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LOCAL PROPERTIES OF ENTROPY FOR FINITE FAMILY OF FUNCTIONS

Ryszard J. Pawlak

Faculty of Mathematics and Computer Science, University of Łódź, Łódź, POLAND

ABSTRACT. In this paper we consider the issues of local entropy for a finite family of generators (that generates the semigroup). Our main aim is to show that any continuous function can be approximated by s-chaotic family of generators.

1. Introduction and preliminaries

Many papers investigate situations where a repetitive action (function, multifunction, etc.) operates in a given space creating an autonomous dynamical system or a sequence of various functions, creating a nonautonomous dynamical system. In this context, the entropies of these systems are considered (e.g., [1], [5]). However, an interesting situation is also when several actions appear (a set of several functions that generate a semigroup). Then, entropy depends on the simultaneous actions of various of these functions ([3], [4]).

In [6], there were introduced three kinds of entropy for finite families of functions (so-called *set of generators*). In a natural way, they gave three methods of measuring chaos for that family. The analysis of the specific examples shows that even considering points accumulating suitable kind of entropy does not always properly illustrate a behavior of a family of functions in the individual parts of the domain (so-called *phase space*). The aim of this paper is to consider a possibility of pointing out the sets with the "dominating kind of entropy", joining entropy points with these sets and showing that any continuous function with a finite entropy can be approximated by a family of functions with separeted sets of entropy points (*s-chaotic functions*).

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Keywords: entropy, semigroup, set of generators, entropy of I,II,III type, (periodic) dynamical system, $\mathcal{A}_J$-invariant set, J-entropy point ($J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$), s-chaotic set of generators.

We will restrict our considerations to functions mapping the unit interval $[0,1]$ into itself.

Throughout the paper, we will use the standard notations ([6]). The symbols $\mathbb{N}$ ($\mathbb{R}$) will stand for the set of all positive integers (real numbers). The interior of a set $A \subset X$ will be denoted by $\operatorname{int}(A)$. If $X \subset [0,1]$ and $x \in X$, $\varepsilon > 0$, then $B_X(x,\varepsilon) = (x-\varepsilon, x+\varepsilon) \cap X$. To simplify the calculations, the base of logarithms is chosen as 2 (we will write $\log x$ instead of $\log_2 x$).

For any function f and a set $A \subset [0,1]$ the symbol $f \restriction A$ will mean the restriction of f to A. Moreover, if f is invertible, then the inverse of f will be denoted by f^{-1}. We will denote by id_A the identity function on A. The cardinality of a set A will be denoted by $\#(A)$.

By a **nonautonomous dynamical system** on a metric space (X,d) we will mean any sequence of functions $(f_{1,\infty}) = \{f_i\}_{i\in\mathbb{N}}$ such that $f_i : X \to X$. We will also write $(f_{1.\infty}) = (f_1, f_2, f_3, \ldots)$. We shall use the symbol $(f_{n,\infty})$ ($n \in \mathbb{N}$) to represent the tail $\{f_i\}_{i=n}^{\infty}$ of the sequence $(f_{1,\infty})$. If $f_i = f$ for $i \in \mathbb{N}$, then we call the **system autonomous** and denote it by (f).

Let us consider

$$(f^n_{1,\infty}) = \{f^n_{(i-1)\cdot n+1}\}_{i\in\mathbb{N}} \qquad \text{for} \qquad n \in \mathbb{N},$$

where

$$f_i^n = f_{n+i-1} \circ f_{n+i-2} \circ \cdots \circ f_{i+1} \circ f_i \quad \text{for any} \quad i, n \in \mathbb{N}.$$

Moreover, let $f_i^0 = \mathrm{id}_X$. If we consider a function $f : X \to X$, then for any $n \in \mathbb{N}$ the symbol f^n will denote the nth iteration of f, i.e., $f^n = f \circ f^{n-1}$ and $f^0 = \mathrm{id}_X$. Moreover, we will write f_0^n instead of id_X^n for $n \in \mathbb{N}$.

We say that a dynamical system $(f_{1,\infty})$ is **periodic with a period** n if $f_k = f_{k \bmod n}$ if $k \bmod n \neq 0$ and $f_k = f_n$ otherwise. The smallest period of a periodic dynamical system is called its **prime period**. If n is a period of $(f_{1,\infty}) = \{f_i\}_{i\in\mathbb{N}}$, then we sometimes write $(f_1, \ldots, f_n)$ instead of $(f_{1,\infty})$.

Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$, where $f_i : [0,1] \to [0,1]$ ($i = 1, \ldots, k$) are continuous functions and $f_0 = \mathrm{id}_{[0,1]}$. The family $\mathcal{A}$ will be called a **set of generators**. Following [3], we assume that the identity function always belongs to $\mathcal{A}$ (and sometimes we denote it by f_0). We say that a semigroup G is **finitely generated** if there exists a set of generators $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ such that

$$G = \bigcup_{n=1}^{\infty} \mathcal{A}^n,$$

where $\mathcal{A}^n = \big\{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} : i_1, \ldots, i_n \in \{0, 1, \ldots, k\}\big\}$. Notice that if $m > n$, then $\mathcal{A}^n \subset \mathcal{A}^m$. To denote that a semigroup G is finitely generated by $\mathcal{A}$, we will use the symbol $G(\mathcal{A})$. Let $X \subset [0,1]$. Then, we will use the notation

$$\mathcal{A} \restriction X = \{f_0 \restriction X, f_1 \restriction X, \ldots, f_k \restriction X\}.$$

Obviously, if the restrictions of several functions to the set X are the same, then they are represented by only one function in the family $\mathcal{A} \upharpoonright X$.

We call a set X invariant for some function f (for $\mathcal{A}$) if $f(X) \subset X$ (for each $f \in \mathcal{A}$).

For a fixed $\varepsilon > 0$, a function f is **ε-approximated by a set of generators** $\mathcal{A} = \{f_0, f_1, \dots, f_k\}$ if $\varrho(f, f_i) < \varepsilon$ for $i = 1, 2, \dots, k$, where ϱ denotes the metric of the uniform convergence (i.e., $\varrho(f, f_i) = \sup\{| f(x) - f_i(x) | : x \in [0,1]\}$, for $i = 1, 2, \dots, k$).

We will also consider the family $T^\infty(\mathcal{A}) = \{g_{1,\infty} = (g_1, g_2, \dots) : g_i \in \mathcal{A}\}$. By $T_p^\infty(\mathcal{A})$ we will denote a subfamily of $T^\infty(\mathcal{A})$ consisting of periodic systems with a prime period not greater than p and consisting of functions belonging to $\mathcal{A} \setminus \{f_0\}$.

Let $\mathcal{A} = \{f_0, f_1, \dots, f_k\}$ be a set of generators. Let $n \in \mathbb{N}$, $\varepsilon > 0$ and $Y \subset [0,1]$. We say that $Z \subset Y$ is **(n, ε)-separated by $G(\mathcal{A})$ in Y** ([3]) if for any two distinct points $p, q \in Z$ there exists a function $g \in \mathcal{A}^n$ such that $| g(p) - g(q) | > \varepsilon$. Let $s(n, \varepsilon, G(\mathcal{A}), Y)$ denote the maximal cardinality of (n,ε)-separated set by $G(\mathcal{A})$ in Y. Then, the **entropy of a semigroup $G(\mathcal{A})$ on Y** is the number ([3]):

$$h\big(G(\mathcal{A}), Y\big) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s\big(n, \varepsilon, G(\mathcal{A}), Y\big).$$

If $Y = [0,1]$, then we will briefly write $h\big(G(\mathcal{A})\big)$ instead of $h\big(G(\mathcal{A}), [0,1]\big)$.

Let $(g_{1,\infty})$ be a dynamical system, $n \in \mathbb{N}$, $\varepsilon > 0$ and $Y \subset [0,1]$. A set $E \subset Y$ is called **(n, ε)-separated for $g_{1,\infty}$ in Y** if for any two distinct points $x, y \in E$ there exists $j \in \{0, \dots, n-1\}$ such that $| g_1^j(x) - g_1^j(y) | > \varepsilon$. Let $s_n(g_{1,\infty}, Y, \varepsilon)$ denote the maximal cardinality of (n, ε)-separated set for $g_{1,\infty}$ in Y. Then, the **entropy of a system $g_{1,\infty}$ on Y** is the number ([5]):

$$h(g_{1,\infty}, Y) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s_n(g_{1,\infty}, Y, \varepsilon).$$

If $Y = [0,1]$, then we will briefly write $h(g_{1,\infty})$ instead of $h(g_{1,\infty}, [0,1])$.

1.1. Three types of entropy for a set of generators

Let $\mathcal{A} = \{f_0, f_1, \dots, f_k\}$ be a set of generators and $Y \subset [0,1]$. Now, we define three types of entropy of $\mathcal{A}$ on Y following [6].

- **Entropy of the first type (briefly I-type entropy) of a set of generators $\mathcal{A}$ on a set** Y is the number

$$h_{\mathrm{I}}(\mathcal{A}, Y) = h\big(G(\mathcal{A}), Y\big).$$

- **Entropy of the second type (briefly II-type entropy) of a set of generators $\mathcal{A}$ on a set** Y is the number

$$h_{\mathrm{II}}(\mathcal{A}, Y) = \sup\{h(g_{1,\infty}, Y) : g_{1,\infty} \in T^\infty(\mathcal{A})\}.$$

- **Entropy of the third type (briefly III-type entropy) of a set of generators $\mathcal{A}$ on a set** Y is the number

$$h_{\mathrm{III}}(\mathcal{A},Y)=\sup\{h(g_{1,\infty},Y):g_{1,\infty}\in T_k^{\infty}(\mathcal{A})\}.$$

If $Y=[0,1]$, then we briefly write $h_J(\mathcal{A})$ instead of $h_J(\mathcal{A},[0,1])$ for $J\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$. It is worth noting that if X is an invariant set (for $\mathcal{A}$), then

$$h_J(\mathcal{A}\restriction X)=h_J(\mathcal{A},X)\quad \text{for}\ \ J\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}.$$

Let us note some facts that are important for our considerations ([4], [6]).

Proposition 1. *Let $\mathcal{A}=\{f_0,f_1,\dots,f_k\}$ be a set of generators and let $Y_1\subset Y_2\subset[0,1]$. Then,*

$$h_J(\mathcal{A},Y_1)\le h_J(\mathcal{A},Y_2)\quad \textit{for}\ \ J\in\{I,II,III\}.$$

Proposition 2. *Let $\mathcal{A}=\{f_0,f_1,\dots,f_k\}$ be a set of generators and $Y\subset[0,1]$. Then,*

$$h_{\mathrm{III}}(\mathcal{A},Y)\le h_{II}(\mathcal{A},Y)\le h_I(\mathcal{A},Y).$$

Proposition 3. *Let $Y\subset[0,1]$. If $\mathcal{A}=\{f_0,f_1\}$, then*

$$h_{\mathrm{III}}(\mathcal{A},Y)=h_{\mathrm{II}}(\mathcal{A},Y)=h_{\mathrm{I}}(\mathcal{A},Y)=h(f_1,Y).$$

For any set of generators $\mathcal{A}$ and invariant set (for $\mathcal{A}$) $X\subset[0,1]$ we will consider three types of functions $E_J^{\mathcal{A}\restriction X}:X\to\mathbb{R}_0\cup\{\infty\}$, where $J\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$, defined in the following way ([6])

$$E_J^{\mathcal{A}\restriction X}(x)=\inf\bigl\{h_J\bigl(\mathcal{A},B_X(x,\varepsilon)\bigr):\varepsilon>0\bigr\}.$$

The fact presented below is a simple modification of Proposition 5 from [6].

Proposition 4. *For any set of generators $\mathcal{A}$, any invariant set (for $\mathcal{A}$) $X\subset[0,1]$ and a point $x\in X$ we have*

$$0\le E_{\mathrm{III}}^{\mathcal{A}\restriction X}(x)\le E_{\mathrm{II}}^{\mathcal{A}\restriction X}(x)\le E_{\mathrm{I}}^{\mathcal{A}\restriction X}(x).$$

2. Local aspects of entropy and chaos — s-chaotic functions

Distinguishing three kinds of sets below will allow us to analyze individual types of entropy on some parts of the domain.

- We say that a set $X\subset[0,1]$ is $\mathcal{A}_{\mathrm{I}}$**-invariant** if it is nonempty, closed, invariant (for $\mathcal{A}$), and $h_{\mathrm{I}}(\mathcal{A}\restriction X)>0=h_{\mathrm{II}}(\mathcal{A}\restriction X)$.
- We say that a set $X\subset[0,1]$ is $\mathcal{A}_{\mathrm{II}}$**-invariant** if it is nonempty, closed and invariant (for $\mathcal{A}$), and either $h_{\mathrm{I}}(\mathcal{A})>h_{\mathrm{II}}(\mathcal{A}\restriction X)>0=h_{\mathrm{III}}(\mathcal{A}\restriction X)$ whenever $h_{\mathrm{II}}(\mathcal{A}\restriction X)<\infty$ or $0=h_{\mathrm{III}}(\mathcal{A}\restriction X)$ whenever $h_{\mathrm{II}}(\mathcal{A}\restriction X)=\infty$.

- We say that a set $X \subset [0,1]$ is $\mathcal{A}_{\mathrm{III}}$-**invariant** if it is nonempty, closed and invariant (for $\mathcal{A}$), and either $h_{\mathrm{II}}(\mathcal{A}) > h_{\mathrm{III}}(\mathcal{A} \upharpoonright X) > 0$ whenever $h_{\mathrm{III}}(\mathcal{A} \upharpoonright X) < \infty$ or $h_{\mathrm{III}}(\mathcal{A} \upharpoonright X) = \infty$.

In order to reach a deep understanding, let us note two observations.

Let $\mathcal{A} = \{f_0, f_1\}$, where f_1 is a continuous function, then

- if $h(f_1) = \infty$, then $[0,1]$ is an $\mathcal{A}_{\mathrm{III}}$-invariant set and it is neither an $\mathcal{A}_{\mathrm{I}}$-invariant set nor an $\mathcal{A}_{\mathrm{II}}$-invariant set;
- if $h(f_1) = 0$, then no set $X \subset [0,1]$ is $\mathcal{A}_J$-invariant set for any $J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.

For further considerations in order to construct a definition of an s-chaotic family, the following remark is important.

Remark 5. There exist a set of generators $\mathcal{A}$; $J, K \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and sets X_1, X_2 such that X_1 is $\mathcal{A}_J$-invariant, X_2 is $\mathcal{A}_K$-invariant and $\mathrm{int}(X_1 \cap X_2) \neq \emptyset$ (see comment after Remark 8).

Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ be a set of generators and $x_0 \in [0,1]$. We say that a point x_0 is a **J-entropy point of** $\mathcal{A}$ ($J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$) if there exists an $\mathcal{A}_J$-invariant nondegenerate interval X containing x_0 (x_0 may be end-point of X) and such that $E_J^{\mathcal{A} \upharpoonright X}(x_0) = h_J(\mathcal{A} \upharpoonright X)$ (we will say that x_0 is connected to X). Let us adopt the following denotations:

$\mathrm{Ent}_J(\mathcal{A}, X)$**:** a set of all J-entropy points of $\mathcal{A}$ connected to $\mathcal{A}_J$-invariant nondegenerate interval X ($J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$).

$\mathrm{Ent}_J(\mathcal{A})$**:** a set of all J-entropy points of $\mathcal{A}$ ($J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$) connected to some invariant nondegenerate interval X .

Lemma 6 ([5]). *Let Z be a compact space, $(f_{1,\infty})$ a nonautonomous dynamical system on Z, $Y \subset Z$, and $Y = \bigcup_{i=1}^{k} K_i$. Then,*

$$h(f_{1,\infty}, Y) = \max\{h(f_{1,\infty}, K_1), \ldots, h(f_{1,\infty}, K_k)\}.$$

Theorem 7. *Let $\mathcal{A} = \{f_0, f_1, \ldots, f_k\}$ be a set of generators. For any $\mathcal{A}_J$-invariant interval $[a,b]$ there exists $x_0 \in \mathrm{Ent}_J(\mathcal{A}, [a,b])$, for $J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$.*

Proof. The proof is analogous to that of Theorem 8 from the paper [6]. The proof of $\mathrm{Ent}_{\mathrm{I}}(\mathcal{A}, [a,b])$ can be rewritten almost word for word. Consequently, we will only present the proof for $\mathrm{Ent}_{\mathrm{II}}(\mathcal{A}, [a,b])$ the existence of which has been only signaled in [6]. The proof in the case of $\mathrm{Ent}_{\mathrm{III}}(\mathcal{A}, [a,b])$ can be carried out in the same way as for $\mathrm{Ent}_{\mathrm{II}}(\mathcal{A}, [a,b])$ or in a slightly modified version from [6].

Put $h_{\mathrm{II}}(\mathcal{A}, [a,b]) = \alpha > 0$. So, for each $m \in \mathbb{N}$ there exists dynamical system $\big(g_{1,\infty}^{[m]}\big) \in T^{\infty}(\mathcal{A} \upharpoonright [a,b])$ (this means that $\big(g_{1,\infty}^{[m]}\big)$ is a system composed of continuous functions mapping $[a,b]$ into itself) such that $h\big(g_{1,\infty}^{[m]}\big) > \alpha - \frac{1}{m}$.

Let us divide $[a,b]$ into disjoint intervals

$$[a,b]=\left[a,a+\frac{b-a}{m}\right]\cup\left(a+\frac{b-a}{m},a+\frac{2(b-a)}{m}\right]\cup\cdots\cup\left(a+\frac{(m-1)(b-a)}{m},b\right].$$

According to Lemma 6, there exists interval

$$I_m\in\left\{\left[a,a+\frac{b-a}{m}\right],\ \left(a+\frac{b-a}{m},\ a\frac{2(b-a)}{m}\right],\ \ldots,\ \left(a+\frac{(m-1)(b-a)}{m},b\right]\right\}$$

such that $h\big(g^{[m]}_{1,\infty},I_m\big)>\alpha-\frac{1}{m}$.

Consider a sequence $\{a_m\}$ consisting of points a_m selected from I_m, for $m\in\mathbb{N}$, and let $\{a_{m_j}\}$ be a subsequence converging to some point x_0. What is left is to show that

$$x_0\in \mathrm{Ent}_{\mathrm{II}}(\mathcal{A},[a,b]). \tag{1}$$

So, let $\eta>0$. We have $h\big(g^{[m_j]}_{1,\infty},I_{m_j}\big)>\alpha-\frac{1}{m_j}$ and diameter $\mathrm{diam}(I_{m_j})=\frac{1}{m_j}$. Therefore, there exists $j_0\in\mathbb{N}$ such that $I_{m_j}\subset(x_0-\eta,x_0+\eta)\cap[a,b]$, for any $j\geq j_0$. To simplify the argument, we may assume, without loss of generality, that the above inclusion occurs for each $j\in\mathbb{N}$. Because of Proposition 1, we have

$$\alpha\geq h_{\mathrm{II}}\big(\mathcal{A},(x_0-\eta,x_0+\eta)\cap[a,b]\big)\geq h_{\mathrm{II}}(\mathcal{A},I_{m_j})\geq h\big(g^{[m_j]}_{1,\infty},I_{m_j}\big)\geq\alpha-\frac{1}{m_j},\quad \text{for } j\in\mathbb{N}.$$

It follows that $h_{\mathrm{II}}\big(\mathcal{A},(x_0-\eta,x_0+\eta)\cap[a,b]\big)=\alpha$, which proves (1), and the proof of the Theorem is complete. □

The following remark is important in the context of introducing the notion of an s-chaotic family in the next part of the paper.

Remark 8. There exist a set of generators $\mathcal{A}$ and $J,K\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$ such that $J\neq K$ and $\mathrm{Ent}_J(\mathcal{A})\cap\mathrm{Ent}_K(\mathcal{A})\neq\emptyset$.

The examples that justify the above remark are quite complex. For this reason, only the sketch for J=II and K=III will be presented below. Let us start by dividing interval $[\frac{1}{2},1]$ into equal parts by points $a=\frac{5}{8},b=\frac{6}{8},c=\frac{7}{8}$. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of points such that

$$0<a_1<b_1<a_2<b_2<\cdots<\frac{1}{2}\quad\text{and}\quad\lim_{n\to\infty}a_n=\frac{1}{2}.$$

Now, we will define a continuous function f_1. Put

$$f_1(x)=x\quad\text{for}\quad x\notin\bigcup_{n=1}^{\infty}[a_n,b_n].$$

On each interval $[a_n,b_n]$ let f_1 be defined in such a way that $f_1([a_n,b_n])=[a_n,b_n]$ and $h(f_1,[a_n,b_n])\geq\log n$ $(n=1,2,\ldots)$.

Before defining a function f_2, we consider an auxiliary function $g\colon [\frac{1}{2}, a] \to [\frac{1}{2}, a]$. Let $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ be sequences of points and d be a point such that

$$a > d > d_1 > c_1 > d_2 > c_2 > \cdots > \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} c_n = \frac{1}{2}.$$

Then, the function g will be defined in the following way: $g(\frac{1}{2}) = \frac{1}{2} = g(a)$ and $g(d) = a$. On each interval $[c_n, d_n]$ let g be defined in such a way that

$$g(c_n) = c_n, \quad g(d_n) = d_n, \quad g([c_n, d_n]) = [c_n, d_n] \quad \text{and} \quad h(g, [c_n, d_n]) \geq \log n$$

(obviously, we require that g is continuous on $[c_n, d_n]$), for $n = 1, 2, \ldots$ Moreover, let g be linear on each of the intervals $[d, a]$, $[d_1, d]$ $[d_{n+1}, c_n]$ $(n = 1, 2, \ldots)$). Finally, we define a function $f_2\colon [0, 1] \to [0, 1]$ in the following way: $f_2(x) = b$, for $x \notin [\frac{1}{2}, a]$ and $f_2(x) = g(x) + \frac{1}{4}$ for $x \in [\frac{1}{2}, a]$.

Let us define one more continuous function f_3 in the following way: $f_3(x) = \frac{1}{2}$ for $x \in [0, a]$, $f_3(b) = a$, $f_3(c) = b$, $f_3(1) = c$ and f_3 is linear on each of the intervals $[a, b]$, $[b, c]$, $[c, 1]$.

Thus, we may define the generator set $\mathcal{A} = \{f_0, f_1, f_2, f_3\}$.

Put

$$X_{\mathrm{II}} = \left[\frac{1}{2}, c\right] \quad \text{and} \quad X_{\mathrm{III}} = [0, c].$$

Note that X_{II}, X_{III} are closed and invariant sets. Obviously, $\mathcal{A} \restriction X_{\mathrm{II}} = \{f_0 \restriction X_{\mathrm{II}}, f_2 \restriction X_{\mathrm{II}}, f_3 \restriction X_{\mathrm{II}}\}$ and $\mathcal{A} \restriction X_{\mathrm{III}} = \{f_0 \restriction X_{\mathrm{III}}, f_1 \restriction X_{\mathrm{III}}, f_2 \restriction X_{\mathrm{III}}, f_3 \restriction X_{\mathrm{III}}\}$. Since $h(f_1 \restriction X_{\mathrm{III}}) = +\infty$, then X_{III} is $\mathcal{A}_{\mathrm{III}}$ - invariant set and consequently,

$$\frac{1}{2} \text{ is III— entropy point of } \mathcal{A} \text{ connected to } X_{\mathrm{III}}\,. \tag{2}$$

One can prove (after detailed calculations) that $h_{\mathrm{III}}(\mathcal{A} \restriction X_{\mathrm{II}}) = 0$ and $h_{\mathrm{II}}(\mathcal{A} \restriction X_{\mathrm{II}}) = +\infty$, which gives that X_{II} is $\mathcal{A}_{\mathrm{II}}$-invariant set. It follows immediately that

$$\frac{1}{2} \text{ is II—entropy point of } \mathcal{A} \text{ connected to } X_{\mathrm{II}}. \tag{3}$$

According to (2) and (3), we have $\mathrm{Ent}_{\mathrm{II}}(\mathcal{A}) \cap \mathrm{Ent}_{\mathrm{III}}(\mathcal{A}) \neq \emptyset$.

In [6], it was pointed out that in many considerations *entropy is treated as some kind of measure of chaos*. Previous considerations led to distinguishing functions having separated sets of entropy (chaos) points. This fact will be emphasized in the next definition.

We say that a family of generators $\mathcal{A}$ is **s-chaotic** (*separately chaotic*) if there exist three nonempty, pairwise disjoint sets

$$Y_{\mathrm{I}} \subset \mathrm{Ent}_{\mathrm{I}}(\mathcal{A}), Y_{\mathrm{II}} \subset \mathrm{Ent}_{\mathrm{II}}(\mathcal{A}), Y_{\mathrm{III}} \subset \mathrm{Ent}_{\mathrm{III}}(\mathcal{A}).$$

Examples of s-chaotic families of generators are easy to establish from the proof of Theorem 14. We now give a quite simple example of a family $\mathcal{A}$ that is not s-chaotic.

Let us consider a family $\mathcal{A} = \{f_0, f_1, \dots, f_k\}$ such that

$$f_1(x) = \frac{3}{4} \quad \text{for } x \in \left[0, \frac{3}{4}\right],$$

$$f_1\left(\left[\frac{3}{4}, 1\right]\right) \subset \left(\left[\frac{3}{4}, 1\right]\right) \quad \text{and} \quad h\left(f_1, \left(\left[\frac{3}{4}, 1\right]\right)\right) = \infty.$$

Let us also assume that f_i are continuous functions such that $f_i([0, \frac{3}{4}]) \subset [\frac{3}{4}, 1]$ and $f_i(x) = x$, for $x \in [\frac{3}{4}, 1]$ and $i \in \{2, \dots, k\}$. This is clearly sufficient to note that all invariant sets for $\mathcal{A}$ are contained in $[\frac{3}{4}, 1]$, and if $Y \subset [\frac{3}{4}, 1]$, then $\mathcal{A} \restriction Y = \{f_0 \restriction Y, f_1 \restriction Y\}$. Of course, Y is neither $\mathcal{A}_{\mathrm{I}}$- nor $\mathcal{A}_{\mathrm{II}}$-invariant.

Before proving the main theorem (Theorem 14), we will present some useful lemmas (sometimes supplemented with comments facilitating their use).

Lemma 9 ([5])**.** *Let $(f_{1,\infty})$ be a dynamical system on compact set $X \subset [0, 1]$. Then, for every $1 \le i \le j < \infty$ the following inequality $h(f_{i,\infty}) \le h(f_{j,\infty})$ occurs.*

It is immediately clear that the above lemma cannot be formulated for entropy on a proper subset of the domain (suitable example has been presented in the picture entitled Figure 2 of [5]). However, it is easy to check that in the case when $A \subset X$ is an invariant set (for each function creating $(f_{1,\infty})$), then it is possible.

Lemma 10 ([5])**.** *Let $(f_{1,\infty})$ be a periodic dynamical system with period n on compact set $X \subset [0, 1]$ and $Y \subset X$. Then, $h(f_1^n, Y) = n \cdot h(f_{1,\infty}, Y)$.*

Lemma 11 ([5])**.** *Let $(f_{1,\infty})$ be a dynamical system consisting of (not necessarily strictly) monotone maps. Then, $h(f_{1,\infty}) = 0$.*

Lemma 12 ([2])**.** *Let $f\colon [0, 1] \to [0, 1]$ be a piecewise monotone map with finitely many pieces of monotonicity, and let c_n be the minimum number of pieces of monotonicity of f^n. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log c_n = h(f).$$

The above lemma requires some comment to be taking into account, among others, Example 4.2.6 from [2]. First of all, this lemma can be written by replacing $[0, 1]$ with any closed nondegenerate interval. Further, if we have function $f\colon [a, b] \to [a, b]$ whose graph goes from a to b or vice versa n times, then $h(f) = \log n$.

Lemma 13. *Let $X \subset [0, 1]$ be a compact and nondegenerate interval and let $(f_{1,\infty})$ be a dynamical system on X such that $h(f_{1,\infty}) > 0$. Moreover, let $\{k_i\}$ be the sequence of all positive integers such that $f_{k_i} \neq \mathrm{id}_X$, $i = 1, 2, \dots$ Put $(g_{1,\infty}) = \{f_{k_i}\}_{i\in\mathbb{N}}$. Then,*

$$h(f_{1,\infty}) \le h(g_{1,\infty}). \tag{4}$$

P r o o f. Note that (according to Lemmas 9 and 11) (k_i) is an infinite sequence.

One can assume that $f_1 \neq \mathrm{id}_X$. Otherwise, we may consider

$$i_* = \min\{i\colon f_i \neq \mathrm{id}_X\};$$

according to Lemma 9, we have $h(f_{1,\infty}) \leq h(f_{i_*,\infty})$ and it would be sufficient to show the inequality $h(f_{i_*,\infty}) \leq h(g_{1,\infty})$ instead of (4).

We will prove that

$$\begin{aligned} &\text{if } E \text{ is an } (n,\varepsilon)\text{−separated set for } (f_{1,\infty}),\\ &\text{then } E \text{ is an } (n,\varepsilon)\text{−separated set for } (g_{1,\infty}),\\ &\qquad\text{for each } n \in \mathbb{N} \text{ and } \varepsilon > 0. \end{aligned} \tag{5}$$

Let $x, y \in E$ be distinct points. It means that there exists $j \in \{0,\dots,n-1\}$ such that $\mid f_1^j(x) - f_1^j(y) \mid > \varepsilon$. If $f_i = \mathrm{id}_X$ for each $0 \leq i \leq j$, then we would have

$$\mid x - y \mid = \mid g_1^0(x) - g_1^0(y) \mid = \mid f_1^j(x) - f_1^j(y) \mid > \varepsilon.$$

Otherwise, we put $t_0 = \max\{j \in \{1,\dots,n-1\}\colon f_j \neq \mathrm{id}_X\}$ and let i_0 be a positive integer such that $k_{i_0} = t_0$ (it is easily seen that $k_{i_0} < n$). Then, $g_1^{k_{i_0}}(z) = f_1^{t_0}(z) = f_1^j(z)$ for each $z \in X$ and, consequently, $\mid g_1^{k_{i_0}}(x) - g_1^{k_{i_0}}(y) \mid > \varepsilon$. This completes the proof of (5). By (5), it is obvious that $h(f_{1,\infty}) \leq h(g_{1,\infty})$ ☐

Let us note that in the above lemma we cannot replace the inequality with an equality. For example, consider a function $f\colon [0,1] \to [0,1]$ with positive but finite entropy. Let us define the functions $f_{2i-1} = \mathrm{id}_{[0,1]}$ and $f_{2i} = f$ for $i = 1, 2, \dots$ In this way we have defined a nonautonomous periodic dynamical system $(f_{1,\infty}) = \{f_i\}_{i=1}^{\infty}$ with period 2. It is obvious that (f_1^2) may by considered as an autonomous system (f) and it can be interpreted as dynamical system $(g_{1,\infty}) = (f_{k_i})$ from the assumption of Lemma 13. Consequently, according to Lemma 10, we have

$$h(f_{1,\infty}) = \frac{1}{2} \cdot h(f_1^2) = \frac{1}{2} \cdot h(f) = \frac{1}{2} \cdot h(g_{1,\infty}).$$

Theorem 14. *For any continuous function f with a finite entropy and any $\varepsilon > 0$, a function f is ε-approximated by an s-chaotic family of generators $\mathcal{A}$ such that $h_{\mathrm{III}}(\mathcal{A}) > h(f)$.*

P r o o f. Fix $\varepsilon > 0$, a fixed point $x_0 \in [0,1]$ of f and a positive integer $\tau > h(f)$.

Obviously, there exists $\delta \in (0, \frac{\varepsilon}{2})$ such that

$$f\big((x_0-\delta, x_0+\delta) \cap [0,1]\big) \subset \Big(f(x_0) - \frac{\varepsilon}{2}, f(x_0) + \frac{\varepsilon}{2}\Big) = \Big(x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\Big).$$

Let $[a_0, b_0] \subset (x_0 - \delta, x_0 + \delta) \cap [0,1]$ $(a_0 < b_0)$. Then,

$$f(x) \in \Big(x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\Big), \quad \text{for } x \in [a_0, b_0]. \tag{6}$$

Next, let us fix six points $a_0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < b_0$.

We will now construct the sets of functions (family of generators) specific to each of the intervals $[a_i, b_i]$ for $i = 1, 2, 3$.

As a first step of the construction, we define some functions connected with the interval $[a_1, b_1]$. First, we choose two points $a_1^1, a_1^2 \in (a_1, b_1)$ such that $a_1^1 < a_1^2$ and next, we divide $[a_1^1, a_1^2]$ into equal parts by points

$$a_1^1 = p_1 < p_2 < \cdots < p_{2^{\tau+3}} = a_1^2.$$

Let us denote by β the length of each of these intervals ($\beta = p_{i+1} - p_i$, $i = 1, 2, \ldots, 2^{\tau+3} - 1$).

This will allow us to define $2^{\tau+2}$ functions $f_i\colon [0,1] \to [0,1]$ $(i = 1, \ldots, 2^{\tau+2})$ as follows.

$$f_i(x) = \begin{cases} x & \text{for } x \in [b_1, b_3], \\ p_{2i-1} & \text{for } x = a_1^1, \\ p_{2i} & \text{for } x = a_1^2, \\ a_1 & \text{for } x = a_1, \\ f(x) & \text{for } x \in [0, a_0] \cup [b_0, 1], \\ \text{linear} & \text{on any interval } [a_0, a_1], [a_1, a_1^1], [a_1^1, a_1^2], [a_1^2, b_1], [b_3, b_0]. \end{cases}$$

For our considerations, it is important to note that $f_i \restriction [a_1, b_3]$ are one to one and therefore, there exist functions

$$(f_i \restriction [a_1, b_3])^{-1} \quad \text{for } i = 1, 2, \ldots, 2^{\tau+2}.$$

Taking into account that $f_i(a_1) = a_1,\ f_i(b_3) = b_3$ and

$$f_i([a_1, b_3]) = [a_1, b_3] = \big(f_i \restriction [a_1, b_3]\big)^{-1}([a_1, b_3]) \qquad i = 1, 2, \ldots, 2^{\tau+2},$$

we may consider the following functions $\mu_i\colon [0,1] \to [0,1]$

$$\mu_i(x) = \begin{cases} \big(f_i \restriction [a_1, b_3]\big)^{-1}(x) & \text{for } x \in [a_1, b_3], \\ f_i(x) & \text{for } x \notin [a_1, b_3], \end{cases} \quad \text{for } i = 1, 2, \ldots, 2^{\tau+2}.$$

In the next step of construction of the sets of generators, we shall consider the interval $[a_2, b_2]$. First, we divide this interval into three equal subintervals by points

$$a_2 = a_2^1 < a_2^2 < a_2^3 < a_2^4 = b_2.$$

Continuing this procedure, let us divide the interval $[a_2^1, a_2^2]$ into $(2^{\tau+1})^3$ equal subintervals by points

$$a_2^1 = q_1 < q_2 < \cdots < q_{(2^{\tau+1})^3} < q_{(2^{\tau+1})^3+1} = a_2^2.$$

Using these divisions, we will define two functions $g_1, g_2\colon [0,1] \to [0,1]$ in the following way.

$$g_1(x) = \begin{cases} x & \text{for } x \in [a_1, b_1] \cup [a_3, b_3], \\ a_2^3 & \text{for } x \in \{q_1, q_3, q_5, \dots, q_{(2^\tau+1)^3+1}\} \cup [a_2^2, b_2], \\ b_2 & \text{for } x \in \{q_2, q_4, \dots, q_{(2^\tau+1)^3}\}, \\ f(x) & \text{for } x \in [0, a_0] \cup [b_0, 1], \\ \text{linear} & \text{on any interval } [a_0,a_1],[b_1,a_2],[q_i,q_{i+1}]\big(i=1,2,\dots,(2^{\tau+1})^3\big), \\ & [b_2, a_3], [b_3, b_0]. \end{cases}$$

$$g_2(x) = \begin{cases} x & \text{for } x \in [a_1, b_1] \cup [a_3, b_3], \\ a_2 & \text{for } x \in [a_2, a_2^2], \\ a_2^3 & \text{for } x = b_2, \\ f(x) & \text{for } x \in [0, a_0] \cup [b_0, 1], \\ \text{linear} & \text{on any interval } [a_0, a_1], [b_1, a_2], [a_2^2, b_2], [b_2, a_3], [b_3, b_0]. \end{cases}$$

Let us note that $g_2(a_2^3) = a_2^2$.

Finally, let us start defining the function related to $[a_3, b_3]$. Similarly as in the previous steps, we start by dividing the interval $[a_3, b_3]$, this time, into 2^τ equal subintervals by points

$$a_3 = r_1 < r_2 < \cdots < r_{2^\tau} < r_{2^\tau+1} = b_3$$

and we define the function $\xi\colon [0,1] \to [0,1]$ by the following formula

$$\xi(x) = \begin{cases} x & \text{for } x \in [a_1, a_3], \\ f(x) & \text{for } x \in [0, a_0] \cup [b_0, 1], \\ a_3 & \text{for } x \in \{r_1, r_3, r_5, \dots, r_{2^\tau+1}\}, \\ b_3 & \text{for } x \in \{r_2, r_4, \dots, r_{2^\tau}\}, \\ \text{linear} & \text{on any interval } [a_0, a_1], [r_i, r_{i+1}] \ (i=1, 2, \dots, 2^\tau), [b_3, b_0]. \end{cases}$$

We have now finished defining all the functions which are necessary to create the generator set

$$\mathcal{A} = \{\mathrm{id}_{[0,1]}, f_1, f_2, \dots, f_{2^\tau+2}, \mu_1, \mu_2, \dots, \mu_{2^\tau+2}, g_1, g_2, \xi\}.$$

Of course, now it is necessary to prove that family $\mathcal{A}$ satisfies the conditions of the theorem.

First, let us note the obvious fact that each of the intervals $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3]$ is an $\mathcal{A}_{\mathrm{I}}, \mathcal{A}_{\mathrm{II}}, \mathcal{A}_{\mathrm{III}}$-invariant set, respectively.

From the definition of the functions belonging to $\mathcal{A}$, one can deduce that $\mathcal{A} \restriction [a_1, b_1] = \{\mathrm{id}_{[a_1,b_1]}, f_1 \restriction [a_1, b_1], f_2 \restriction [a_1, b_1], \dots, f_{2^\tau+2} \restriction [a_1, b_1], \mu_1 \restriction [a_1, b_1], \mu_2 \restriction [a_1, b_1], \dots, \mu_{2^\tau+2} \restriction [a_1, b_1]\}$.

We first prove
$$h_{\mathrm{I}}(\mathcal{A} \restriction [a_1, b_1]) \geq \tau + 2. \tag{7}$$

Taking into account the fact that the considerations in this part of the proof concern only the interval $[a_1, b_1]$, in order to increase the readability in further denotations (connected with this part of the proof), we will omit the symbol of the restriction: "$\restriction [a_1, b_1]$".

Let $\sigma \in (0, \beta)$ and let $y_0 \in (a_1^1, a_1^2)$. Set

$$Z_k = \{ (f_{s_k} \circ \cdots \circ f_{s_1})(y_0) \colon s_i \in \{1, \ldots, 2^{\tau+2}\},\ i = 1, \ldots, k \},\ k \in \mathbb{N} \setminus \{1\}.$$

We shall show that
$$\#(Z_k) = (2^{\tau+2})^k. \tag{8}$$

In order to prove (8), we first show that for each $f_{s_k} \circ \cdots \circ f_{s_1}, f_{j_k} \circ \cdots \circ f_{j_1}$, where $f_{s_i}, f_{j_i} \in \{f_1, \ldots, f_{2^{\tau+2}}\}$ for $i = 1, \ldots, k$, the following implication is obtained:

$$\begin{array}{lrcl} \text{if} & f_{s_k} \circ \cdots \circ f_{s_1} & \neq & f_{j_k} \circ \cdots \circ f_{j_1}, \\ \text{then} & (f_{s_k} \circ \cdots \circ f_{s_1})(y_0) & \neq & (f_{j_k} \circ \cdots \circ f_{j_1})(y_0) \end{array} \tag{9}$$

(note that $f_{s_k} \circ \cdots \circ f_{s_1} \neq f_{j_k} \circ \cdots \circ f_{j_1}$ iff $f_{s_k} \restriction [a_1, b_1] \circ \cdots \circ f_{s_1} \restriction [a_1, b_1] \neq f_{j_k} \restriction [a_1, b_1] \circ \cdots \circ f_{j_1} \restriction [a_1, b_1]$.)

Denote $k_0 = \max\{i \colon f_{s_i} \neq f_{j_i}\} \geq 1$. Define an auxiliary function t in the following way

$$t = \begin{cases} f_{s_k} \circ \cdots \circ f_{s_{k_0+1}} = f_{j_k} \circ \cdots \circ f_{j_{k_0}} & \text{if } k_0 < k, \\ \mathrm{id}_{[0,1]} & \text{if } k_0 = k. \end{cases}$$

According to the definitions of f_{s_i}, f_{j_i} $(i = 1, 2, \ldots, k)$, it follows that

$$(f_{s_{k_0}} \circ \ldots \circ f_{s_1})(y_0) \in [p_{2s_{k_0}-1}, p_{2s_{k_0}}] \text{ and } (f_{j_{k_0}} \circ \cdots \circ f_{j_1})(y_0) \in [p_{2j_{k_0}-1}, p_{2j_{k_0}}].$$

Since $f_{s_{k_0}} \neq f_{j_{k_0}}$, then $[p_{2s_{k_0}-1}, p_{2s_{k_0}}] \cap [p_{2j_{k_0}-1}, p_{2j_{k_0}}] = \emptyset$ which enables us to write
$$(f_{s_{k_0}} \circ \cdots \circ f_{s_1})(y_0) \neq (f_{j_{k_0}} \circ \cdots \circ f_{j_1})(y_0).$$

On account of the fact that $t \restriction [a_1, b_1]$ is one to one, we have

$$\begin{aligned} &(f_{s_k} \circ \cdots \circ f_{s_1})(y_0) = \\ &\qquad (t \circ f_{s_{k_0}} \circ \cdots \circ f_{s_1})(y_0) \neq (t \circ f_{j_{k_0}} \circ \cdots \circ f_{j_1})(y_0) = \\ &\qquad\qquad (f_{j_k} \circ \cdots \circ f_{j_1})(y_0). \end{aligned}$$

The proofs of (9) and thus of (8) are completed.

Continuing the proof of (7), we will show that

$$Z_k \text{ is a } (k,\sigma)-\text{separated set.} \tag{10}$$

Indeed, let $x = (f_{s_k} \circ \cdots \circ f_{s_1})(y_0)$, $y = (f_{j_k} \circ \cdots \circ f_{j_1})(y_0) \in Z$ be the points such that $x \neq y$, where $f_{s_i}, f_{j_i} \in \{f_1, \ldots, f_{2^{\tau+2}}\}$ for $i = 1, \ldots, k$. Hence, $(f_{s_k} \circ \cdots \circ f_{s_1})(y_0) \neq (f_{j_k} \circ \cdots \circ f_{j_1})(y_0)$. Put $\gamma = \max\{i \colon f_{s_i} \neq f_{j_i}\}$. We now apply previous considerations again to obtain

$$(f_{s_\gamma} \circ \cdots \circ f_{s_1})(y_0) \in [p_{2s_\gamma - 1}, p_{2s_\gamma}] \text{ and } (f_{j_\gamma} \circ \cdots \circ f_{j_1})(y_0) \in [p_{2j_\gamma - 1}, p_{2j_\gamma}].$$

There is no loss of generality in assuming that $\gamma < k$ (if $\gamma = k$, the simplification of considerations would be obvious). Let us consider a function

$$\varphi = \mathrm{id}^{\gamma}_{[0,1]} \circ \mu_{s_{\gamma+1}} \circ \cdots \circ \mu_{s_k}.$$

Obviously, $\varphi \in \mathcal{A}^k$ and $\mu_{s_{\gamma+1}} = \mu_{j_{\gamma+1}}, \ldots, \mu_{s_k} = \mu_{j_k}$. One can check immediately that

$$\begin{aligned}\varphi(x) &= \mathrm{id}^{\gamma}_{[0,1]} \circ \mu_{s_{\gamma+1}} \circ \cdots \circ \mu_{s_k}\big((f_{s_k} \circ \cdots \circ f_{s_1})(y_0)\big)\\ &= (f_{s_\gamma} \circ \cdots \circ f_{s_1})(y_0) \in [p_{2s_\gamma - 1}, p_{2s_\gamma}].\end{aligned}$$

The same reasoning leads us to the conclusion

$$\varphi(y) \in [p_{2j_\gamma - 1}, p_{2j_\gamma}].$$

On account of $f_{s_\gamma} \neq f_{j_\gamma}$, we have $[p_{2s_\gamma - 1}, p_{2s_\gamma}] \cap [p_{2j_\gamma - 1}, p_{2j_\gamma}] = \emptyset$. Hence, $\mid \varphi(x) - \varphi(y) \mid > \sigma$, which proves (10). Consequently, according to (8), we may conclude that $s\big(k, \sigma, G(\mathcal{A} \restriction [a_1, b_1])\big) \geq \#(Z_k) = (2^{\tau+2})^k$, which allows us to calculate

$$h_{\mathrm{I}}(\mathcal{A} \restriction [a_1, b_1]) = \lim_{\sigma \to 0+} \limsup_{k \to \infty} \frac{1}{k} \log\ s\big(k, \sigma, G(\mathcal{A} \restriction [a_1, b_1])\big) \geq \tau + 2.$$

This completes the proof of (7).

At the end of this part of the proof, let us now note that $\mathcal{A} \restriction [a_1, b_1]$ consists of monotone functions. Applying Lemma 11, it is clear that $h_{\mathrm{II}}(\mathcal{A} \restriction [a_1, b_1]) = 0$ which means that $[a_1, b_1]$ is an $\mathcal{A}_{\mathrm{I}}$-invariant set.

Let $Y_{\mathrm{I}} = \mathrm{Ent}_{\mathrm{I}}(\mathcal{A} \restriction [a_1, b_1]) \subset \mathrm{Ent}_{\mathrm{I}}(\mathcal{A})$. Theorem 7 guarantees that $Y_{\mathrm{I}} \neq \emptyset$.

Let us now turn to the considerations related to the interval $[a_2, b_2]$. It is easily seen that $\mathcal{A} \restriction [a_2, b_2] = \{\mathrm{id}_{[a_2,b_2]}, g_1 \restriction [a_2, b_2], g_2 \restriction [a_2, b_2]\}$.

Similarly to the previous part of the proof, taking into account the fact that the present considerations (in this part of the proof) concern only the interval $[a_2, b_2]$, we will omit the symbol of the restriction $'' \restriction [a_2, b_2]''$ connected with functions g_1 and g_2.

Let us start by noticing that $h(g_1^2) = 0 = h(g_1)$, which according to Lemma 10 implies that $h(g_1) = 0 = h(g_2^2)$. Since $(g_1 \circ g_2)^2, (g_2 \circ g_1)^2$ are constant functions, then (Lemma 10) $h(g_1 \circ g_2) = 0 = h(g_2 \circ g_1)$. This observation enables us to the conclusion that there is no periodic dynamical system with period not greater than 2 with positive entropy. This gives $h_{\mathrm{III}}(\mathcal{A} \restriction [a_2, b_2])$=0.

Next, we will consider entropy of successive dynamical systems consisting of funtions belonging to $\mathcal{A} \restriction [a_2, b_2]$ with positive entropy. Lemma 13 indicates that it suffices to consider systems that do not contain $\mathrm{id}_{[a_2,b_2]}$.

Let us start with periodic system $(g^{[1]}_{1,\infty}) = (g_1, g_2, g_2)$. Clearly, (g_1^3) is an autonomous system and (using Lemma 10 and Lemma 12)

$$h(g^{[1]}_{1,\infty}) = \frac{1}{3} h(g_1^3) = \frac{1}{3} \log 2^{3\cdot(\tau+1)} = \tau + 1 > 0.$$

So, we conclude that $h_{\mathrm{II}}(\mathcal{A} \restriction [a_2, b_2]) > 0$.

For $\mathcal{A}_{\mathrm{II}}$-invariance we shall show that

$$h_{\mathrm{I}}(\mathcal{A}) > h_{\mathrm{II}}(\mathcal{A} \restriction [a_2, b_2]). \tag{11}$$

For this purpose, we will consider entropy of all possible dynamical systems with positive entropy consisting of funtions belonging to $\mathcal{A} \restriction [a_2, b_2]$. We will prove that in no case these entropies are greater than $\tau+1$. Lemma 13 indicates that it suffices to consider systems that do not contain $\mathrm{id}_{[a_2,b_2]}$ (in other words, we will only consider systems consisting of functions g_1 and g_2).

The assumption of considering only systems with positive entropy eliminates all the systems for which the composition of the initial functions is a constant function.

Let us start with the case when g_1 is the first function in our system. Taking into account the previous statements, g_1 cannot be a second function in this system. Thus, at the beginning of this system, there are functions g_1 and g_2. Now, the function g_1 cannot appear either, so the beginning is composed of a sequence (g_1, g_2, g_2). The fourth function in this system cannot be g_2, so the beginning will be a sequence (g_1, g_2, g_2, g_1). Continuing this reasoning, it is not difficult to notice that we will get the periodic system considered earlier $(g^{[1]}_{1,\infty})$ with entropy equal to $\tau + 1$.

Generally, every dynamical system composed of functions g_1 and g_2 with positive entropy in which the first function is g_1 has an entropy equal to $\tau + 1$.

Let us consider the case when g_2 is the first function in the dynamical system. In this case, we should consider two options:

- **The second function in our system is $\mathbf{g_1}$.**
  Then, the third function must be g_2 and the next g_2. The initial sequence of our system is of the form (g_2, g_1, g_2, g_2). Then, the following functions will have to appear one by one: $g_1, g_2, g_2, g_1, \dots$ In this way we create the periodic dynamical system $\big(g^{[2]}_{1,\infty}\big) = (g_2, g_1, g_2)$. Let us use Lemma 9 to simplify the calculations. Note that $\big(g^{[2]}_{2,\infty}\big) = \big(g^{[1]}_{1,\infty}\big)$, which gives $h\big(g^{[2]}_{2,\infty}\big) \le h\big(g^{[1]}_{1,\infty}\big) = \tau + 1$.

- **The second function in our system is $\mathbf{g_2}$.**
  In this case, we start with the sequence (g_2, g_2). Continuing this way as before, we get a dynamical system $\big(g^{[3]}_{1,\infty}\big) = (g_2, g_2, g_1, g_2, g_2, g_1, g_2, g_2, \ldots)$. And again, it can be seen that $\big(g^{[3]}_{3,\infty}\big) = \big(g^{[1]}_{1,\infty}\big)$. Consequently, using Lemma 9, we will have $h\big(g^{[3]}_{1,\infty}\big) \leq \tau + 1$.

The above considerations cover all possibilities of dynamical systems formed by the functions g_1 and g_2 with positive entropy. Moreover, we have $h_{\mathrm{I}}(\mathcal{A}) \geq h_{\mathrm{I}}(\mathcal{A} \restriction [a_1, b_1]) > \tau + 1 = h_{\mathrm{II}}(\mathcal{A} \restriction [a_2, b_2])$ which completes the proof of (11).

Let $Y_{\mathrm{II}} = \mathrm{Ent}_{\mathrm{II}}(\mathcal{A} \restriction [a_2, b_2]) \subset \mathrm{Ent}_{\mathrm{II}}(\mathcal{A})$. Theorem 7 guarantees that $Y_{\mathrm{II}} \neq \emptyset$.

Finally, let us now turn to the considerations related to the interval $[a_3, b_3]$. One can observe that $\mathcal{A} \restriction [a_3, b_3] = \{\mathrm{id}_{[a_3,b_3]}, \xi \restriction [a_3, b_3]\}$. An easy computation (using Lemma 12) shows that $h_{\mathrm{III}}(\mathcal{A} \restriction [a_3, b_3]) = \tau$. Hence, $h_{\mathrm{II}}(\mathcal{A}) \geq h_{\mathrm{II}}(\mathcal{A} \restriction [a_2, b_2]) > \tau = h_{\mathrm{III}}(\mathcal{A} \restriction [a_3, b_3])$. So, $[a_3, b_3]$ is an $\mathcal{A}_{\mathrm{III}}$-invariant set. Moreover, we have $\emptyset \neq Y_{\mathrm{III}} = \mathrm{Ent}_{\mathrm{III}}(\mathcal{A} \restriction [a_2, b_2]) \subset \mathrm{Ent}_{\mathrm{III}}(\mathcal{A})$.

From the disjointness of $[a_1, b_1], [a_2, b_2], [a_3, b_3]$ we conclude that also Y_{I}, $Y_{\mathrm{II}}, Y_{\mathrm{III}}$ are pairwise disjoint.

In concluding the consideration connected with entropy, let us note that

$$h_{\mathrm{III}}(\mathcal{A}) \geq h_{\mathrm{III}}(\mathcal{A} \restriction [a_3, b_3]) = \tau > h(f).$$

What is left is to show that $\varrho(f, \psi) < \varepsilon$, for each $\psi \in \mathcal{A} \setminus \mathrm{id}_{[0,1]}$. For this purpose, applying (6), it is sufficient to observe that

$$\{u \in [0, 1] \colon \psi(u) \neq f(u)\} \subset [a_0, b_0]. \qquad \square$$

After Remark 5, we signaled that *entropy is treated as some kind of measure of chaos*. For this reason, the demand that $h_{\mathrm{III}}(\mathcal{A}) > h(f)$ in the above theorem is very important. Note, however, that it is not difficult to see that every continuous function f (including function with infinite entropy) is a uniform limit of functions with finite entropy (piecewise monotone maps with finitely many pieces of monotonicity). The entropies of these functions converge to the entropy of function f (entropy regarded as a function mapping space of all continuous functions into $\mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function). Then, we can apply Theorem 14 to each of the previously mentioned functions having a finite entropy, and we get an ε-approximation by an s-chaotic family of generators. Of course, we will not receive the important condition

$$h_{\mathrm{III}}(\mathcal{A}) > h(f).$$

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Faculty of Mathematics and
Computer Science,
University of Łódź
Banacha 22,
PL-90-238 Łódź
POLAND
E-mail: ryszard.pawlak@wmii.uni.lodz.pl