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# A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY GEGENBAUER POLYNOMIALS

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ABSTRACT. In this paper, we introduce a new subclass of analytic functions with negative coefficients defined by Gegenbauer polynomials. We obtain coefficient bounds, growth and distortion properties, extreme points and radii of starlikeness, convexity and close-to-convexity for functions belonging to the class $TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$. Furthermore, we obtained the Fekete-Szego problem for this class.

## 1. Introduction

Let $A$ denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let $S$ be a subclass of $A$ consisting of univalent functions and let satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by $S$ the subclass of $A$ consisting of functions $u(z)$ which are all univalent in $E$.

A function $u \in A$ is a starlike function of the order $\upsilon, 0 \le \upsilon < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

This class will be denoted by $S^*(\upsilon)$ .


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A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfies

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{3}$$

$K(\upsilon)$ will denote this class.

Note that $S^*(0) = S^*$ and $K(0) = K$ are usual classes of starlike and convex functions in $E$, respectively.

Let $T$ denote the class of functions analytic in $E$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0,\ z \in E) \tag{4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon),\ C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and the allied classes possess some interesting properties and have been extensively studied by Silverman [13].

The class $\mathcal{T}(\lambda), \lambda \geq 0$, was introduced and investigated by Szynal [16] as a subclass of $\mathcal{A}$ consisting of functions of the form

$$u(z) = \int_{-1}^{1} k(z,m)\mathrm{d}\mu(m), \tag{5}$$

where

$$k(z,m) = \frac{z}{(1 - 2mz + z^2)^{\lambda}} \qquad (z \in E, m \in [-1,1]) \tag{6}$$

and $\mu$ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$.

The Taylor series expansion of the function in (6) gives

$$k(z,m) = z + c_1^{\lambda}(m)z^2 + c_2^{\lambda}(m)z^3 + \cdots \tag{7}$$

and the coefficients for (7) were given below:

$$\begin{gathered} c_0^{\lambda}(m) = 1; \quad c_1^{\lambda}(m) = 2\lambda m; \quad c_2^{\lambda}(m) = 2\lambda(\lambda+1)m^2 - \lambda; \\ c_3^{\lambda}(m) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)m^3 - 2\lambda(\lambda+1)m \ldots, \end{gathered} \tag{8}$$

where $c_n^{\lambda}(m)$ denotes the Gegenbauer polynomial of degree $n$. Varying the parameter $\lambda$ in (7), we obtain the class of typically real functions studied by [1, 2, 6, 9, 10, 12, 14] and [17].

Let $\mathcal{G}_{\lambda}^{m} : A \to A$ be defined in terms of convolution by

$$\mathcal{G}_{\lambda}^{m} u(z) = k(z,m) * u(z).$$

We have

$$\mathcal{G}_{\lambda}^{m} u(z) = z + \sum_{n=2}^{\infty} \phi(\lambda, m, n) a_n z^n \tag{9}$$

where

$$\phi(\lambda, m, n) = c_{n-1}^{\lambda}(m).$$

Now, by making use of the linear operator $\mathcal{G}_\lambda^m$, we define a new subclass of functions belonging to the class $A$.

**Definition 1.1.** For $0 \le \gamma \le 1,\ \varrho \ge 1, k \ge 0$ and $0 \le \vartheta < 1$, a function $u \in A$ is said to be in the class $S_\lambda^m(\gamma, \varrho, k, \vartheta)$ if it satisfies condition

$$\Re\left\{\varrho\frac{zF'(z)}{F(z)} - (\varrho - 1)\right\} > k\left|\varrho\frac{zF'(z)}{F(z)} - \varrho\right| + \vartheta, \tag{10}$$

where

$$F(z) = (1-\gamma)\mathcal{G}_\lambda^m u(z) + \gamma z\big(\mathcal{G}_\lambda^m u(z)\big)'. \tag{11}$$

We also define

$$TS_\lambda^m(\gamma, \varrho, k, \vartheta) = S_\lambda^m(\gamma, \varrho, k, \vartheta) \cap T.$$

By suitably specializing the parameters involved, the class $S_\lambda^m(\gamma, \varrho, k, \vartheta)$ and if it satisfies the condition $TS_\lambda^m(\gamma, \varrho, k, \vartheta)$ can be reduced to new or to known much simpler classes of functions which were studied in earlier works (see [3–5, 7, 11, 15]).

The object of this paper is to study various properties for functions belonging to the class $S_\lambda^m(\gamma, \varrho, k, \vartheta)$ and $TS_\lambda^m(\gamma, \varrho, k, \vartheta)$, respectively.

## 2. Coefficient estimates

In order to prove our results from this section, we need the following lemma.

**Lemma 2.1.** *Let $\vartheta$ be a real number and $w$ be a complex number. Then, $\Re(w) \ge \vartheta$ if and only if*

$$|w + (1-\vartheta)| - |w - (1+\vartheta)| \ge 0.$$

*First, we give a sufficient coefficient bound for functions in the class*

$$S_\lambda^m(\gamma, \varrho, k, \vartheta).$$

**Theorem 2.2.** *Let $u \in A$ be given by* (1). *If*

$$\sum_{n=2}^{\infty}[1 - \vartheta + \varrho(n-1)(1+k)]A_n(\lambda, \gamma, m)|a_n| \le 1 - \vartheta \tag{12}$$

*where*

$$A_n(\lambda, \gamma, m) = [1 + \gamma(m-1)]\ \phi(\lambda, m, n). \tag{13}$$

*Then, $u \in S_\lambda^m(\gamma, \varrho, k, \vartheta)$.*

P r o o f. In virtue of Definition 1.1 and Lemma 2.1, it is sufficient to show that

$$\left|\varrho\frac{zF'(z)}{F(z)}-(\varrho-1)-k\left|\varrho\frac{zF'(z)}{F(z)}-\varrho\right|-(1+\vartheta)\right| \leq \left|\varrho\frac{zF'(z)}{F(z)}-(\varrho-1)-k\left|\varrho\frac{zF'(z)}{F(z)}-\varrho\right|+(1+\vartheta)\right|. \quad (14)$$

For the right hand and left hand side of (14), respectively, we may write

$$\begin{aligned} R &= \left|\varrho\frac{zF'(z)}{F(z)}-(\varrho-1)-k\left|\varrho\frac{zF'(z)}{F(z)}-\varrho\right|+(1-\vartheta)\right| \\ &= \frac{1}{|F(z)|}\left|\varrho zF'(z)-(\varrho-1)F(z)-ke^{i\theta}\left|\varrho zF'(z)-\varrho F(z)\right|+(1-\vartheta)F(z)\right| \\ &> \frac{|z|}{|F(z)|}\left[2-\vartheta-\sum_{n=2}^{\infty}2-\vartheta+\varrho(n-1)(k+1)\right]A_n(\lambda,\gamma,m)|a_n| \end{aligned}$$

and similarly,

$$\begin{aligned} L &= \left|\varrho\frac{zF'(z)}{F(z)}-(\varrho-1)-k\left|\varrho\frac{zF'(z)}{F(z)}-\varrho\right|-(1+\vartheta)\right| \\ &= \frac{1}{|F(z)|}\left|\varrho zF'(z)-(\varrho-1)F(z)-ke^{i\theta}\left|\varrho zF'(z)-\varrho F(z)\right|-(1+\vartheta)F(z)\right| \\ &< \frac{|z|}{|F(z)|}\left[\vartheta+\sum_{n=2}^{\infty}\left|\varrho(n-1)(1+k)-\vartheta\right|A_n(\lambda,\gamma,m)|a_n|\right] \end{aligned}$$

since

$$R-L > \frac{|z|}{|F(z)|}\left[2(1-\vartheta)-2\sum_{n=2}^{\infty}[1-\vartheta+\varrho(n-1)(1+k)]A_n(\lambda,\gamma,m)|a_n|\right] \geq 0,$$

the required condition (12) is satisfied. □

In the next theorem, we obtain a necessary and sufficient condition for a function $u\in T$ to be in the class $TS_\lambda^m(\gamma,\varrho,k,\vartheta)$.

**Theorem 2.3.** *Let $u\in T$ be given by (4). Then, $u\in TS_\lambda^m(\gamma,\varrho,k,\vartheta)$ if and only if*

$$\sum_{n=2}^{\infty}[1-\vartheta+\varrho(n-1)(1+k)]A_n(\lambda,\gamma,m)a_n \leq 1-\vartheta, \quad (15)$$

*where $A_n(\lambda,\gamma,m)$ is defined by (13). The result is sharp.*

P r o o f. Assume that inequality (15) holds true. In virtue of Theorem 2.2 and the definition of $TS_{\lambda}^{m}(\gamma,\varrho,k,\vartheta)$, choosing the values of $z$ on the positive real axis, inequality (10) reduces to

$$\frac{1-\sum\limits_{n=2}^{\infty}[1+\varrho(n-1)]A_n(\lambda,\gamma,m)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}A_n(\lambda,\gamma,m)a_nz^{n-1}}-\vartheta > k\left|\frac{\sum\limits_{n=2}^{\infty}\varrho(n-1)A_n(\lambda,\gamma,m)a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}A_n(\lambda,\gamma,m)a_nz^{n-1}}\right|. \quad (16)$$

Letting $z\to 1^-$, we obtain the desired inequality. Finally, equality holds for the function $u$ defined by

$$u(z)=z-\frac{1-\vartheta}{[1-\vartheta+\varrho(n-1)(1+k)]A_n(\lambda,\gamma,m)}z^n, \quad (n\geq 2). \quad (17)$$

□

**Corollary 2.4.** *If $u\in TS_{\lambda}^{m}(\gamma,\varrho,k,\vartheta)$ then,*

$$a_n\leq\frac{1-\vartheta}{[1-\vartheta+\varrho(n-1)(1+k)]A_n(\lambda,\gamma,m)}, \quad (n\geq 2). \quad (18)$$

*Equality is obtained for the function $u$ given by* (17).

## 3. Growth and Distortion theorem

**Theorem 3.1.** *Let $u\in TS_{\lambda}^{m}(\gamma,\varrho,k,\vartheta)$. Then, for $|z|=r<1$*

$$r-\frac{(1-\vartheta)}{B_2(\lambda,\gamma,m,\varrho,k,\vartheta)}r^2\ \leq\ |u(z)|\ \leq r+\frac{(1-\vartheta)}{B_2(\lambda,\gamma,m,\varrho,k,\vartheta)}r^2 \quad (19)$$

*and*

$$1-\frac{2(1-\vartheta)}{B_2(\lambda,\gamma,m,\varrho,k,\vartheta)}r^2\ \leq\ |u'(z)|\ \leq 1+\frac{2(1-\vartheta)}{B_2(\lambda,\gamma,m,\varrho,k,\vartheta)}r, \quad (20)$$

*where*

$$B_n(\lambda,\gamma,m,\varrho,k,\vartheta)=[1-\vartheta+\varrho(n-1)(1+k)]A_n(\lambda,\gamma,m), \quad (n\geq 2). \quad (21)$$

*Inequalities* (19) *and* (20) *are sharp for the function $u$ given by*

$$u(z)=z-\frac{(1-\vartheta)}{B_2(\lambda,\gamma,m,\varrho,k,\vartheta)}z^2.$$

Proof. Since $u \in TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$ and from Theorem 2.3, it follows

$$\sum_{n=2}^{\infty} B_n(\lambda, \gamma, m, \varrho, k, \vartheta)a_n \leq (1-\vartheta),$$

where $B_n(\lambda, \gamma, m, \varrho, k, \vartheta)$ is given by (21), we have

$$\begin{aligned} B_2(\lambda, \gamma, m, \varrho, k, \vartheta)\sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} B_2(\lambda, \gamma, m, \varrho, k, \vartheta)a_n \\ &\leq \sum_{n=2}^{\infty} B_n(\lambda, \gamma, m, \varrho, k, \vartheta)a_n \leq 1-\vartheta \end{aligned}$$

and therefore,

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)}. \tag{22}$$

Since $u$ is given by (3), we obtain

$$|u(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n |z|^{n-2} \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)} r^2$$

and

$$|u(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n |z|^{n-2} \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)} r^2.$$

In view of Theorem 2.3, we also have

$$\begin{aligned} \frac{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)}{2}\sum_{n=2}^{\infty} na_n &= \sum_{n=2}^{\infty} \frac{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)}{2}\ na_n \\ &\leq \sum_{n=2}^{\infty} (B_n, \lambda, \gamma, m, \varrho, k, \vartheta)a_n \leq (1-\vartheta) \end{aligned}$$

which yields

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)}.$$

Thus,

$$|u'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 + r\sum_{n=2}^{\infty} na_n \leq 1 + \frac{2(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)} r$$

and

$$|u'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \geq 1 - r\sum_{n=2}^{\infty} na_n \geq 1 - \frac{2(1-\vartheta)}{B_2(\lambda, \gamma, m, \varrho, k, \vartheta)} r.$$

Now, the proof of our theorem is completed. □

## 4. Extreme points

Next, we examine the extreme points for the function class $TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$.

**Theorem 4.1.** *Let the functions $u_1(z) = z$ and*

$$u_n(z) = z - \frac{(1-\vartheta)}{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)} z^n, \tag{23}$$

*where*

$$0 \le \lambda \le 1, \quad 0 \le \gamma \le 1, \quad m \in N, \quad \varrho \ge 1, \quad k \ge 0, \quad 0 \le \vartheta < 1, \quad n \ge 2.$$

*Then, $u \in TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$ if and only if*

$$u(z) = \sum_{n=2}^{\infty} \lambda_n u_n(z), \ (z \in E), \tag{24}$$

*where*

$$\lambda_n \ge 0 \ (n \ge 1) \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1. \tag{25}$$

P r o o f. Assume that $u$ can be written as in (24). Then,

$$u(z) = \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left[ z - \frac{(1-\vartheta)}{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)} z^n \right] = z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\vartheta)}{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)} z^n.$$

Since

$$\sum_{n=2}^{\infty} B_n(\lambda, \gamma, m, \varrho, k, \vartheta) \lambda_n \frac{(1-\vartheta)}{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)} = (1-\vartheta) \sum_{n=2}^{\infty} \lambda_n = (1-\vartheta)(1-\lambda_1) \le (1-\vartheta),$$

in virtue of Theorem 2.3 it follows that

$$u \in TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta).$$

Conversely, suppose $u \in TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$ and consider

$$\lambda_n = \frac{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{(1-\vartheta)} a_n, \quad (n \ge 2) \quad \text{and} \quad \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then,

$$u(z) = \sum_{n=1}^{\infty} \lambda_n u_n(z).$$

Hence, the proof is completed. □

## 5. Radii of starlikeness, convexity and close to convexity

We begin this section with the following theorem.

**Theorem 5.1.** *Let the function $u$ given by (4) be in the class $TS_{\lambda}^{m}(\gamma, \varrho, k, \vartheta)$. Then, $u$ is starlike of order $\rho (0 \le \rho < 1)$ in $|z| < r_1(\lambda, \gamma, m, \varrho, k, \vartheta)$, where*

$$r_1(\lambda, \gamma, m, \varrho, k, \vartheta) = \inf_{n \ge 2} \left[ \frac{(1-\rho) B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{(n-\rho)(1-\vartheta)} \right]^{\frac{1}{n-1}}.$$

Proof. To prove the theorem, we must show that

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \le 1 - \rho, \quad \text{for } z \in E,\ 0 \le \rho < 1 \text{ with } |z| < r_1(\lambda, \gamma, m, \varrho, k, \vartheta).$$

We have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| = \left| \frac{-\sum\limits_{n=2}^{\infty} (n-1) a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \right| \le \frac{\sum\limits_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus,

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \le 1 - \rho \quad \text{if} \quad \sum_{n=2}^{\infty} \frac{(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \le 1. \tag{26}$$

In virtue of (15), we have

$$\frac{\sum\limits_{n=2}^{\infty} B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{1-\vartheta} a_n \le 1.$$

Hence, inequality (26) will be true if

$$\frac{(n-\rho)}{(1-\rho)} |z|^{n-1} \le \frac{B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{1-\vartheta}$$

or if

$$|z| \le \left[ \frac{(1-\rho) B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{(n-\rho)(1-\vartheta)} \right]^{\frac{1}{n-1}}, \qquad (n \ge 2).$$

Thus, the proof of the theorem is completed. □

Proofs of the following Theorem 5.2 and Theorem 5.3 are analogous to that of Theorem 5.1, so we omit them.

**Theorem 5.2.** *Let the function $u$ given by (4) be in the class $TS_\lambda^m(\gamma, \varrho, k, \vartheta)$. Then, $u$ is convex of order $\rho$ $(0 \leq \rho < 1)$ in $|z| < r_2(\lambda, \gamma, m, \varrho, k, \vartheta)$, where*

$$r_2(\lambda, \gamma, m, \varrho, k, \vartheta) = \inf_{n \geq 2} \left[ \frac{(1-\rho)B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{n(n-\rho)(1-\vartheta)} \right]^{\frac{1}{n-1}}.$$

**Theorem 5.3.** *Let the function $u$ given by (4) be in the class $TS_\lambda^m(\gamma, \varrho, k, \vartheta)$. Then, $u$ in close-to-convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_3(\lambda, \gamma, m, \varrho, k, \vartheta)$, where*

$$r_3(\lambda, \gamma, m, \varrho, k, \vartheta) = \inf_{n \geq 2} \left[ \frac{(1-\rho)B_n(\lambda, \gamma, m, \varrho, k, \vartheta)}{n(1-\vartheta)} \right]^{\frac{1}{n-1}}.$$

## 6. The Fekete-Szego problem for the function class $S_\lambda^m(\gamma, \varrho, k, \vartheta)$

In this section, we obtain the Fekete-Szego inequality for the functions in the class $S_\lambda^m(\gamma, \varrho, k, \vartheta)$. In the order to prove our main result, we need the following lemma.

**Lemma 6.1** ([8]). *If $p(z) = 1 + c_1 z + c_2 z + c_3 z^2 + \cdots$ is an analytic function with positive real part in $E$, then*

$$|c_2 - \nu c_1^2| = \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu \geq 1, \end{cases}$$

*when $\nu < 0$ or $\nu > 1$ the inequality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if*

$$p(z) = \frac{1+z^2}{1-z^2}$$

*or one of rotations. If $\nu = 0$, the equality holds if and only if*

$$p(z) = \left(\frac{1+\delta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\delta}{2}\right)\frac{1-z}{1+z}, \quad (0 \leq \delta \leq 1) \text{ or one of its rotations.}$$

*If $\nu = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$.*

**Theorem 6.2.** *Let $\varrho \geq 1$, $0 \leq k \leq \vartheta < 1$. If $u \in S_\lambda^m(\gamma, \varrho, k, \vartheta)$ is given by (1), then*

$$|a_3-\mu a_2^2| = \begin{cases} \frac{(1-\vartheta)}{\varrho^2(1-k)^2 A_3(\lambda,\gamma,m)}\left[\varrho(1-k)+2(1-\vartheta)-4\mu(1-\vartheta)\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)}\right], & \mu \leq \sigma_1, \\ \frac{(1-\vartheta)}{\varrho(1-k)A_3(\lambda,\gamma,m)}, & \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{-(1-\vartheta)}{\varrho^2(1-k)^2 A_3(\lambda,\gamma,m)}\left[\varrho(1-k)+2(1-\vartheta)-4\mu(1-\vartheta)\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)}\right], & \mu \geq \sigma_2, \end{cases}$$

*where*

$$\sigma_1 = \frac{A_2^2(\lambda,\gamma,m)}{2A_3(\lambda,\gamma,m)} \qquad \textit{and} \qquad \sigma_2 = \frac{A_2^2(\lambda,\gamma,m)[1-\vartheta+\varrho(1-k)]}{2A_3(\lambda,\gamma,m)(1-\vartheta)}.$$

*The result is sharp.*

Proof. Since $\Re(w) \leq |w|$ for any complex numbers, $u \in S_\lambda^m(\gamma, \varrho, k, \vartheta)$ implies that

$$\Re\left[\varrho\frac{zF'(z)}{F(z)} - (\varrho-1)\right] > k\Re\left[\varrho\frac{zF'(z)}{F(z)} - \varrho\right] + \vartheta$$

or that

$$\Re\left(\frac{zF'(z)}{F(z)}\right) > \frac{\vartheta - 1 + \varrho(1-k)}{\varrho(1-k)}.$$

Hence,

$$G \in S^*\left(\frac{\vartheta - 1 + \varrho(1-k)}{\varrho(1-k)}\right).$$

Let

$$p(z) = \frac{\frac{zF'(z)}{F(z)} - \frac{\vartheta-1+\varrho(1-k)}{\varrho(1-k)}}{\frac{1-\vartheta}{\varrho(1-k)}} = 1 + c_1 z + c_2 z^2 + \cdots$$

Then, by virtue of (9) and (11), we have

$$a_2 = \frac{(1-\vartheta)}{\varrho(1-k)A_2(\lambda,\gamma,m)} c_1$$

and

$$a_3 = \frac{(1-\vartheta)}{2\varrho(1-k)A_2(\lambda,\gamma,m)}\left[c_2 + \frac{1-\vartheta}{\varrho(1-k)}c_1^2\right].$$

Therefore, we obtain

$$a_3 - \mu a_2^2 = \frac{(1-\vartheta)}{2\varrho(1-k)A_3(\lambda,\gamma,m)}\left[c_2 - \frac{1-\vartheta}{\varrho(1-k)}c_1^2\right] - \mu\frac{(1-\vartheta)^2}{\varrho^2(1-k)^2A_2^2(\lambda,\gamma,m)}c_1^2$$

$$= \frac{(1-\vartheta)}{2\varrho(1-k)A_3(\lambda,\gamma,m)}\left[c_2 - \frac{1-\vartheta}{\varrho(1-k)}c_1^2\Big(2\mu\frac{A_3(\lambda,\gamma,m)}{A_1^2(\lambda,\gamma,m)} - 1\Big)\right].$$

We write

$$a_3 - \mu a_2^2 = \frac{(1-\vartheta)}{2\varrho(1-k)A_3(\lambda,\gamma,m)}(c_2 - \rho c_1^2),$$

where,

$$\rho = \frac{(1-\vartheta)}{\varrho(1-k)}\left[2\mu\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)} - 1\right].$$

Our result follows by application of the above lemma. Denote

$$\xi = \frac{\vartheta - 1 + \varrho(1-k)}{\varrho(1-k)}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if

$$F(z) = \frac{z}{(1-e^{i\theta}z)^{2(1-\xi)}}, \quad (\theta \in R).$$

When $\sigma_1 < \mu < \sigma_2$, the equality holds true if and only if

$$F(z) = \frac{z}{(1-e^{i\theta}z^2)^{(1-\xi)}}, \quad (\theta \in R).$$

If $\mu = \sigma_1$ , then the equality holds true if and only if

$$F(z) = \left[\frac{z}{(1-e^{i\theta}z)^{2(1-\xi)}}\right]^{\frac{1+\delta}{2}}\left[\frac{z}{(1+e^{i\theta}z)^{2(1-\xi)}}\right]^{\frac{1-\delta}{2}} = \frac{z}{\left[(1-e^{i\theta}z)^{1+\delta}(1+e^{i\theta}z)^{1-\delta}\right]^{1-\xi}}, \quad (0 \le \delta \le 1,\ \theta \in R).$$

Finally, when $\mu = \sigma_2$, the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that equality holds true in the case of $\mu = \sigma_2$. □

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