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HAHN-BANACH-TYPE THEOREMS AND SUBDIFFERENTIALS FOR INVARIANT AND EQUIVARIANT ORDER CONTINUOUS VECTOR LATTICE-VALUED OPERATORS WITH APPLICATIONS TO OPTIMIZATION

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ABSTRACT. We give some versions of Hahn-Banach, sandwich, duality, Moreau-Rockafellar-type theorems, optimality conditions and a formula for the subdifferential of composite functions for order continuous vector lattice-valued operators, invariant or equivariant with respect to a fixed group G of homomorphisms. As applications to optimization problems with both convex and linear constraints, we present some Farkas and Kuhn-Tucker-type results.

1. Introduction

The Hahn-Banach theorem is one of the most important results in Functional Analysis, and has played a fundamental role, both because it has several deep theoretical consequences and since there have been a lot of studies and applications in various branches of Mathematics and other sciences, e.g., Subdifferential

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Calculus, Optimization, Operations Research, Numerical Analysis, Differential Equations, Calculus of Variations, Measure theory, Probability, Image Restoration (see, e.g., [2, 3, 8, 11, 12, 16]). Some other related topics are, for instance, amenable (semi)groups, invariant and equivariant functionals, which are widely used in Probability, Statistics and Machine Learning (see, e.g. [13, 14, 22, 24]). In the literature there have been many studies on these subjects also when it is dealt with functionals, taking values in partially ordered vector spaces. This structure is very important, for instance, in investigating conditional expected values, measures or operators which can depend not only on events, but also on the state of knowledge and/or on the time (see also [7, 9, 25]).

This paper is a free continuation of [4] and [5]. We extend some Hahn-Banach, sandwich, Fenchel duality and Moreau-Rockafellar-type theorems, getting the existence of linear order continuous vector lattice-valued operators, invariant or equivariant with respect to a given group of homomorphisms. We extend previous results proved in [6, 10, 15, 26–30]. Note that, in our context, no topological structure is required, and in the spaces $L^0(\Omega, \mathcal{A}, \mu)$, where μ is a σ-finite and non-atomic countably additive extended real-valued positive measure (with identification up to μ-null sets), order convergence coincides with almost everywhere convergence, which does not have a topological nature (see also [23]). In this setting, we formulate a condition, involving only order boundedness, which is an extension to our context of "continuity"-type conditions given in [16] and [28]. Moreover, we present a formula on the subdifferential of composite functions. This result is given for invariant or equivariant convex and linear vector lattice-valued operators, order continuous or not necessarily order bounded. As applications to Optimization and Operations Research, we give some Farkas and Kuhn-Tucker-type theorems in which we consider both convex and linear constraints, extending some theorems proved in [17] and [18] in the vector lattice context and some results proved in [29] for linear continuous operators in the setting of locally convex ordered vector spaces, and formulating constraint qualifications in which no topological structure is needed, but involving only algebraic properties.

2. Preliminaries

Let X be a real vector space. An *affine combination* of elements $x_1, x_2, \dots, x_n$ of X is a linear combination of the type $\sum_{i=1}^n \lambda_i x_i$, with $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^n \lambda_i = 1$. An *affine manifold* of X is a nonempty subset of X, closed under affine combinations.

Let $\emptyset \neq D \subset X$. We denote by $\mathrm{span}(D)$ (resp., $\mathrm{span}_{\mathrm{aff}}(D)$) the smallest linear subspace (resp., affine manifold) of X which contains D.

A point $x_0 \in D$ is an *algebraic interior* (resp. *algebraic relative interior*) *point of* D iff for any $x \in X$ (resp. $x \in \mathrm{span}_{\mathrm{aff}}(D)$) there exists a $\lambda_0 > 0$ such that $(1-\lambda)x_0 + \lambda\, x \in D$ for each $\lambda \in [-\lambda_0, \lambda_0]$. We denote by D^0 and $int(D)$ the sets of all algebraic interior points and of all algebraic relative interior points of D, respectively.

A nonempty subset D of a real vector space X is said to be *convex* iff $\lambda x_1 + (1-\lambda)x_2 \in D$ for every $x_1,\, x_2 \in D$ and $\lambda \in [0,1]$.

Given a real vector space X, a partially ordered vector space Y and a convex subset $D \subset X$, we say that a function $U : D \to Y$ is *convex* (*on* D) iff $U\big(\lambda x_1 + (1-\lambda)x_2\big) \le \lambda U(x_1) + (1-\lambda)U(x_2)$ for every $x_1,\, x_2 \in D$ and $\lambda \in [0,1]$. We set $Y^+ = \{y \in Y : y \ge 0\}$.

A nonempty set $A \subset Y$ is *upper* (resp., *lower*) *order bounded* iff there exists $y_0 \in Y$ such that $y \le y_0$ (resp., $y \ge y_0$) whenever $y \in A$. We say that A is *order bounded* iff A is both upper and lower bounded. A partially ordered vector space Y is called a *vector lattice* iff for every $y_1,\, y_2 \in Y$ there exists in Y the supremum $y_1 \vee y_2$. We say that Y is *Dedekind complete* iff every nonempty and upper order bounded subset of Y admits a supremum in Y. A Dedekind complete vector lattice Y is *super Dedekind complete* iff every nonempty subset $A \subset Y$ having a supremum $y^* \in Y$ contains a countable subset A' such that $\bigvee A' = y^*$.

If X and Y are two vector lattices and $\emptyset \ne D \subset X$, then we say that a function $U : D \to Y$ is *locally order bounded at a point* $\overline{x} \in D$ iff for every $r \in X^+$, $r \ne 0$, there exist a positive real number δ_r and an element $\gamma_r \in Y^+$, $\gamma_r \ne 0$, such that $|U(\overline{x} + \delta_r\, x)| \le \gamma_r$ whenever $x \in D$, $\overline{x} + \delta_r\, x \in D$ and $|x| \le r$.

Let X, R be two vector lattices. An *R-functional* (*on* X) is any function $\varphi : X \to R$. A *real functional* is an $\mathbb{R}$-functional.

An R-functional φ is said to be *positive* iff $\varphi(x) \in R^+$ whenever $x \in X^+$. A *positive order continuous R-functional* is a positive R-functional φ such that, for each upward directed increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in X with

$$X \ni x = \bigvee_{\lambda} x_\lambda, \quad \text{it is} \quad \varphi(x) = \bigvee_{\lambda} \varphi(x_\lambda).$$

A linear R-functional is *order bounded* (resp., *order continuous*) if and only if it can be expressed as a difference of two positive (resp., positive order continuous) R-functionals. Observe that, if R is a Dedekind complete vector lattice, then an R-functional on X is order bounded if and only if it maps order bounded subsets of X into order bounded subsets of R (see, e.g., [1]).

A (D)-*sequence* in a vector lattice X is any double sequence $(a_{i,j})_{i,j}$ of elements of X such that, for every $i \in \mathbb{N}$, the sequence $(a_{i,j})_j$ is decreasing and

$$\bigwedge_{j=1}^{\infty} a_{i,j} = 0.$$

A vector lattice X has the *d-property* if and only if for every (D)-sequence $(a_{i,j})_{i,j}$ in X there exists $w \in X^+$ such that, for every $i \in \mathbb{N}$, there is $k = k(i) \in \mathbb{N}$ with $a_{i,k(i)} \leq w$ (see also [20, Definition 68.3]).

Remark 1.
(a) Note that the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences is super Dedekind complete and has the d-property. Moreover, if $(\Omega, \mathcal{A}, \mu)$ is a measure space, where $\mu : \mathcal{A} \to \widetilde{\mathbb{R}}$ is a σ-finite positive countably additive measure, then the space $L^0(\Omega, \mathcal{A}, \mu)$ of all real-valued μ-measurable functions on Ω, with the identification of μ-null sets, is super Dedekind complete and has the d-property (see also [20]).
(b) Observe that, if X is a super Dedekind complete vector lattice with the d--property and R is a Dedekind complete vector lattice, then every order bounded R-valued linear functional on X is order continuous too (see also [20, Theorem 70.2], [23, § 1, Proposition 5.16 and Corollary 5.21]).

The *algebraic dual* of a vector lattice R is the ordered vector space R^* whose elements are the linear real functionals on R, where the addition and the scalar multiplication are those inherited by $\mathbb{R}^R$, and in which the order $\leq$ is defined by $\varphi_1 \leq \varphi_2$ iff $\varphi_2 - \varphi_1$ is a positive linear $\mathbb{R}$-functional on R. The *order dual* (resp., *order continuous dual*) of R is the vector space $R^\sim$ (resp., $R^\times$) of all order bounded (resp., order continuous) linear functionals on R, where the addition, the scalar multiplication and the order are those inherited by R^*.

A subset I of R is said to be *order dense* in R iff for each $r \in R^+$, $r \neq 0$, there is $u \in I$ with $0 \leq u \leq r$ and $u \neq 0$. A subspace I of R is called an *ideal of* R, if and only if, r_1, $r_2 \in I$, $r_3 \in R$ and $r_1 \leq r_3 \leq r_2$ imply $r_3 \in I$.

Let Υ be the set of all order dense ideals of R, and put $\Phi = \bigcup_{I \in \Upsilon} I^\times$. Then, a function $\varphi \in \mathbb{R}^R$ belongs to Φ iff there exists $I \in \Upsilon$ such that φ is an order continuous linear functional on I. If $\varphi \in \Phi$, let I_φ be its domain. Given $\varphi \in \Phi$ and $r \in R^+$, let $|\varphi|(r) = \sup\{|\varphi(s)| : 0 \leq s \leq r\}$, and for $r \in R$, set $|\varphi|(r) = |\varphi|(r^+) - |\varphi|(r^-)$, where $r^+ = r \vee 0$, $r^- = (-r) \vee 0$. If $D_\varphi = \{r \in R : |\varphi|(r) < +\infty\}$, then D_φ is the largest order dense ideal of R on which $|\varphi|$ can be extended finitely.

Given φ_1, $\varphi_2 \in \Phi$, we say that $\varphi_1 \approx \varphi_2$ iff the set $\{r \in R : \varphi_1(r) = \varphi_2(r)\}$ contains an order dense ideal of R. Note that $\approx$ is an equivalence relation. We denote by the symbol $[\varphi]$ a generic class of equivalence with respect to $\approx$.

Let $R^\rho = \Phi/\approx$. Given $[\varphi_i] \in R^\rho$, $i = 1, 2, 3$, we say that $[\varphi_1] + [\varphi_2] = [\varphi_3]$ iff there exist $\varphi_i' \in [\varphi_i]$, $i = 1, 2, 3$, such that the set $\{r \in R : \varphi_1'(r) + \varphi_2'(r) = \varphi_3'(r)\}$ contains an order dense ideal. With a similar technique, it is possible to endow R^ρ with structures of order and product with real numbers, in such a way that R^ρ is a vector lattice, and the lattice supremum in R^ρ corresponds to the pointwise supremum on an order dense ideal of R. Note that for any $[\varphi] \in R^\rho$ there is $\varphi_* \in [\varphi]$ with the property that, for each $\varphi' \in [\varphi]$, it is $I_{\varphi'} \subset I_{\varphi_*}$ and $\varphi' = \varphi_*$ on $I_{\varphi'}$.

Given $\varphi \in \Phi$, we say that φ_* is the *maximal element* determined by φ, and we identify φ with φ_*. If $y = [\varphi] \in R^\rho$, then we set $\mathcal{I}_y = D_{\varphi_*}$.

Now we define the *evaluation map* $c : R \to R^{\rho\rho} = (R^\rho)^\rho$. For every $r \in R^+$, put $I_r = \{y \in R^\rho : r \in \mathcal{I}_y\}$. For any $r \geq 0$, $y \in I_r$ and $\varphi \in y$, set $c(r)(y) = \varphi(r)$, and for any $r \in R$ put $c(r) = c(r^+) - c(r^-)$. Note that this definition makes sense (see also [19]).

A vector lattice R is called a *ρ-space* iff the evaluation map c is one-to-one.

For example, if $(\Omega, \mathcal{A}, \mu)$ is a measure space, where $\mu : \mathcal{A} \to \widetilde{\mathbb{R}}$ is a σ-finite positive countably additive measure, then the space $L^0(\Omega, \mathcal{A}, \mu)$ is a Dedekind complete ρ-space; moreover, there are ρ-spaces, which are not Dedekind complete (see also [5]).

Let G be a group, $\mathcal{P}(G)$ be the class of all subsets of G, and R be a vector lattice. A finitely additive measure $\nu : \mathcal{P}(G) \to R$ is a *G-invariant mean* iff $\nu(G) \in R^+ \setminus \{0\}$ and $\nu(\{gh : g \in E\}) = \nu(E)$ whenever $E \subset G$ and $h \in G$. We say that G is *amenable* iff there exists a G-invariant mean $\nu : \mathcal{P}(G) \to \mathbb{R}$, with $\nu(G) = 1$.

From now on, let X and R be two Dedekind complete vector lattices, and $G \subset X^X$ be an amenable group of positive linear X-functionals on X. A set $\emptyset \neq D \subset X$ is said to be *G-invariant* iff $gx \in D$ whenever $x \in D$ and $g \in G$. When $R \subset X$, we always assume that R is G-invariant. A function $L : X \to R$ is *G-invariant* (resp. *G-equivariant*) iff $L(gx) = L(x)$ (resp., $g^{-1}\big(L(gx)\big) = L(x)$ or equivalently $g\big(L(x)\big) = L(gx)$) whenever $x \in X$ and $g \in G$. We denote by $l_b(G, R)$ the set of all bounded R-valued functions defined on G, by $\mathcal{L}(X, R)$ the set of all linear R-functionals on X, by $\mathcal{L}_{\text{inv}}(X, R)$ (resp., $\mathcal{L}_{\text{equiv}}(X, R)$) the set of all G-invariant (resp., G-equivariant) R-functionals of $\mathcal{L}(X, R)$. Let $\mathrm{v} \in \{\text{inv}, \text{equiv}\}$, and denote by $\mathcal{L}_{+,\mathrm{v}}(X, R)$ (resp., $\mathcal{L}_{\text{oc},\mathrm{v}}(X, R)$, $\mathcal{L}_{+,\text{oc},\mathrm{v}}(X, R)$) the set of all positive (resp., order continuous, positive order continuous) R--functionals of $\mathcal{L}_{\mathrm{v}}(X, R)$.

Let $U : D(U) \to R$, $V : D(V) \to R$ be two convex and G-equivariant functions, where $D(U)$, $D(V)$ are convex and G-invariant subsets of X. The *order continuous G-invariant* (resp., *G-equivariant*) *conjugate* (shortly, *conjugate*) of U is the function U^c defined by

$$\begin{aligned} U^c(L) &= \bigvee \{L(x) - U(x) : x \in D(U)\}, \quad L \in D(U^c), \\ D(U^c) &= \Big\{L \in \mathcal{L}_{\text{oc},\mathrm{v}}(X, R) : \bigvee \{L(x) - U(x) : x \in D(U)\} \text{ exists in } R\Big\}. \end{aligned}$$

If $x_0 \in D(U)$, then the *order continuous G-invariant* (resp., *G-equivariant*) *subdifferential* (briefly, *subdifferential*) *at x_0 of U*, $\partial_{\text{oc},\mathrm{v}} U(x_0)$, is defined by

$$\partial_{\text{oc},\mathrm{v}} U(x_0) = \{L \in \mathcal{L}_{\text{oc},\mathrm{v}}(X, R) : L(x) - L(x_0) \leq U(x) - U(x_0) \text{ for all } x \in D(U)\},$$

and similarly, we define the set $\partial_{\mathrm{v}} U(x_0)$ as the G-invariant (resp., G-equivariant) subdifferential.

An element $L \in \partial_{\rm oc,v} U(x_0)$ (resp., $\partial_{\rm v} U(x_0)$) is called *subgradient of U at x_0*. When $x_0 = 0$, we denote by $\partial_{\rm oc,v} U$ and $\partial_{\rm v} U$ the sets $\partial_{\rm oc,v} U(0)$ and $\partial_{\rm v} U(0)$, respectively.

We will deal with the following problems.

Problem I. Find $r = \bigwedge \{U(x) + V(x) : x \in D(U) \cap D(V)\}$ in R.

Problem II. Find $s = \bigvee \{-U^c(L) - V^c(-L) : L \in D(U^c) \cap D(V^c)\}$ in R, where $D(U^c) \cap D(V^c) \neq \emptyset$.

We recall the next result, which will be useful later.

Proposition 2.1 ([5], Proposition 2.5). *For any $g \in G$ and every order bounded family $(r_\xi)_{\xi \in \Xi}$ in X it is*

$$g\Big(\bigvee_\xi r_\xi\Big) = \bigvee_\xi g(r_\xi) \quad \text{and} \quad g\Big(\bigwedge_\xi r_\xi\Big) = \bigwedge_\xi g(r_\xi).$$

3. The main results

From now on, in the context of G-invariance, we assume that R is an arbitrary Dedekind complete vector lattice, while in the setting of G-equivariance we suppose that R is a Dedekind complete ρ-space and that R is contained in the domain of all involved functions. The problem of finding linear equivariant functionals with values in an arbitrary Dedekind complete vector lattice is still an open problem (see also [4, 5]). Moreover, when we deal with order continuous functionals, we always assume that X is super Dedekind complete and has the d-property, and we will not write it explicitly. We call v-*convex* a convex and G-invariant (resp., convex and G-equivariant) function, according to the studied context. We begin with recalling a Hahn-Banach-type theorem on the existence of linear functionals (not necessarily order continuous), proved in [4, Theorem 6] in the context of G-invariance and in [5, Theorem 3.2] in the setting of G-equivariance, which will be useful later.

Theorem 3.1. *Let $U : D(U) \to R$ be* v-*convex, $D(U) \subset X$ be convex and G-invariant, $0 \in int\big(D(U)\big)$ and $U(0) = 0$. Then there exists $L \in \mathcal{L}_{\rm v}(X, R)$, with $L(x) \leq U(x)$ for any $x \in D(U)$.*

We give some sandwich, duality and Moreau-Rockafellar-type theorems and optimality conditions for linear order continuous invariant or equivariant functionals, in which no topological structure is required. So, we replace the continuity conditions existing in the classical literature (when it is dealt with continuous linear operators, see also [28]) with hypotheses involving only "local boundedness", which in the classical case is weaker than continuity. We begin with the following

Theorem 3.2 (Sandwich theorem). *Let $U : D(U) \to R$, $V : D(V) \to R$ be two* v*-convex functions, where*

$$D(U), D(V) \subset X \quad \text{are convex and} \quad G\text{-invariant,} \tag{1}$$

and suppose that

$$U(x) + V(x) \geq 0 \text{ for any } x \in D(U) \cap D(V).$$

Assume that

3.2.1) there is $\overline{x} \in int\big(D(U)\big) \cap int\big(D(V)\big)$, such that either U or V is locally order bounded at $\overline{x}$.

Then there are $L_0 \in \mathcal{L}_{\mathrm{oc,v}}(X, R)$ and $u_0 \in R$ such that:

$$L_0(x) - u_0 \leq U(x) \qquad \text{for all} \qquad x \in D(U); \tag{2}$$

$$L_0(x') - u_0 \geq -V(x') \quad \text{for each} \quad x' \in D(V). \tag{3}$$

Proof. Without loss of generality, we can suppose that U is locally bounded at $\overline{x}$. We observe that from 3.2.1) it follows that $0 \in int\big(D(U) - D(V)\big)$. By [4, Theorem 1] and [5, Theorem 3.2], we find a linear functional $L_0 \in \mathcal{L}_{\mathrm{v}}(X, R)$ and an element $u_0 \in R$, satisfying (2) and (3). Note that, for every $x_1 \in D(U)$ and $x_2 \in D(V)$, it is

$$L_0(x_1) - L_0(x_2) = L_0(x_1 - x_2) \leq U(x_1) + V(x_2). \tag{4}$$

Now, choose arbitrarily $r \in X^+$, $r \neq 0$. By the local order boundedness of U at $\overline{x}$, there are a positive real number δ_r and $\gamma_r \in R^+$, $\gamma_r \neq 0$, such that $|U(\overline{x} + \delta_r\, x)| \leq \gamma_r$ whenever $x \in D$, $\overline{x} + \delta_r\, x \in D$ and $|x| \leq r$. From this and (4) it follows that

$$L_0(x) = L_0\Big(\overline{x} + \delta_r\, \frac{x}{\delta_r} - \overline{x}\Big) \leq U\Big(\overline{x} + \delta_r\, \frac{x}{\delta_r}\Big) + V(\overline{x}) \leq \gamma_{r/\delta_r} + V(\overline{x}) \tag{5}$$

since, of course, $|x| \leq r$ if and only if $\Big|\dfrac{x}{\delta_r}\Big| \leq \dfrac{r}{\delta_r}$. Changing x with $-x$, proceeding analogously as in (5), we get

$$L_0(x) = -L_0(-x) \geq -\gamma_{r/\delta_r} - V(\overline{x}). \tag{6}$$

From (5) and (6) we obtain that L_0 maps the order bounded interval $[-r, r] \subset X$ into a bounded subset of R. By the arbitrariness of r, we deduce that L_0 is order bounded. Since X is super Dedekind complete and has the d-property, by Remark 1 (b), L_0 is order continuous, too. This ends the proof. □

As consequences of Theorem 3.2, arguing similarly as in [4] and [5], it is possible to prove the following results, in which we assume condition 3.2.1).

Theorem 3.3 (Duality theorem). *Let $U : D(U) \to R$, $V : D(V) \to R$ be* v*-convex functions, where $D(U)$ and $D(V)$ satisfy* (1)*. Let*

$$r = \bigwedge\{U(x) + V(x) : x \in D(U) \cap D(V)\}$$

exist in R, where r is as in Problem I) , and let U^c, V^c be as in Problem II).

Then Problem II) has a solution L_0, such that $-U^c(L_0) - V^c(-L_0) = r$.

Theorem 3.4 (Optimality condition). *Let U, V, $D(U)$, $D(V)$ be as in Theorem 3.3, and let $x_0 \in D(U) \cap D(V)$ be a solution of Problem I). Then, $\partial_{\mathrm{oc,v}} U(x_0) \cap \big(-\partial_{\mathrm{oc,v}} V(x_0)\big) \neq \emptyset$.*

Theorem 3.5 (Moreau-Rockafellar formula). *Let U, V, $D(U)$, $D(V)$ be as in Theorem 3.3, $x_0 \in D(U) \cap D(V)$, and suppose that $\partial_{\mathrm{oc,v}} U(x_0) \neq \emptyset$ and $\partial_{\mathrm{oc,v}} V(x_0) \neq \emptyset$. Then, $\partial_{\mathrm{oc,v}}(U + V)(x_0) = \partial_{\mathrm{oc,v}}(U)(x_0) + \partial_{\mathrm{oc,v}}(V)(x_0)$.*

Arguing analogously as in [4] and [5], it is possible to prove the next results.

Theorem 3.6. *Let $U : D(U) \to R$ be* v*-convex, $D(U) \subset X$ be convex and G--invariant, and assume that $U(0) = 0$. Suppose that there exists $\overline{x} \in int\big(D(U)\big)$ such that U is locally order bounded at $\overline{x}$. Then there is $L \in \mathcal{L}_{\mathrm{oc,v}}(X, R)$, with $L(x) \le U(x)$ for any $x \in D(U)$.*

Theorem 3.7. *Let U and $D(U)$ be as in Theorem* 3.6 and $Z \subset X$ be a G--invariant subspace. Suppose that there exists $\overline{x} \in int\big(D(U)\big) \cap Z$ such that U is locally order bounded at $\overline{x}$. Let $L' \in \mathcal{L}_{\mathrm{oc,v}}(Z, R)$ be such that $L'(z) \le U(z)$ for all $z \in D(U) \cap Z$. Then L' has an extension $L \in \mathcal{L}_{\mathrm{oc,v}}(X, R)$, with $L(x) \le U(x)$ for every $x \in D(U)$.

The next result is new in the context of invariant/equivariant vector lattice-valued functionals, even when the involved operators are not necessarily order continuous. It deals with the subdifferential of composite functions and extends [16, 1.4.14 (4)].

Theorem 3.8. *Let R, X, Y be three Dedekind complete vector lattices with $R \subset X \subset Y$, and $G \subset Y^Y$ be a group of positive linear Y-functionals on Y. Assume that R, X are G-invariant, and let $T : Y \to X$ be a linear G-equivariant functional, $U : D(U) \to R$ be a* v*-convex function, and $D(U) \subset X$ be a convex and G-invariant set. Suppose that $U(0) = 0$ and*

$$0 \in \big(D(U)\big)^0. \tag{7}$$

Then

$$\partial_{\mathrm{v}}(U \circ T) = \big(\partial_{\mathrm{v}} U\big) \circ T, \tag{8}$$

where

$$(\partial_{\mathrm{v}} U) \circ T = \{L \circ T : L \in \partial_{\mathrm{v}} U\}.$$

Moreover, if 3.8.1) X *is super Dedekind complete and has the d-property;*

3.8.2) $T^{-1}(A)$ *is an order bounded subset of* X *whenever* A *is an order bounded subset of* Y*;*

3.8.3) *there is* $\overline{x} \in int\big(D(U)\big) \cap T(Y)$ *such that* U *is locally order bounded at* $\overline{x}$*, then*

$$\partial_{\mathrm{oc,v}}(U \circ T) = (\partial_{\mathrm{oc,v}} U) \circ T, \tag{9}$$

where $(\partial_{\mathrm{oc,v}} U) \circ T = \{L \circ T : L \in \partial_{\mathrm{oc,v}} U\}$*.*

P r o o f. We begin with proving the inclusions "$\subset$" in (8) and (9). Pick arbitrarily $S \in \partial_{\mathrm{v}}(U \circ T)$. We claim that

$$\mathrm{Ker}\; T \subset \mathrm{Ker}\; S. \tag{10}$$

From (7), for each $y \in Y$ there is $\lambda > 0$ such that $T(\lambda\, y) = \lambda\, T(y)$ and $T(-\lambda\, y) = -\lambda\, T(y)$ belong to $D(U)$, and hence

$$-U\big(T(-\lambda\, y)\big) \le -S(-\lambda\, y) = S(\lambda\, y) \le U\big(T(\lambda\, y)\big). \tag{11}$$

So, if $T(y) = 0$, then $T(\lambda\, y) = T(-\lambda\, y) = 0$. From this and (11), since $U(0) = 0$, it follows that $S(\lambda\, y) = 0$, and hence $S(y) = \frac{1}{\lambda} S(\lambda\, y) = 0$. Thus, we obtain (10).

Now we define $L' : T(Y) \to R$ by setting, for every $t \in T(Y)$, $L'(t) = S(y)$, where y is any element arbitrarily chosen in $T^{-1}(t)$. We claim that L' is well-defined, that is it does not depend on the choice of y. Indeed, let y_1, $y_2 \in Y$ be such that $t = T(y_1) = T(y_2)$. Then, $0 = T(y_1) - T(y_2) = T(y_1 - y_2)$. By (10), we obtain $0 = S(y_1 - y_2)$, and hence $S(y_1) = S(y_2)$ thanks to the linearity of S, getting the claim.

Now we prove that the G-invariance (resp.G-equivariance) of S implies the G--invariance (resp. G-equivariance) of L'. First, note that for each $g \in G$, $t \in T(Y)$ and $y \in T^{-1}(t)$ it is

$$gt = g\big(T(y)\big) = T(gy), \tag{12}$$

thanks to the G-equivariance of T. From (12) it follows that

$$L'(gt) = S(gy) = S(y) = L'(t)$$

when S is G-invariant, and

$$L'(gt) = S(gy) = g\big(S(y)\big) = g\big(L'(t)\big)$$

when S is G-equivariant, getting the G-invariance (resp., G-equivariance) of L'.

Now we claim that L' is linear. For each $\alpha_i \in \mathbb{R}$, $t_i \in T(Y)$ and $y_i \in T^{-1}(t_i)$, $i = 1, 2$, it is

$$\alpha_1 t_1 + \alpha_2 t_2 = \alpha_1 T(y_1) + \alpha_2 T(y_2) = T(\alpha_1 y_1 + \alpha_2 y_2)$$

thanks to the linearity of T, and hence

$$\begin{aligned} L'(\alpha_1 t_1 + \alpha_2 t_2) &= S(\alpha_1 y_1 + \alpha_2 y_2) \\ &= \alpha_1 S(y_1) + \alpha_2 S(y_2) = \alpha_1 L'(t_1) + \alpha_2 L'(t_2) \end{aligned} \tag{13}$$

thanks to the linearity of S, getting the claim.

Moreover, since $S \in \partial(U \circ T)$, then for every $t \in D(U) \cap T(Y)$ and $y \in T^{-1}(t)$, we get $L'(t) = S(y) \leq U\big(T(y)\big) = U(t)$.

Now we show that, if S is order bounded and 3.8.j), j=1,2, hold, then L' is order continuous. Indeed in this case, if $A \subset X$ is order bounded, then $T^{-1}(A)$ is, too, and hence, thanks to 3.8.3), $L'(A) = S\big(T^{-1}(A)\big)$ is order bounded. Thus, L' is order bounded. By 3.8.1) and Remark 1 (b), L' is order continuous too.

By Theorem 3.7 (resp., by its analogous versions for G-invariant or G-equivariant functionals, not necessarily order bounded or order continuous, see [4, Theorem 5] and [5, Theorem 4.4]) and thanks to 3.8.3), L' has an extension $L \in \mathcal{L}_{\mathrm{oc,v}}(X, R)$ (resp., $L \in \mathcal{L}_{\mathrm{v}}(X, R)$), with $L(x) \leq U(x)$ for every $x \in D(U)$. Thus, $L \in \partial_{\mathrm{oc,v}} U$ (resp., $L \in \partial_{\mathrm{v}} U$). Moreover, for each $y \in Y$ it is $L\big(T(y)\big) = L'\big(T(y)\big) = S(y)$, that is $S = L \circ T$. Therefore, $S \in (\partial_{\mathrm{oc,v}} U) \circ T$ (resp., $S \in (\partial_{\mathrm{v}} U) \circ T$). This proves the inclusion "$\subset$".

Now we prove the converse inclusion. Let

$$S \in (\partial_{\mathrm{oc,v}} U) \circ T \quad \text{(resp., } S \in (\partial_{\mathrm{v}} U) \circ T.\text{)}$$

Then, there is $L \in \partial_{\mathrm{oc,v}} U$ (resp., $L \in \partial_{\mathrm{v}} U$) with $S(y) = L\big(T(y)\big)$ for any $y \in Y$ and $L(x) \leq U(x)$ for all $x \in X$. Hence, $S(y) = L\big(T(y)\big) \leq U\big(T(y)\big)$ for any $y \in Y$, namely, $S \in \partial_{\mathrm{oc,v}}(U \circ T)$ (resp., $S \in \partial_{\mathrm{v}}(U \circ T)$). □

4. Applications to Optimization Problems

In [4] and [5], we gave some Farkas and Kuhn-Tucker-type theorems, in which we studied optimization problems with convex constraints. Here we extend these results to order continuous linear G-invariant or G-equivariant vector lattice-valued operators, in which we investigate optimization problems where both convex and linear constraints are present. To this aim, we will apply the Hahn-Banach-type theorem 3.1. We extend to our context, in which no topology structure is required, some results given in [17] and [18] in the vector lattice setting and some theorems proved in [29] for linear continuous operators in the context of locally convex ordered vector spaces. When some linear constraints are present, our results are new in the context of invariance/equivariance, even when it is dealt with functionals, not necessarily order continuous. We begin with giving the following

Assumptions 4.1. Let D_0, D_1, $D_2, \ldots, D_q$ be convex and G-invariant subsets of X, assume that $D = \bigcap_{i=0}^{q} D_i \neq \emptyset$, let $U_i : D_i = D(U_i) \to X$, $i = 0, 1, 2, \ldots, q$, be convex functions, $U_{q+j} : X \to X$, $j = 1, 2, \ldots, n$, be linear functions. Assume that U_0 is G-invariant (resp., G-equivariant) and U_i is G-equivariant for all $i = 1, 2, \ldots, q + n$. Let $\mathcal{W} = X^q$ be endowed with the "componentwise" order, given by $w = (w_1, w_2, \ldots, w_q) \geq w' = (w'_1, w'_2, \ldots, w'_q)$ if and only if $w_i \geq w'_i$ for any $i = 1, 2, \ldots, q$, and set $\mathcal{K} = (X^+)^q$.

Let $\mathcal{Z} = \prod_{j=1}^n Z_{q+j}$, where $Z_{q+j} = U_{q+j}(X)$, endowed with the "component-wise" order defined analogously as above, and put

$$\begin{aligned} H(x) &= \big(U_1(x), U_2(x), \ldots, U_q(x)\big), & x \in \textstyle\bigcap_{i=1}^q D_i, \\ K(x) &= \big(U_{q+1}(x), U_{q+2}(x), \ldots, U_{q+n}(x)\big), & x \in X. \end{aligned} \tag{14}$$

It is not difficult to check that H is convex, K is linear and the range of K is $\mathcal{Z}$.

For every $g \in G$, $w \in \mathcal{W}$, $w = (w_1, w_2, \ldots, w_q)$ and $z \in \mathcal{Z}$, $z = (z_1, z_2, \ldots, z_n)$, set $gw = (gw_1, gw_2, \ldots, gw_q)$ and $gz = (gz_1, gz_2, \ldots, gz_n)$. We say that H (resp., K) is *G-equivariant* iff $H(gx) = g\big((H(x)\big)$ for every $g \in G$ and $x \in D$ (resp., $K(gx) = g\big(K(x)\big)$ for every $g \in G$ and $x \in X$). Note that this property is equivalent to the G-equivariance of the U_i's, $i = 1, 2, \ldots, q$ (resp., $i = q+1, q+2, \ldots, q+n$).

We consider the following optimization problem

Problem III. Find $x_0 \in D$ such that $U_0(x_0) = \min\{U_0(x) : x \in D, U_i(x) \leq 0$ for all $i = 1, 2, \ldots, q$, and $U_{q+j}(x) = 0$ for all $j = 1, 2, \ldots, n\}$.

We now give the following condition, which is a "constraint qualification" and extends to our setting a condition formulated in [29] in the context of locally convex ordered vector spaces. Note that in our setting, since no topological structure is required, we use only algebraic properties. For a related literature on constraint qualifications see, e.g., [3, 16, 21] and the references therein. We assume that

4.1.1) there is $\overline{x} \in D$ with $U_{q+j}(\overline{x}) = 0$ for all $j = 1, 2, \ldots, n$, and such that for every $x \in X$ there are a positive real number λ_x and an element $c^{(x)} \in \mathcal{W}$, $c^{(x)} = \big(c_1^{(x)}, c_2^{(x)}, \ldots, c_q^{(x)}\big)$, with

$$\overline{x} + \lambda x \in D, \quad H(\overline{x} + \lambda x) \leq c^{(x)} \ \text{ and } \ 0 \in (c^{(x)} + \mathcal{K})^0 \tag{15}$$

for all $\lambda \in [-\lambda_x, \lambda_x]$.

We prove the following version of the Farkas theorem, extending [4, Theorem 11] and [29, Theorem 3] to G-invariant or G-equivariant linear order continuous operators.

Theorem 4.2. *Under Assumptions 4.1, suppose that, for every $x \in D$, it is*

$$U_0(x) \geq 0 \textit{ whenever } U_i(x) \leq 0 \textit{ for any } i = 1, 2, \ldots, q \tag{16}$$

and

$$U_{q+j}(x) = 0 \textit{ for all } j = 1, 2, \ldots, n,$$

and that 4.1.1) *holds.*

Then there exist $L_i \in \mathcal{L}_{+,\mathrm{oc},\mathrm{v}}(X,R)$, $i = 1, 2, \dots, q+n$, *with*

$$U_0(x) + \sum_{i=1}^{q+n} L_i\big(U_i(x)\big) \geq 0 \quad \textit{for each } x \in D. \tag{17}$$

P r o o f. Set $\mathcal{X} = \mathcal{W} \times \mathcal{Z}$. If $g \in G$ and $z \in \mathcal{Z}$, then, since K is G-equivariant, we get $K(gz) = g(K(z)) = 0$, that is $gz \in \mathcal{Z}$. Hence, $\mathcal{Z}$ is G-invariant, and so $\mathcal{X}$ is G-invariant, too. Moreover, it is not difficult to see that $\mathcal{X}$ is convex. Now, set

$$\begin{aligned} A &= \{(w,z,y) \in \mathcal{X} \times R : \exists\, x \in D \text{ with } w \geq H(x),\ z = K(x),\ y \geq U_0(x)\}; \\ B &= \bigcup_{\lambda > 0} \lambda\, A. \end{aligned} \tag{18}$$

By proceeding analogously as in [4, 11.3)], it is possible to see that the sets A and B defined in (18) are convex, and

$$\big(w^{(1)} + w^{(2)}, z^{(1)} + z^{(2)}, y^{(1)} + y^{(2)}\big) \in B \tag{19}$$

whenever

$$\big(w^{(1)}, z^{(1)}, y^{(1)}\big), \ \ \big(w^{(2)}, z^{(2)}, y^{(2)}\big) \in B.$$

We will construct a convex and G-invariant (resp., G-equivariant) function $p : \mathcal{X} \to R$, in order to apply Theorem 3.1. First, for every $w \in \mathcal{W}$ and $z \in \mathcal{Z}$, put $E_{w,z} = \{y \in R : (w,z,y) \in B\}$. We claim that

$$E_{w,z} \neq \emptyset \quad \text{for each } \ w \in \mathcal{W} \quad \text{and} \quad z \in \mathcal{Z}. \tag{20}$$

Fix arbitrarily $w \in \mathcal{W}$, $w = (w_1, w_2, \dots, w_q)$, and $z \in \mathcal{Z}$, $z = (z_1, z_2, \dots, z_n)$. There exists $x' \in X$ with $z = K(x')$. Let $\overline{x}$ be as in 4.1.1). In correspondence with $\overline{x}$ and x' there are a positive real number $\lambda_{x'}$ and an element $c^{(x')} \in \mathcal{W}$, $c^{(x')} = \big(c_1^{(x')}, c_2^{(x')}, \dots, c_q^{(x')}\big)$, satisfying (15). Thus, for every $i \in \{1, 2, \dots, q\}$, in correspondence with w_i, $\overline{x}$, x' and $c^{(x')}$, there are positive real numbers λ_i, $i = 1, 2, \dots, q$, with $c_i^{(x')} \leq \lambda\, w_i$ for every $\lambda \in [-\lambda_i, \lambda_i]$. Let $\lambda_0 = \min\{\lambda_{x'}; \lambda_i$: $i = 1, 2, \dots, q\}$. We get: $\overline{x} + \lambda_0 x' \in D$; $U_i(\overline{x} + \lambda_0 x') \leq c_i^{(x')} \leq \lambda_0\, w_i$, $i = 1, 2, \dots, q$. Hence, we obtain: $H(\overline{x} + \lambda_0 x') \leq \lambda_0\, w$; $K(\overline{x} + \lambda_0 x') = K(\overline{x}) + \lambda_0 K(x') = \lambda_0\, z$. From this, it follows that $\big(\lambda_0\, w, \lambda_0\, z, U_0(\overline{x} + \lambda_0 x')\big) \in A$, and hence

$$\left(w, z, \frac{1}{\lambda_0}\, U_0(\overline{x} + \lambda_0\, x')\right) = \frac{1}{\lambda_0}\big(\lambda_0\, w, \lambda_0\, z, U_0(\overline{x} + \lambda_0\, x')\big) \in B.$$

Thus, we obtain (20).

Furthermore, by proceeding analogously as in [4, 11.5)], it is not difficult to check that $E_{w,z} + E_{w',z'} \subset E_{w+w',z+z'}$ for every $w, w' \in \mathcal{W}$ and $z, z' \in \mathcal{Z}$.

Now we claim that

4.2.1) for each $w \in \mathcal{W}$ and $z \in \mathcal{Z}$, the set $E_{w,z}$ is lower order bounded, and

$$E_{0,0} \subset R^+. \tag{21}$$

Choose arbitrarily $y \in E_{w,z}$. As K is linear, we get $K(\lambda\, z) = \lambda\, K(z) = 0$, and hence $\lambda\, z \in \mathcal{Z}$, for all $\lambda \in \mathbb{R}$ and $z \in \mathcal{Z}$. By (20), $E_{-w,-z} \neq \emptyset$. Let $y_0 \in R$ be such that $-y_0 \in E_{-w,-z}$. We get:

$$y - y_0 \in E_{w,z} + E_{-w,-z} \subset E_{0,0} = \{\zeta \in R : (0,0,\zeta) \in B\}. \tag{22}$$

Hence, there are $\lambda_* > 0$, $\zeta \in R$ and $x_0 \in D$ such that $H(x_0) \le 0$, $K(x_0) = 0$ and $\lambda_*\, \zeta \ge U_0(x_0)$. From this, (16) and (22) we obtain $U_0(x_0) \ge 0$. This implies that $\xi \ge 0$ whenever $(0,0,\xi) \in B$, that is (21). Moreover, we get $y - y_0 \ge 0$. By the arbitrariness of y, we deduce that the element y_0 is a lower order bound for the set $E_{w,z}$, getting 4.2.1).

Thus, it makes sense to define a function $p : \mathcal{X} \to R$, by putting

$$p(w,z) = \bigwedge\{y \in R : y \in E_{w,z}\}, \quad w \in \mathcal{W},\ z \in \mathcal{Z}.$$

Proceeding analogously as in the proof of [4, Theorem 11] and [29, Theorem 3], it is not difficult to see that $p(0,0) = 0$ and p is convex on $\mathcal{X}$.

Now we demonstrate that

4.2.2) p is G-invariant (resp., G-equivariant).

Before proving 4.2.2), we claim that, if U_0 is G-invariant (resp., G-equivariant), then for every $g \in G$ we get

$$(gw, gz, y) \in A \quad \text{if and only if} \quad (w,z,y) \in A, \tag{23}$$

respectively,

$$(gw, gz, gy) \in A \quad \text{if and only if} \quad (w,z,y) \in A.$$

We prove only the "if" part, since the "only if" part is analogous, by changing g with g^{-1}. Pick arbitrarily $g \in G$ and $(w,z,y) \in A$. Then, there exists an element $x \in D$ with $w \ge H(x)$, $z = K(x)$ and $y \ge U_0(x)$. Note that $gx \in D$, because D is G-invariant. Since, by hypothesis, H and K are G-equivariant and the elements of G are increasing homomorphisms, we have $H(gx) = g\big((H(x)\big) \le gw$, $K(gx) = g\big((K(x)\big) = gz$. Moreover, $U_0(gx) = U_0(x) \le y$ when U_0 is G-invariant, and $U_0(gx) = g\big((U_0(x)\big) \le gy$ when U_0 is G-equivariant. This implies $(gw, gz, y) \in A$ when U_0 is G-invariant and $(gw, gz, gy) \in A$ when U_0 is G--equivariant, getting the claim. Note that (23) holds also when A is replaced by B.

Now we turn to 4.2.2). By (23) used with B instead of A, and taking into account Proposition 2.1, for each $g \in G$, $w \in \mathcal{W}$ and $z \in \mathcal{Z}$ it is

$$\begin{aligned} p\big(g(w,z)\big) &= p(gw,gz) = \bigwedge\{y \in R : (gw,gz,y) \in B\} \\ &= \bigwedge\{y \in R : (w,z,y) \in B\} = p(w,z) \end{aligned} \tag{24}$$

when U_0 is G-invariant, and

$$\begin{aligned} p\big(g(w,z)\big) &= \bigwedge\{y \in R : (gw,gz,y) \in B\} = \bigwedge\{y \in R : (w,z,g^{-1}y) \in B\} \\ &= \bigwedge\{gy \in R : (w,z,y) \in B\} = \bigwedge g(E_{w,z}) \\ &= g\Big(\bigwedge E_{w,z}\Big) = g\big(p(w,z)\big) \end{aligned} \tag{25}$$

when U_0 is G-equivariant. Thus we get the G-invariance or the G-equivariance of p, respectively. This proves 4.2.2).

So, analogously as in Theorem 3.1, we find a linear and G-invariant (resp., G-equivariant) function $L : \mathcal{W} \times \mathcal{Z} \to R$ with $L(w,z) \le p(w,z)$ for every $w \in \mathcal{W}$ and $z \in \mathcal{Z}$. The existence of G-invariant, linear and positive functions L_i satisfying (17) follows by proceeding analogously as in the proofs of [4, Theorem 11] and [29, Theorem 3]. Since X is a super Dedekind complete vector lattice and has the d-property, the L_i's are order continuous. This ends the proof. □

When $n = 0$, namely when there are no linear constraints, we deal with the following problem:

Problem IV. Find $x_0 \in D$ such that $U_0(x_0) = \min\{U_0(x) : x \in D,\ U_i(x) \le 0,$ $i = 1, 2, \dots, q\}$.

In this case, it is possible to replace the constraint qualification 4.1.1) with the following weaker condition

4.1.2) $0 \in int\big(H(D) + \mathcal{K}\big)$

(see also [4]), to prove the next Farkas-type theorem.

Theorem 4.3. *Assume that, for each $x \in D$, it is*

$$U_0(x) \ge 0 \quad \textit{whenever} \quad U_i(x) \le 0 \quad \textit{for all } i = 1,2,\dots,q, \tag{26}$$

and that 4.1.2) *holds. Then there are $L_i \in \mathcal{L}_{+,\mathrm{oc},\mathrm{v}}(X,R)$, $i = 1, 2, \dots, q$, with*

$$U_0(x) + \sum_{i=1}^{q} L_i\big(U_i(x)\big) \ge 0 \quad \textit{for any } x \in D.$$

Proof. Let $\mathcal{Y} = H(D) + \mathcal{K}$. Proceeding analogously as in the proof of [4, Theorem 11], it is possible to see that $\mathcal{Y}$ is convex and G-invariant. Set

$$A = \{(w, y) \in \mathcal{Y} \times R : \text{ there is } x \in D \text{ with } w \geq H(x) \text{ and } y \geq U_0(x)\};$$
$$B = \bigcup_{\lambda > 0} \lambda A.$$

Note that A and B are convex (see also [4, 11.3)]). For each $w \in \mathcal{Y}$, put $S_w = \{y \in R : (w, y) \in B\}$. We claim that

4.3.1) $$S_w \neq \emptyset \quad \text{for all } w \in \mathcal{Y}. \tag{27}$$

Pick arbitrarily $w \in \mathcal{Y}$. As $0 \in int(\mathcal{Y})$, we find a positive real number λ_0 such that $\lambda w \in \mathcal{Y}$ for any $\lambda \in [-\lambda_0, \lambda_0]$. Thus, there is $x_0 \in D$ with

$$0 \leq \lambda_0 w - H(x_0) = \lambda_0 \Big(w - \frac{1}{\lambda_0} H(x_0) \Big).$$

Since $\big(\lambda_0 w, U_0(x_0)\big) \in A$, we have

$$\Big(w, \frac{1}{\lambda_0} U_0(x_0) \Big) = \frac{1}{\lambda_0} \big(\lambda_0 w, U_0(x_0)\big) \in B,$$

getting (27).

Now we prove that

$$S_0 \subset R^+. \tag{28}$$

Let $y \in S_0$. Then, $(0, y) \in B$, namely there are a positive real number λ_0 and an element $x_0 \in D$ such that $U_i(x_0) \leq 0$ for all $i = 1, 2, \ldots, q$ and $\lambda_0 y \geq U_0(x_0)$. By (26), $U_0(x_0) \geq 0$, and hence $\lambda_0 y \geq 0$. Thus, $y \geq 0$, and (28) follows from the arbitrariness of y.

Now we prove that

4.3.2) the set S_w is lower order bounded for every $w \in \mathcal{Y}$.

Fix $w \in \mathcal{Y}$, and choose arbitrarily $y \in S_w$. Since $0 \in int(\mathcal{Y})$, there is a real number $\lambda_w \in (0, 1)$ with $\lambda w \in \mathcal{Y}$ whenever $|\lambda| \leq \lambda_w$. Thus, the set $S_{-\lambda_w w}$ is well-defined and nonempty. Let $y_w \in S_{-\lambda_w w}$. Since $(-\lambda_w w, y_w) \in B$, $(w, y) \in B$ and B is convex, we obtain

$$\Big(0, \frac{1}{1+\lambda_w} y_w + \frac{\lambda_w}{1+\lambda_w} y \Big) = \frac{1}{1+\lambda_w} (-\lambda_w w, y_w) + \frac{\lambda_w}{1+\lambda_w} (w, y) \in B,$$

and hence, thanks to (28),

$$\frac{1}{1+\lambda_w} y_w + \frac{\lambda_w}{1+\lambda_w} y \in S_0 \subset R^+.$$

Therefore, we get $y \geq -\frac{y_w}{\lambda_w}$, and 4.3.2) follows from the arbitrariness of y. Thus, it is possible to define a function $p : \mathcal{Y} \to R$, by setting

$$p(w) = \bigwedge S_w, \quad w \in \mathcal{Y}. \tag{29}$$

Arguing analogously as in [4, Theorem 3.11], one sees that p is well-defined and convex, and $p(0) = 0$. Proceeding similarly as in (24) (resp. (25)), it is possible to check that p is G-invariant (resp., G-equivariant). Thus, analogously as in Theorem 3.1, we find a linear and G-invariant (resp., G-equivariant) function $L : X^q \to R$ with $L(w) \leq p(w)$ for every $w \in \mathcal{Y}$. The assertion follows by proceeding analogously as at the end of the proof of Theorem (4.2). □

As a consequence of Theorems 4.2 and 4.3, we give a Kuhn-Tucker-type result on existence of saddle points related with Problems III) and IV), respectively, whose proof is analogous of those of [4, Theorem 13], [5, Theorem 4.7] and [29, Theorem 5].

Corollary 4.4. *Under the same hypotheses as in Theorem 4.2 (resp., Theorem 4.3), if x_0 is a solution of Problem III (resp., Problem IV), then there are $L_{0,i} \in \mathcal{L}_{+,\mathrm{oc},\mathrm{v}}(X, R)$, with*

$$\begin{aligned} U_0(x_0) + \sum_{i=1}^{q+n} L_i\big(U_i(x_0)\big) &\leq U_0(x_0) + \sum_{i=1}^{q+n} L_{0,i}\big(U_i(x_0)\big) \\ &\leq U_0(x) + \sum_{i=1}^{q+n} L_{0,i}\big(U_i(x)\big), \end{aligned}$$

(respectively,

$$\begin{aligned} U_0(x_0) + \sum_{i=1}^{q} L_i\big(U_i(x_0)\big) &\leq U_0(x_0) + \sum_{i=1}^{q} L_{0,i}\big(U_i(x_0)\big) \\ &\leq U_0(x) + \sum_{i=1}^{q} L_{0,i}\big(U_i(x)\big) \;) \end{aligned}$$

for any $x \in D$ and $L_i \in \mathcal{L}_{+,\mathrm{oc},\mathrm{v}}(X, R)$, $i = 1, 2, \ldots, q+n$ (resp., $i = 1, 2, \ldots, q$).

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