well-known Egoroff's theorem [7] says that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ as $n \to +\infty$ for all $x \in E$, where E is a Lebesgue measurable subset of n-dimensional Euclidean space, then for each $\varepsilon > 0$ there exists a set $A \subseteq E$ with $\lambda(A) < \varepsilon$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $E \setminus A$ (λ denotes the n-dimensional Lebesgue measure).

S. J. Taylor in [8] formulated a new form of Egoroff's theorem

Theorem 1. *Suppose that E is a Lebesgue measurable subset of n-dimensional Euclidean space and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions such that*

$$f_n(x) \to f(x) \quad \textit{as} \quad n \to +\infty \quad \textit{for all } x \in E.$$

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Then, there exist a decreasing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ *of positive real numbers tending to* 0 *and a subset* $A\subseteq E$ *with* $\lambda(E\setminus A)=0$ *such that for all* $x\in A$

$$\lim_{n\to+\infty}\frac{|f_n(x)-f(x)|}{\delta_n}=0.$$

In the present paper, we discuss some properties of the sequences $\{\delta_n\}_{n\in\mathbb{N}}$ satisfying conditions from Theorem 1, so we define a family of such sequences.

Definition 2. Let $\mathscr{S}_0$ denote the family of sequences of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ tending to 0. For a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued functions pointwise converging to a function f on a set A, let $J(A,\{f_n\})$ denote the set of sequences $\{\delta_n\}_{n\in\mathbb{N}}\in\mathscr{S}_0$ such that for every $x\in A$, $\lim_{n\to+\infty}\frac{|f_n(x)-f(x)|}{\delta_n}=0$.

We are mainly interested in the case when the functions f_n are measurable and the set A is measurable of positive measure as in the case of Theorem 1, which can now be written as follows:

Theorem 1. *Suppose that* E *is a Lebesgue measurable subset of* n*-dimensional Euclidean space and* $\{f_n\}_{n\in\mathbb{N}}$ *is a sequence of measurable functions such that* $f_n(x)\to f(x)$ *as* $n\to+\infty$ *for all* $x\in E$*. Then, there exists a subset* $A\subseteq E$ *with* $\lambda(E\setminus A)=0$ *such that* $J(A,\{f_n\})\neq\emptyset$*.*

The next lemma shows that for a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real functions converging to f on a set A we can equivalently consider the existence of arbitrary or decreasing sequences belonging to $J(A,\{f_n\})$ or the equal convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f on A (defined in [4] or in [2] under the name "quasi-normal convergence").

Lemma 3. *The following conditions are equivalent:*

1) *There exists a decreasing sequence* $\{\delta_n\}\in J(A,\{f_n\})$*.*
2) $J(A,\{f_n\})\neq\emptyset$*.*
3) $\{f_n\}_{n\in\mathbb{N}}$ *converges equally [4] (or quasi-normally – [2]) to* f *on* A*, i.e., there exists a sequence* $\{\delta_n\}_{n\in\mathbb{N}}$ *of non-negative numbers tending to zero such that for every* $x\in A$ *there is an index* k *such that* $|f_n(x)-f(x)|\leq\delta_n$ *for* $n\geq k$*.*

P r o o f. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious.
3) $\Rightarrow$ 1): Let $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy (3). For every $n\in\mathbb{N}$ define

$$\delta'_n:=\max\{\sqrt{\delta_k}:k\geq n\}+2^{-n}.$$

Then, $\delta_n\leq\delta'^2_n$ and $\lim_{n\to+\infty}\delta'_n=0$. The sequence $\{\delta'_n\}_{n\in\mathbb{N}}$ satisfies 1) because $\{\delta'_n\}_{n\in\mathbb{N}}$ is decreasing and positive and for every $x\in A$:

$$\lim_{n\to+\infty}\frac{|f_n(x)-f(x)|}{\delta'_n}\leq\lim_{n\to+\infty}\frac{\delta_n}{\delta'_n}\leq\lim_{n\to+\infty}\delta'_n=0.$$

$\square$

Observe that for a sequence of Borel functions pointwise convergent on any uncountable Borel set E it is possible that $J(E,\{f_n\})=\emptyset$.

EXAMPLE 4. Let E be an uncountable Borel subset of n-dimensional Euclidean space. Since $\mathscr{S}_0$ is uncountable Borel subset of the Polish space $\mathbb{R}^{\mathbb{N}}$, it is Borel isomorphic to E [6, 15.3 and 15.6]. Hence, there exists a Borel bijective function $F:\mathscr{S}_0\to E$. Explicitly, for every $\{\delta_n\}_{n\in\mathbb{N}}\in\mathscr{S}_0$ there exists a unique point $x_{\{\delta_n\}}\in E$ such that $F(\{\delta_n\})=x_{\{\delta_n\}}$. Define a function f_n putting $f_n(x):=\delta_n$, where $x=x_{\{\delta_n\}}$, i.e., $f_n=\mathrm{proj}_n\circ F^{-1}$, so f_n is a Borel function for all $n\in\mathbb{N}$.

For a fixed $x\in E$ the sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ are the same, so

$$\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty}\delta_n=0.$$

Thus, $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to 0.

On the other hand, $\{\delta_n\}\notin J(E,\{f_n\})$ for any $\{\delta_n\}_{n\in\mathbb{N}}\in\mathscr{S}_0$. Take an arbitrary $\{\delta_n\}_{n\in\mathbb{N}}\in\mathscr{S}_0$ and consider $x_{\{\delta_n\}}$. Then,

$$\lim_{n\to+\infty}\frac{|f_n(x_{\{\delta_n\}})-0|}{\delta_n}=\lim_{n\to+\infty}\frac{\delta_n}{\delta_n}=1.$$

Let E be a fixed measurable subset of an Euclidean space of positive measure. All functions that we consider from now on are measurable functions defined on the set E.

Define $\frac{1}{f}(x):=\frac{1}{f(x)}$, if $f(x)\neq 0$ and $\frac{1}{f}(x):=1$, if $f(x)=0$. Obviously, $\frac{1}{f}$ is measurable whenever f is measurable. If $f(x)\neq 0$, then $\lim_{n\to+\infty}f_n(x)=f(x)$ if and only if $\lim_{n\to+\infty}\frac{1}{f_n}(x)=\frac{1}{f}(x)$.

PROPOSITION 5. *Let $f_n(x)\to f(x)$ for all $x\in E$.*

1) *If $\{\alpha_n\}\in J(E,\{f_n\})$, then $\{a\alpha_n\}\in J(E,\{f_n\})$ for all $a>0$.*
2) *If $f_n=f$ for all $n\in\mathbb{N}$, then $J(E,\{f_n\})=\mathscr{S}_0$.*
3) *$J(E,\{af_n\})=J(E,\{f_n\})$ for every $a\neq 0$.*
4) *If $f(x)\neq 0$ for all $x\in E$, then $J(E,\{\frac{1}{f_n}\})=J(E,\{f_n\})$.*

Proof. The proof of 1), 2) and 3) is obvious.

4) By the assumption, for every $x\in E$ for all but finitely many $n\in\mathbb{N}$, $f_n(x)\neq 0$, and for all such n and all positive δ_n we have

$$\frac{|f(x)-f_n(x)|}{\delta_n}=\frac{|\frac{1}{f_n}(x)-\frac{1}{f}(x)|}{\delta_n}\cdot|f_n(x)f(x)|.$$

Since $\lim_{n\to+\infty}f_n(x)f(x)=\big(f(x)\big)^2\neq 0$, then $J(E,\{\frac{1}{f_n}\})=J(E,\{f_n\})$. □

Proposition 6. *Let $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for all $x \in E$. Let $\{\alpha_n\} \in J(A_f, \{f_n\})$ and $\{\beta_n\} \in J(A_g, \{g_n\})$ for some sets A_f, $A_g \subseteq E$ and let $\{\gamma_n\} = \max\{\alpha_n, \beta_n\}$ for all $n \in \mathbb{N}$. Then*

1) *$\{\gamma_n\} \in J(A_f \cap A_g, \{af_n + bg_n\})$ for all a, $b \in \mathbb{R}$.*

2) *$\{\gamma_n\} \in J(A_f \cap A_g, \{f_n \cdot g_n\})$.*

(Obviously, if A_f and A_g are complements of null-sets, then also the set $A_f \cap A_g$ is the complement of a null-set.)

P r o o f. Obviously,

$$\gamma_n > 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to+\infty} \gamma_n = 0.$$

Due to Proposition 5 2) – 3), it suffices to prove the assertion 1) for $a = b = 1$. For all $x \in A_f \cap A_g$ we have

$$\begin{aligned}
0 &\leq \frac{|f_n(x) + g_n(x) - (f(x) + g(x))|}{\gamma_n} = \frac{|f_n(x) - f(x) + g_n(x) - g(x)|}{\gamma_n} \\
&\leq \frac{|f_n(x) - f(x)| + |g_n(x) - g(x)|}{\gamma_n} = \frac{|f_n(x) - f(x)|}{\gamma_n} + \frac{|g_n(x) - g(x)|}{\gamma_n} \\
&\leq \frac{|f_n(x) - f(x)|}{\alpha_n} + \frac{|g_n(x) - g(x)|}{\beta_n} \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Hence, using the squeeze theorem, we get that

$$\frac{|f_n(x) + g_n(x) - (f(x) + g(x))|}{\gamma_n} \underset{n\to+\infty}{\longrightarrow} 0.$$

2) For a fixed $x \in A_f \cap A_g$ we have

$$\begin{aligned}
0 &\leq \frac{|f_n(x) \cdot g_n(x) - f(x) \cdot g(x)|}{\gamma_n} \\
&= \frac{|f_n(x) \cdot g_n(x) - f(x) \cdot g(x) - f_n(x) \cdot g(x) + f_n(x) \cdot g(x)|}{\gamma_n} \\
&= \frac{|f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))|}{\gamma_n} \\
&\leq \frac{|f_n(x) \cdot (g_n(x) - g(x))| + |g(x) \cdot (f_n(x) - f(x))|}{\gamma_n} \\
&= \frac{|f_n(x) \cdot (g_n(x) - g(x))|}{\gamma_n} + \frac{|g(x) \cdot (f_n(x) - f(x))|}{\gamma_n} \\
&= |f_n(x)| \cdot \frac{|g_n(x) - g(x)|}{\gamma_n} + |g(x)| \cdot \frac{|f_n(x) - f(x)|}{\gamma_n} \\
&\leq |f_n(x)| \cdot \frac{|g_n(x) - g(x)|}{\alpha_n} + |g(x)| \cdot \frac{|f_n(x) - f(x)|}{\beta_n}.
\end{aligned}$$

Since $f_n(x) \to f(x)$ as $n \to +\infty$ for all $x \in A_f \cap A_g \subset E$, then for each $x \in A_f \cap A_g$ there exists $M(x) > 0$ such that $|f_n(x)| \le M(x)$ for each $x \in A_f \cap A_g$. Therefore,

$$
\begin{aligned}
&|\, f_n(x)\,| \cdot \frac{|g_n(x) - g(x)|}{\alpha_n} + |g(x)| \cdot \frac{|f_n(x) - f(x)|}{\beta_n} \\
&\le M(x) \cdot \frac{|g_n(x) - g(x)|}{\alpha_n} + |g(x)| \cdot \frac{|f_n(x) - f(x)|}{\beta_n} \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

Hence, using the squeeze theorem again, we get that

$$
\frac{|f_n(x) \cdot g_n(x) - (f(x) \cdot g(x))|}{\gamma_n} \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{for all } \ x \in A_f \cap A_g.
$$

□

Corollary 7. *Let $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for all $x \in E$. Then*

1) $J(E, \{f_n\}) \cap J(E, \{g_n\}) = J(E, \{f_n + g_n\}) \cap J(E, \{g_n\})$.
2) $J(E, \{f_n\}) \cap J(E, \{g_n\}) \subseteq J(E, \{f_n \cdot g_n\})$.
3) $J(E, \{f_n\}) \cap J(E, \{g_n\}) = J(E, \{f_n \cdot g_n\}) \cap J(E, \{g_n\})$ *provided that $g(x) \neq 0$ for all $x \in E$.*

P r o o f. Inclusions "$\subset$" are the straightforward consequence of Proposition 6. Inclusions "$\supset$" in 1) and 3) come from the fact that by Proposition 5 3) and 4), $J(E, \{-g_n\}) = J(E, \{g_n\})$ and $J(E, \{\frac{1}{g_n}\}) = J(E, \{g_n\})$, and from Proposition 6. □

Despite the fact that the connection between the uniform and the equal convergence has already been known (see e.g. [3, Theorem 1.1]), we would like to formulate an analogous theorem in our setting.

Theorem 8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions uniformly convergent to a function f on a set E. Then, $J(E, \{f_n\}) \neq \emptyset$.*

P r o o f. Let

$$
a_n := \sup_{x\in E} |f_n(x) - f(x)| \quad \text{for all} \quad n \in \mathbb{N}.
$$

Obviously, $a_n \ge 0$ for all $n \in \mathbb{N}$ and, by assumption, $\lim_{n\to+\infty} a_n = 0$. As in the proof of Lemma 3, we show that $J(E, \{f_n\}) \neq \emptyset$. □

The converse form of the above theorem need not be true. To see this, consider the following example:

Example 9. Let $f_n(x) = x^n$, where $x \in [0, 1]$. We know that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to the function $f(x)$ which is equal to 0 for $x \in [0, 1)$ and is equal to 1 for $x = 1$ and is not uniformly convergent to this function on $[0, 1]$. Let $\delta_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Obviously, $\{\delta_n\}_{n\in\mathbb{N}} \in \mathscr{S}_0$.

Now, we show that $\{\delta_n\} \in J([0,1], f_n)$. Indeed, for $x \in \{0,1\}$ we have

$$\lim_{n\to+\infty} \frac{|f_n(x) - f(x)|}{\delta_n} = \lim_{n\to+\infty} \frac{0}{\frac{1}{n}} = 0.$$

For $x \in (0,1)$ we have

$$\lim_{n\to+\infty} \frac{|f_n(x) - f(x)|}{\delta_n} = \lim_{n\to+\infty} \frac{x^n}{\frac{1}{n}} = \lim_{n\to+\infty} \frac{n}{(\frac{1}{x})^n}.$$

Consider $f(y) = y$ and $g(y) = (\frac{1}{x})^y$, where x is a fixed number from (0,1). By L'Hospital's rule, we get

$$\lim_{y\to+\infty} \frac{f(y)}{g(y)} = \lim_{y\to+\infty} \frac{y}{(\frac{1}{x})^y} = \lim_{y\to+\infty} \frac{1}{(\frac{1}{x})^y \cdot \ln\frac{1}{x}} = 0.$$

Hence, $\lim_{n\to+\infty} \dfrac{n}{(\frac{1}{x})^n} = 0.$

In Example 9, we have a sequence of functions which is not uniformly convergent on the whole interval $[0,1]$. However, it is also possible to construct a sequence of functions for which a suitable sequence $\{\delta_n\}_{n\in\mathbb{N}}$ satisfies condition $J(A, f_n)$ for a certain set $A \subset [0,1]$ of the full measure but is not uniformly convergent on any subset of A of the full measure.

Example 10. Let C be a Cantor set of positive measure on $[0,1]$ and let $f : [0,1] \to \mathbb{R}$ be an indicator function of this set. We have

$$f^{-1}(-\infty, a) = \begin{cases} C', & \text{if } 0 < a \le 1, \\ [0,1], & \text{if } a > 1, \\ \emptyset, & \text{if } a \le 0, \end{cases}$$

$$f^{-1}(a, +\infty) = \begin{cases} \emptyset, & \text{if } a \ge 1, \\ C, & \text{if } 0 \le a < 1, \\ [0,1], & \text{if } a < 0. \end{cases}$$

The sets $\emptyset, C, C'$ and $[0,1]$ are of F_σ-type. Thus, f is a function of the first Baire class (see e.g. [1, Section 10.4]), so it can be represented as a limit of a sequence of continuous functions. By the Taylor's theorem, there exist a set $A \subset [0,1]$, $\lambda(A) = 1$ and the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n \in J(A, \{f_n\})$.

On the other hand, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not uniformly convergent to the function f on any subset $T \subset A$ such that $\lambda(T) = 1$, since f is not continuous on T. Indeed, there exists $x \in T \cap C$ such that $f(x) = 1$. Since C is a nowhere dense set and T is of full measure, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \in T \cap C'$ such that $x_n \to x$ and $f(x_n) = 0$ for all $n \in \mathbb{N}$. It means that $f(x_n) \nrightarrow f(x)$, so f is not continuous on T.

Let us come back to the properties of the set $J(A, \{f_n\})$.

Proposition 11. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f on a set E of positive measure and $A \subseteq E$. If $\{\delta_n\} \in J(A, \{f_n\})$ and the sequence $\{\beta_n\} \in \mathscr{S}_0$ is such that $\overline{\lim}_{n\to+\infty} \frac{\delta_n}{\beta_n} < +\infty$, then $\{\beta_n\} \in J(A, \{f_n\})$.*

Proof. Since $\overline{\lim}_{n\to+\infty} \frac{\delta_n}{\beta_n} < +\infty$, then there exists $M > 0$ such that

$$\frac{\delta_n}{\beta_n} \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

We have, for all $x \in A$

$$0 \leq \frac{|f_n(x) - f(x)|}{\beta_n} = \frac{|f_n(x) - f(x)|}{\delta_n} \cdot \frac{\delta_n}{\beta_n} \leq \frac{|f_n(x) - f(x)|}{\delta_n} \cdot M \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence, using the squeeze theorem, we get

$$\lim_{n\to+\infty} \frac{|f_n(x) - f(x)|}{\beta_n} = 0$$

which means that $\{\beta_n\} \in J(A, \{f_n\})$. $\square$

Note that Proposition 11 generalizes Proposition 5 1) because it includes every sequence $\{\beta_n\}$ defined by $\beta_n = a \cdot \delta_n$ for $a > 0$.

Proposition 12. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions uniformly convergent to a function f on a set E. Then, for every $\{\delta_n\} \in J(E, \{f_n\})$ there is $\{\beta_n\} \in J(E, \{f_n\})$ such that $\lim_{n\to+\infty} \frac{\delta_n}{\beta_n} = +\infty$.*

Proof. Let

$$a_n := \sup_{x\in E} |f_n(x) - f(x)| \quad \text{for all} \quad n \in \mathbb{N}.$$

Then, $\{\gamma_n\} \in J(E, \{f_n\})$ if and only if $\lim_{n\to+\infty} \frac{a_n}{\gamma_n} = 0$. Let $\{\delta_n\} \in J(E, \{f_n\})$. Put $\beta_n := \delta_n \cdot \sqrt{\frac{a_n}{\delta_n}}$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to+\infty} \frac{a_n}{\beta_n} = \lim_{n\to+\infty} \sqrt{\frac{a_n}{\delta_n}} = 0$, so $\beta_n \in J(E, \{f_n\})$. Moreover, $\lim_{n\to+\infty} \frac{\delta_n}{\beta_n} = \lim_{n\to+\infty} \sqrt{\frac{\delta_n}{a_n}} = +\infty$. $\square$

We would like to ask a more general question.

Question 13. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f on a set E of positive measure. Under what conditions can one show that for every $\{\delta_n\} \in J(E, \{f_n\})$ there exist a set $A \subseteq E$ of full measure and $\{\beta_n\} \in J(A, \{f_n\})$ such that $\overline{\lim}_{n\to+\infty} \frac{\delta_n}{\beta_n} = +\infty$?

Let S_∞ denote a set of permutations of $\mathbb{N}$.
We would like to consider $J(E, \{f_{\sigma(n)}\})$ for $\sigma \in S_\infty$. It is easily seen that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions converging to f on E, $\{\delta_n\} \in J(E, \{f_n\})$ and $\sigma \in S_\infty$ is such that $\sigma(n) = n$ for $n > n_0$, then $\{\delta_n\} \in J(E, \{f_{\sigma(n)}\})$.

Proposition 14. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f on a set E of positive measure. If there exists an increasing sequence of natural numbers $\{p_n\}_{n\in\mathbb{N}}$ such that $\mathbb{N}\setminus\{p_n : n\in\mathbb{N}\}$ is infinite, and a sequence $\{y_{p_n}\}_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to+\infty}\frac{y_{p_n}}{p_n}=0$ and $\lambda\big(\limsup_{n\to+\infty}\{x\in E : |f_{p_n}(x)-f(x)|>y_{p_n}\}\big)>0$, then for every $\{\delta_n\}\in J(E,\{f_n\})$ there exists $\sigma\in S_\infty$ such that $\{\delta_n\}\notin J(A,\{f_{\sigma(n)}\})$ for any $A\subset E$, $\lambda(E\setminus A)=0$.*

P r o o f. For simplicity, we assume that $f\equiv 0$. Denote

$$A_{p_n} := \{x\in E : |f_{p_n}(x)|>y_{p_n}\} \quad \text{for} \quad n\in\mathbb{N}.$$

Let $\{\delta_n\}\in J(E,\{f_n\})$. For $n\in\mathbb{N}$ we choose $k(n)$ such that $\delta_{k(n)}<\frac{y_{p_n}}{p_n}$ and $k(n)<k(n)+1<k(n+1)$. Let $K:=\{k(n) : n\in\mathbb{N}\}$. Then, $\mathbb{N}\setminus K$ is an infinite set and by the assumption also $\mathbb{N}\setminus\{p_n : n\in\mathbb{N}\}$ is infinite. Define a one-to-one function $\pi : K\to\{p_n : n\in\mathbb{N}\}$ such that $\pi(k(n))=p_n$ for $n\in\mathbb{N}$. Let σ be any permutation of $\mathbb{N}$ extending π. Then, for $x\in A_{p_n}$ we have

$$\left|\frac{f_{\pi(k(n))}(x)}{\delta_{k(n)}}\right| = \left|\frac{f_{p_n}(x)}{\delta_{k(n)}}\right| > \frac{y_{p_n}}{y_{p_n}}\cdot p_n = p_n,$$

so

$$A_{p_n}\subset\left\{x\in E : \left|\frac{f_{\pi(k(n))}(x)}{\delta_{k(n)}}\right|>p_n\right\} \quad \text{for every} \quad n\in\mathbb{N}.$$

Therefore, if $x\in\limsup_{n\to+\infty}A_{p_n}$, then for every $m\in\mathbb{N}$ there exists $p_n>m$ such that $\frac{f_{\pi(k(n))}(x)}{\delta_{k(n)}}>p_n$, so $\overline{\lim}_{n\to+\infty}\frac{f_{\pi(n)}(x)}{\delta_n}=+\infty$. Since $\lambda(\limsup_{n\to+\infty}A_{p_n})>0$, the proof is complete. □

Observe that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of constant functions converging to f on a set of positive measure and $f_n\neq f$ for infinitely many $n\in\mathbb{N}$, then it satisfies the assumptions of the last proposition. Indeed, for simplicity, we assume that $f\equiv 0$ and $f_n\neq f$ for every $n\in\mathbb{N}$. Then, for every $n\in\mathbb{N}$ we have

$$\lambda(E)=\lambda(\{x\in E : |f_n(x)|\neq 0\})=\lambda\left(\bigcup_{k\in\mathbb{N}}\left\{x\in E : |f_n(x)|>\frac{1}{k}\right\}\right).$$

There exists $k(n)\in\mathbb{N}$ such that $\lambda(\{x\in E : |f_n(x)|>\frac{1}{k(n)}\})>\frac{\lambda(E)}{2}$. It suffices to define p_n and y_{p_n} as follows:

$$p_n := k(2n) \quad \text{and} \quad y_{p_n} := \frac{1}{p_n} \quad \text{for every} \quad n\in\mathbb{N}.$$

Additionally, we would like to underline that in Proposition14 it is not enough to assume that $\lambda\big(\{x\in E : f_n(x)\neq f(x)\}\big)>0$ for every $n\in\mathbb{N}$. It is shown by the sequence $\{f_n\}_{n\in\mathbb{N}}$ of characteristic functions of intervals $[0,\frac{1}{n}]$ defined on $[0,1]$.;Then, $\{f_n\}_{n\in\mathbb{N}}$ converges to 0 on $(0,1]$ and to 1 at the point $x=0$ and

$\lambda\big(\{x \in [0,1] : f_n(x) \neq 0\}\big) = 1 - \frac{1}{n}$. For every sequence $\{\delta_n\} \in \mathscr{S}_0$ we have $\lim_{n\to+\infty} \frac{|f_n(x)-f(x)|}{\delta_n} = 0$ for every $x \in [0,1]$, so for any $A \subset [0,1]$ with $\lambda(A) > 0$ there is no $\{\delta_n\} \in \mathscr{S}_0 \setminus J(A, \{f_n\})$. Therefore, for any $\{\delta_n\} \in J([0,1], \{f_n\})$ there is no $\sigma \in S_\infty$ such that $\{\delta_n\} \notin J(A, \{f_{\sigma(n)}\})$, where $A \subset [0,1]$ has a positive measure.

Question 15. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f on a set E of positive measure. Under what conditions do any of the next assertions hold?

1) For every $\{\delta_n\} \in J(E, \{f_n\})$ there is $\sigma \in S_\infty$ such that for every set $A \subseteq E$ of full measure, $\{\delta_n\} \notin J(A, \{f_{\sigma(n)}\})$.
2) For every $\sigma \in S_\infty$ there exists a set $A \subseteq E$ of full measure such that $J(E, \{f_n\}) \subseteq J(A, \{f_{\sigma(n)}\})$.

Question 16. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions converging to f on a set E of positive measure. Obviously, for every finite set $B \subseteq S_\infty$ there exists a set $A \subseteq E$ of full measure such that $\bigcap_{\sigma\in B} J(A, \{f_{\sigma(n)}\}) \neq \emptyset$ (apply Theorem 1 for the sequence of functions $\{\max_{\sigma\in B} |f_{\sigma(n)}(x) - f(x)|\}_{n\in\mathbb{N}}$). Is there a set $B \subseteq S_\infty$ for which this is not true? If yes, what is the least cardinality of a set $B \subseteq S_\infty$ for which this is not true?

Question 16 has a simple answer for countable sets $B \subseteq S_\infty$. It is given in Proposition 17. Before making this assertion, we need to recall a few facts:

- A space X is called a QN-space if every sequence of continuous functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, pointwise converging to 0 quasi-normally converges to 0.
- Recall that $\mathfrak{b}$ is the bounding number, i.e., $\mathfrak{b}$:=minimal cardinality of a set of functions $B \subseteq {}^{\mathbb{N}}\mathbb{N}$ without an upper bound in the eventual partial ordering $\varphi \leq^* \psi \Leftrightarrow$ there exists $k \in \mathbb{N}$ such that $\varphi(n) \leq \psi(n)$ for every $n \geq k$, $n \in \mathbb{N}$;
- $\mathfrak{b}$ is a regular cardinal and $\omega_1 \leq \mathfrak{b} \leq \mathfrak{c}$ (see [5]).
- Note that $\mathfrak{b}$ is the minimal cardinality of a space that is not a QN-space [3]!

Proposition 17. *If $J(A, \{f_n\}) \neq \emptyset$, then $\bigcap_{\sigma\in B} J(A, \{f_{\sigma(n)}\}) \neq \emptyset$ for every $B \subseteq S_\infty$ of cardinality less than $\mathfrak{b}$.*

Proof. If $J(A, \{f_n\}) \neq \emptyset$, then by Lemma 3 there is $\{\varepsilon_n\} \in \mathscr{S}_0$ such that for every $x \in A$ there is an index k such that $|f_n(x) - f(x)| \leq \varepsilon_n$ for $n \geq k$. For $n \in \mathbb{N}$ let $g_n : S_\infty \to \mathbb{R}$ be defined by $g_n(\sigma) := \varepsilon_{\sigma(n)}$. The functions g_n are continuous (if S_∞ is considered as a subspace of the Baire space ${}^{\mathbb{N}}\mathbb{N}$ or if S_∞ has the discrete topology) and the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ pointwise converges to 0. If $B \subseteq S_\infty$ is of cardinality less than $\mathfrak{b}$, then B is a QN-space and therefore, $\{g_n\}_{n\in\mathbb{N}}$ quasi-normally converges to 0 on B. By Lemma, 3 there is $\{\delta_n\} \in J(B, \{g_n\})$. Then, $\{\delta_n\} \in \bigcap_{\sigma\in B} J(A, \{f_{\sigma(n)}\})$ because

$$\lim_{n\to+\infty} \frac{|f_{\sigma(n)}(x) - f_n(x)|}{\delta_n} leq \lim_{n\to+\infty} \frac{\varepsilon_{\sigma(n)}}{\delta_n} = \lim_{n\to+\infty} \frac{g_n(\sigma)}{\delta_n} = 0$$

for every $x \in A$ and for every $\sigma \in B$. $\square$

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