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A FIXED POINT APPROACH TO THE HYERS-ULAM-RASSIAS STABILITY PROBLEM OF PEXIDERIZED FUNCTIONAL EQUATION IN MODULAR SPACES

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ABSTRACT. In this paper, we consider pexiderized functional equations for studying their Hyers-Ulam-Rassias stability. This stability has been studied for a variety of mathematical structures. Our framework of discussion is a modular space. We adopt a fixed-point approach to the problem in which we use a generalized contraction mapping principle in modular spaces. The result is illustrated with an example.

1. Introduction

The study of stabilities of functional equations is an active area of research in mathematics. The kind of stability for which the results are established here is Hyers-Ulam-Rassias stability. This stability problem was first raised by Ulam [36] and thereafter it was further generalized in the works of Hyers [5] and Rassias [39]. It is a general concept which is applicable in diverse frameworks of mathematics like those in problems of differential equations [25, 32, 37], fixed points [20], isometrics [42], etc. Particularly for the functional equations, the Hyers-Ulam-Rassias stability has been considered in a good number of papers in different types of spaces [2, 3, 7, 10, 15, 21, 26, 27, 30, 33–35, 38, 43–45]. Several books like [6, 13, 28, 29, 31, 40] provide a comprehensive account of development of this line of research.

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Our present work is in the framework of modular spaces [12, 17]. A modular is a function defined on a linear space. It defines a corresponding space which is known as modular space. It has a general structure and several studies in other domains of functional analysis have been very legitimately extended to this space. For an in detail study and for the motivations behind the development of this important concept of mathematics, we refer to the references [23,41].

The study of functional equations in modular spaces was initiated in the work of W. M. Kozlowski [41] in 1988 following which several other works on this line of research were published in modular spaces [1, 11, 18, 22].

Recently in 2014, G. Sadeghi [9] by applying the fixed point methods examined Hyers-Ulam-Rassias stability of the generalized Jensen functional equation in the frame work of modular spaces with Fatou property [18] satisfying Δ_2- conditions.

We prove generalized Hyers-Ulam-Rassias stability for an additive pexiderized functional equation

$$f(x+y) = g(x) + h(y) \tag{1.1}$$

which is a generalization of Jensen functional equation [16].

The problem of Hyers-Ulam-Rassias stability in its most general formalism seeks the answer to the question whether an approximation of a mathematical object is possible from a class of entities in case the concerned mathematical object has an approximate behavior like the entities belonging to that class. Our results also indicate how such approximations are possible for functional equations in modular spaces. In many of the problems considered in modular spaces, Δ_2- condition has been used [9, 11, 19, 24]. In those works, it is pivotal to the proofs of the results established therein. We do not use this condition in our proof. For this reason our proofs are more complicated. It is a remarkable feature of our work. Our approach to this problem of stability is through a fixed point methodology for which we use a contraction mapping theorem appearing in [1].

2. Preliminaries

The following is a definition of pexiderized additive functional equation [14]. A mapping $f : R \to R$ is said to be an additive form if

$$f(x) = ax \quad \text{for all } x, a \in R.$$

If X and Y are assumed to be a real vector space and a Banach space, respectively, then for a mapping $f : X \to Y$, consider

$$f(x+y) = f(x) + f(y) \cdots \tag{1}$$

known as a Cauchy functional equation. Any solution of (1) is termed as an additive mapping. Particularly, if $X = Y = R$, the additive form $f(x) = ax$ is a solution of (1). The form

$$f(x+y) = g(x) + h(y) \cdots \tag{2}$$

is called a pexiderized additive functional equation.

In this section, we recall some definitions and results concerning modular spaces.

Definition 2.1. Let X be a vector space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $\rho : X \to [0, \infty]$ is called a *modular* is for arbitrary $x, y \in X$

i) $\rho(x) = 0$ if and only if $x = 0$,
ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
iii) $\rho(z) \le \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.
iii′) If iii) is replaced by $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$, then we say that ρ is a convex modular. The corresponding *modular space*, denoted by X_ρ, is then defined by

$$X_\rho := \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

The modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$||X||_\rho := \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\}.$$

Remark 2.2.

i) For a fixed $x \in X_\rho$, the valuation [8] $\gamma \in K \to \rho(\gamma x)$ is increasing.
ii) $\rho(x) \le \delta\rho\big((1/\delta)x\big)$ for all $x \in X_\rho$, provided that ρ is a convex modular and $0 < \delta \le 1$.
iii) Every norm defined on X is a modular on X. In general, the modular ρ does not behave as norm or distance because it is not sub additive [17].

Definition 2.3.

Let X_ρ be a modular space and let $\{x_n\}$ be a sequence in X_ρ. Then,

i) $\{x_n\}$ is ρ-convergent to a point $x \in X_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.
ii) $\{x_n\}$ is called ρ-Cauchy if for any $\epsilon > 0$ one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$.

ii) A subset $K \subset X_\rho$ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

iv) The modular ρ has the Fatou property if $\rho(x) \leq \lim_{n\to\infty} \inf \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x.

v) A modular ρ is said to satisfy Δ_2-condition if there exists $\kappa \geq 2$ such that

$$\rho(2x) \leq \kappa\rho(x) \qquad \text{for all } \; x \in X_\rho.$$

Definition 2.4. Given a modular space X_ρ, a nonempty subset $C \subset X_\rho$, and a mapping $T : C \to C$. The orbit of T around a point $x \in X_\rho$ is the set

$$\mathbb{O}(x) := \{x, Tx, T^2x, \dots\}.$$

The quantity $\delta_\rho(x) := \sup\{\rho(u-v) : u, v \in \mathbb{O}(x)\}$ is then associated and is called the *orbit diameter* of T at x. In particular, if $\delta_\rho(x) < \infty$, one says that T has a bounded orbit at x.

Definition 2.5. Let ρ be a modular defined on a vector space X. Let $C \subset X_\rho$ be nonempty. A mapping $T : C \to C$ is called ρ-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\rho\big(T(x) - T(y)\big) \leq L\rho(x-y), \ \forall\, x, y \in C.$$

If $L < 1$, then T is called ρ-contraction.

The first result is the modular version of the Banach Contraction Principle.

Theorem 2.6 ([1]). *Assume X_ρ is ρ-complete. Let C be a nonempty ρ-closed subset of X_ρ. Let $T : C \to C$ be a ρ-contraction mapping. Then, T has a fixed point z if and only if T has a ρ-bounded orbit. Moreover, if $\rho(x-z) < \infty$, then $\{T^n(x)\}$ ρ-converges to z, for any $x \in C$.*

The above theorem allows us to conclude that if z_1 and z_2 are two fixed points of T such that $\rho(z_1 - z_2) < \infty$, then we have $z_1 = z_2$. In particular, if C is ρ-bounded, then T has a unique fixed point in C.

3. The generalized Hyers-Ulam stability of (1.1) in modular spaces

Throughout this paper, X is considered to be linear space and X_ρ-complete convex modular space and also the convex modular ρ has the Fatou property.

Theorem 3.1. *Let $f : X \to X_\rho$ is a mapping which satisfied the functional inequality*

$$\rho\big(4f(x+y) - 4g(x) - 4h(y)\big) \le \phi(x,y) \tag{3.1}$$

for all $x, y \in X$, where $\phi : X^2 \to [0,\infty)$ is a mapping satisfying

$$\phi(3x,3y) \le 3L\phi(x,y) \tag{3.2}$$

for all $x, y \in X$ and some L with $0 < L < 1$.
Then, there exists a unique additive mapping $A : X \to X_\rho$ such that

$$\begin{aligned} \rho\left(A(x) - \frac{f(x)-f(0)}{4}\right) &\le \frac{1}{4.3(1-L)}\psi(x,x), \\ \rho\left(A(x) - \frac{g(x)-g(0)}{4}\right) &\le \frac{1}{4.3(1-L)}\psi(x,x) + \frac{1}{4^2}\phi(x,0), \end{aligned} \tag{3.3}$$

and,

$$\rho\left(A(x) - \frac{h(x)-h(0)}{4}\right) \le \frac{1}{4.3(1-L)}\psi(x,x) + \frac{1}{4^2}\phi(0,x),$$

for all $x \in X$, where $\psi(x,x) = \psi_1(x,x) + \psi_2(x,x)$ and

$$\begin{aligned} \psi_1(x,x) &= \tfrac{1}{4}\phi\left(\tfrac{x}{2}, \tfrac{-x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{-x}{2}, \tfrac{x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{x}{2}, \tfrac{x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{-x}{2}, \tfrac{-x}{2}\right), \\ \psi_2(x,x) &= \tfrac{1}{4}\phi\left(\tfrac{-x}{2}, \tfrac{3x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{3x}{2}, \tfrac{-x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{-x}{2}, \tfrac{-x}{2}\right) + \tfrac{1}{4}\phi\left(\tfrac{3x}{2}, \tfrac{3x}{2}\right). \end{aligned}$$

Proof. Consider the set $M = \{g : X \to X_\rho : g(0) = 0\}$ and define a mapping $\tilde{\rho}$ on M by

$$\tilde{\rho}(g) = \inf\big\{c > 0 : \rho\big(g(x)\big) \le c\psi(x,x)\big\}, \qquad g \in M.$$

It is easy to prove that $\tilde{\rho}$ is complete convex modular on M [9]. Also, consider a mapping $J : M_{\tilde{\rho}} \to M_{\tilde{\rho}}$ and define

$$Jg(x) = \frac{1}{3}g(3x) \qquad \text{for all } g \in M_{\tilde{\rho}} \quad \text{and} \quad x \in X.$$

We now prove that J is a $\tilde{\rho}$-strict contractive. Let $g, h \in M_{\tilde{\rho}}$ and let $c \in [0,\infty)$ be a constant with $\tilde{\rho}(g-h) \le c$. Then,

$$\rho\big(g(x) - h(x)\big) \le c\psi(x,x) \qquad \text{for all } x \in X.$$

Now,

$$\begin{aligned} \rho\big(Jg(x) - Jh(x)\big) &= \rho\left(\frac{1}{3}g(3x) - \frac{1}{3}h(3x)\right) \\ &\le \frac{1}{3}\rho\big(g(3x) - h(3x)\big) \\ &\le \frac{1}{3}c\psi(3x,3x) \\ &\le cL\psi(x,x) \qquad \text{for all } x \in X. \end{aligned}$$

Therefore,
$$\tilde{\rho}(Jg - Jh) \le cL.$$
Hence,
$$\tilde{\rho}(Jg - Jh) \le L\tilde{\rho}(g - h) \qquad \text{for all } g, h \in M_{\tilde{\rho}}.$$
That is, J is a $\tilde{\rho}$-strict contractive.

Now, we prove
$$\delta_{\tilde{\rho}} = \sup\big\{\tilde{\rho}\big(J^n(f) - J^m(f)\big) : m, n \in \mathbb{N}\big\} < \infty.$$

Putting $x = \frac{x}{2}$ and $y = \frac{y}{2}$ in (3.1), we get
$$\rho\left(4f\Big(\frac{x+y}{2}\Big) - 4g\Big(\frac{x}{2}\Big) - 4h\Big(\frac{y}{2}\Big)\right) \le \phi\Big(\frac{x}{2}, \frac{y}{2}\Big). \tag{3.4}$$
Now,
$$\begin{aligned}
&\rho\bigg(2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y)\bigg)\\
&= \rho\Bigg(\frac{1}{4}\bigg(4f\Big(\frac{x+y}{2}\Big) - 4g\Big(\frac{x}{2}\Big) - 4h\Big(\frac{y}{2}\Big)\bigg) + \frac{1}{4}\bigg(4f\Big(\frac{x+y}{2}\Big) - 4g\Big(\frac{y}{2}\Big) - 4h\Big(\frac{x}{2}\Big)\bigg)\\
&\quad - \frac{1}{4}\bigg(4f(x) - 4g\Big(\frac{x}{2}\Big) - 4h\Big(\frac{x}{2}\Big)\bigg) - \frac{1}{4}\bigg(4f(y) - 4g\Big(\frac{y}{2}\Big) - 4h\Big(\frac{y}{2}\Big)\bigg)\Bigg)\\
&\le \frac{1}{4}\phi\Big(\frac{x}{2}, \frac{y}{2}\Big) + \frac{1}{4}\phi\Big(\frac{y}{2}, \frac{x}{2}\Big) + \frac{1}{4}\phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \frac{1}{4}\phi\Big(\frac{y}{2}, \frac{y}{2}\Big).
\end{aligned} \tag{3.5}$$

Again, putting $y = -x$ in (3.5) and setting $F(x) = f(x) - f(0)$, we have
$$\begin{aligned}
&\rho\big(2f(0) - f(x) - f(-x)\big)\\
&\le \frac{1}{4}\phi\Big(\frac{x}{2}, \frac{-x}{2}\Big) + \frac{1}{4}\phi\Big(\frac{-x}{2}, \frac{x}{2}\Big) + \frac{1}{4}\phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \frac{1}{4}\phi\Big(\frac{-x}{2}, \frac{-x}{2}\Big),\\
&\rho\big(-F(x) - F(-x)\big)\\
&\le \frac{1}{4}\phi\left(\frac{x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{x}{2}\right) + \frac{1}{4}\phi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{-x}{2}\right),
\end{aligned}$$
that is,
$$\begin{aligned}
&\rho\big(F(x) + F(-x)\big)\\
&\le \frac{1}{4}\phi\left(\frac{x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{x}{2}\right) + \frac{1}{4}\phi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{-x}{2}\right)\\
&= \psi_1(x, x).
\end{aligned} \tag{3.6}$$

Again, putting $y = 3x$ and $x = -x$ in (3.5)

$$\rho\big(2f(x) - f(-x) - f(3x)\big)$$
$$\leq \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{3x}{2}\right) + \frac{1}{4}\phi\left(\frac{3x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{3x}{2}, \frac{3x}{2}\right),$$

or,

$$\rho\big(2F(x) - F(-x) - F(3x)\big)$$
$$\leq \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{3x}{2}\right) + \frac{1}{4}\phi\left(\frac{3x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{-x}{2}, \frac{-x}{2}\right) + \frac{1}{4}\phi\left(\frac{3x}{2}, \frac{3x}{2}\right) \quad (3.7)$$
$$= \psi_2\,(x,\, x).$$

Now,

$$\rho\left(\frac{3F(x)}{2} - \frac{F(3x)}{2}\right) = \rho\left(\frac{F(3x)}{2} - \frac{3F(x)}{2}\right)$$
$$= \rho\left(\frac{1}{2}\big(2F(x) - F(-x) - F(3x)\big) + \frac{1}{2}\big(F(x) + F(-x)\big)\right) \quad (3.8)$$
$$= \frac{1}{2}\psi_2\,(x,\, x)\frac{1}{2}\psi_1\,(x,\, x) = \frac{1}{2}\psi\,(x,\, x).$$

Also,

$$\rho\left(\frac{F(x)}{2} - \frac{F(3x)}{2\cdot 3}\right) = \rho\left(\frac{1}{3}\left(\frac{3F(x)}{2} - \frac{F(3x)}{2}\right)\right)$$
$$= \frac{1}{2\cdot 3}\psi\,(x,\, x). \quad (3.9)$$

Therefore,

$$\rho\left(\frac{F(x)}{2} - \frac{F(3^2\,x)}{2\cdot 3^2}\right)$$
$$= \rho\left(\frac{1}{3}\left(\frac{3F(x)}{2} - \frac{F(3x)}{2}\right) + \frac{1}{3}\left(\frac{F(3x)}{2} - \frac{F(3^2\,x)}{2\cdot 3}\right)\right)$$
$$\leq \frac{1}{2\cdot 3}\psi(x,\, x) + \frac{1}{2\cdot 3^2}\psi(3x,\, 3x),$$

by using (3.8) and (3.9),

$$= \frac{1}{2}\sum_{i=1}^{2}\frac{1}{(3)^i}\psi(3^{i-1}x, 3^{i-1}x) \qquad \text{for all } x \in X. \quad (3.10)$$

Therefore, by the method of mathematical induction, we have

$$\rho\left(\frac{F(3^n x)}{2\cdot 3^n} - \frac{F(x)}{2}\right) \le \frac{1}{2}\sum_{i=1}^{n}\frac{1}{3^i}\,\psi(3^{i-1}x,\,3^{i-1}x) \tag{3.11}$$

Hence, we have

$$\begin{aligned}\rho\left(\frac{F(3^n x)}{2\cdot 3^n} - \frac{F(x)}{2}\right) &\le \frac{1}{2}\sum_{i=1}^{n}\frac{3^{i-1}\psi(x,x)}{3^i}L^{i-1} \qquad \text{[by (3.2)]}\\ &\le \frac{\psi(x,x)}{2\cdot 3}\sum_{i=1}^{n}L^{i-1} \qquad (3.12)\\ &\le \frac{\psi(x,x)}{2\cdot 3(1-L)},\end{aligned}$$

as $0 < L < 1$ for all $x \in X$ and $n \in \mathbb{N}$.

Thus, it follows from 3.12 that for any $n, m \in N$, we have

$$\begin{aligned}&\rho\left(\frac{F(3^n x)}{4.3^n} - \frac{F(3^m x)}{4.3^m}\right)\\ &\le \frac{1}{2}\rho\left(\frac{F(3^n x)}{2\cdot 3^n} - \frac{F(x)}{2}\right) + \frac{1}{2}\rho\left(\frac{F(3^m x)}{2\cdot 3^m} - \frac{F(x)}{2}\right)\rho\left(\frac{F(3^n x)}{4.3^n} - \frac{F(3^m x)}{4.3^m}\right)\\ &\le \frac{1}{2}\times\frac{\psi(x,x)}{2\cdot 3(1-L)} + \frac{1}{2}\times\frac{\psi(x,x)}{2\cdot 3(1-L)}\rho\left(\frac{F(3^n x)}{4.3^n} - \frac{F(3^m x)}{4.3^m}\right)\\ &\le \frac{\psi(x,x)}{2\cdot 3(1-L)} \qquad \text{for all } x\in X \qquad \text{[by (3.12)]}.\end{aligned}$$

This implies that $\tilde{\rho}\big(J^n(\frac{1}{4}F) - J^m(\frac{1}{4}F)\big) \le \frac{1}{2\cdot 3(1-L)} < \infty$ for all $m, n \in N$. This shows that J has a bounded orbit at $\frac{1}{4}F$.

Therefore by an application of Theorem 2.6 we have

i) J has a fixed point $A \in M$ at $\frac{1}{4}F$ that is, $JA = A$ that is, $A(x) = \frac{1}{3}A(3x)$ for all $x \in X$.

ii) The sequence $\left\{J^n\left(\frac{1}{4}F\right)\right\}$ $\tilde{\rho}$-converges to A.

Therefore, $\lim_{n\to\infty}\rho\Big(\big(\frac{1}{4.3^n}F(3^n x)\big) - A(x)\Big) = 0$.

So, we can define

$$A(x) := \frac{1}{4}\lim_{n\to\infty}\frac{F(3^n x)}{3^n} = \frac{1}{4}\lim_{n\to\infty}\frac{f(3^n x) - f(0)}{3^n}. \tag{3.13}$$

Again, replacing x and y by $3^n x$ and $3^n y$, respectively, in (3.5), we get

$$\rho\left(\frac{1}{4.3^n}\left(2\,f\Big(3^n\Big(\frac{x+y}{2}\Big)\Big)-f(3^nx)-f(3^ny)\right)\right)$$
$$=\rho\left(\frac{1}{4.3^n}\left(2F\Big(3^n\Big(\frac{x+y}{2}\Big)\Big)-F(3^nx)-F(3^ny)\right)\right)$$
$$\leq\frac{1}{4.3^n}\psi_1(3^nx,3^ny)\leq\frac{1}{4}L^n\psi_1(x,y)\qquad\text{for all}\quad x,\,y\in X\text{ and }n\in\mathbb{N}.$$

Now, taking limit as $n\to\infty$ and using Fatou property and $0<L<1$, we get

$$2A\left(\frac{x+y}{2}\right)=A(x)+A(y).$$

Also, from (3.13) we have that $A(0)=0$. Therefore, A is an additive mapping [16]. Also, using the fact that ρ has Fatou property, from (3.12) we get

$$\rho\left(2A(x)-\frac{F(x)}{2}\right)\leq\frac{1}{2\cdot 3(1-L)}\psi(x,x)\qquad\text{for all}\ \ x\in X.$$

That is,

$$\rho\left(A(x)-\frac{F(x)}{4}\right)\leq\frac{1}{2}\rho\left(2A(x)-\frac{F(x)}{2}\right)$$
$$\leq\frac{1}{2\cdot 2\cdot 3(1-L)}\psi(x,x)\qquad\text{for all}\ \ x\in X,$$

that is,

$$\rho\left(A(x)-\frac{f(x)-f(0)}{4}\right)\leq\frac{1}{2\cdot 2\cdot 3(1-L)}\psi(x,x)\qquad\text{for all}\ \ x\in X.\tag{3.14}$$

Also, putting $y=0$ in (3.1), we have,

$$\rho\big(4f(x)-4g(x)-4h(0)\big)\leq\phi(x,0),$$

that is,

$$\rho\big(4f(x)-4g(x)-4f(0)+4g(0)\big)\leq\phi(x,0),$$

as

$$f(0)=g(0)+h(0),\qquad\rho\big(4F(x)-4G(x)\big)\leq\phi(x,0).$$

Therefore,

$$\rho\big(F(x)-G(x)\big)=\rho\left(\frac{1}{4}\big(4F(x)-4G(x)\big)\right)$$
$$\leq\frac{1}{4}\phi(x,0).$$

Now,

$$\rho\left(A(x)-\frac{G(x)}{4}\right)=\rho\left(\frac{1}{2}\left(2A(x)-\frac{F(x)}{2}\right)+\frac{1}{4}\big(F(x)-G(x)\big)\right)$$
$$\leq\frac{1}{2}\times\frac{1}{2\cdot 3(1-L)}\psi(x,x)+\frac{1}{4}\times\frac{1}{4}\phi(x,0).$$

That is,

$$\rho\left(A(x)-\frac{g(x)-g(0)}{4}\right)\leq\frac{1}{2}\times\frac{1}{2\cdot 3(1-L)}\psi(x,x)+\frac{1}{4}\times\frac{1}{4}\phi(x,0),\quad \text{for all } x\in X.$$

Similarly,

$$\rho\left(A(x)-\frac{h(x)-h(0)}{4}\right)\leq\frac{1}{2}\times\frac{1}{2\cdot 3(1-L)}\psi(x,x)+\frac{1}{4}\times\frac{1}{4}\phi(0,x),\quad \text{for all } x\in X.$$

To prove the uniqueness, let $A' : X \to X_\rho$ be another additive mapping satisfying (3.3). Then, we have

$$\rho\left(\frac{A(x)}{2}-\frac{A'(x)}{2}\right)$$
$$\leq\frac{1}{2}\rho\left(A(x)-\frac{F(x)}{4}\right)+\frac{1}{2}\rho\left(A'(x)-\frac{F(x)}{4}\right)$$
$$\leq\frac{\psi(x,x)}{4\cdot 3(1-L)}<\infty \qquad \text{for all } x\in X.$$

Also, since A and A' are two fixed points of J and $\rho\left(\frac{A(x)}{2}-\frac{A'(x)}{2}\right)<\infty$, so by Theorem 2.6 we conclude that $A(x)=A'(x)\ \forall\ x\in X$.

This completes the proof of the theorem. □

Corollary 3.2. *Let $\theta\geq 0$ and X be a normed linear space and a mapping $f : X\to X_\rho$ with $f(0)=g(0)=0$ satisfying inequality*

$$\rho(f(x+y)-g(x)-h(y))\leq\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in$ and $0\leq p<1$. Then, there exists a unique additive mapping $A : X\to X_\rho$ such that

$$\rho\left(A(x)-\frac{f(x)}{4}\right)\leq\frac{(3+3^p)\,\theta}{3\cdot 2^{p+2}(1-2^{p-1})}\|x\|^p$$

and

$$\rho\left(A(x)-\frac{g(x)}{4}\right)\leq\frac{(3+3^p)\,\theta}{3\cdot 2^{p+2}(1-2^{p-1})}\|x\|^p+\frac{\theta}{4^2}\|x\|^p\quad \textit{for all } x\in X.$$

P r o o f. Define $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ and take $L = 2^{p-1}$; the proof of the result is the same as above. □

Corollary 3.3. *Let $\epsilon \geq 0$ and X be a normed linear space and a mapping $f : X \to X_\rho$ with $f(0) = g(0) = 0$ satisfying inequality*

$$\rho\big(f(x+y) - g(x) - h(y)\big) \leq \epsilon$$

for all $x, y \in$. Then, there exists a unique additive mapping $A : X \to X_\rho$ such that

$$\rho\left(A(x) - \frac{f(x)}{4}\right) \leq \frac{\epsilon}{3} \qquad \text{and} \qquad \rho\left(A(x) - \frac{g(x)}{4}\right) \leq \frac{19}{48}\epsilon \qquad \text{for all } x \in X.$$

P r o o f. Define $\phi(x,y) = \epsilon$ for all $x, y \in X$ and take $L = \frac{1}{2}$, and the proof of the result is the same as the Theorem 3.1. □

Example 3.4. Let $(X, \|\cdot\|)$ be a normed linear space and X_ρ a ρ-complete convex modular space where $\rho(x) = \|x\|$. Define $f : X \to X_\rho$ by

$$f(x) = ax + A\|x\|x_0,\ g(x) = ax + B\|x\|x_0 \qquad \text{and} \qquad h(x) = ax + C\|x\|x_0$$

for all $x \in X$, where $A, B, C \in \mathbb{R}^+$ and x_0 is a unit vector in X. Then,

$$\rho\big(f(x+y) - g(x) - h(y)\big) \leq (A-B)\|x\| + (A-C)\|y\| \qquad \text{for all } x, y \in X.$$

Let

$$\phi(x,y) = (A-B)\|x\| + (A-C)\|y\| \qquad \text{for all } x, y \in X$$

and take $L = \frac{1}{2}$. Thus, all the conditions of Theorem 3.1 are satisfied. Then, there exists a unique additive mapping $A : X \to X_\rho$ such that

$$\rho\left(A(x) - \frac{f(x)}{4}\right) \leq \frac{1}{4}(2A - B - C)\|x\|,$$

$$\rho\left(A(x) - \frac{g(x)}{4}\right) \leq \frac{1}{4^2}(9A - 5B - 4C)\|x\|$$

and

$$\rho\left(A(x) - \frac{h(x)}{4}\right) \leq \frac{1}{4^2}(9A - 4B - 5C)\|x\| \qquad \text{for all } x \in X.$$

4. Conclusion

It is interesting to observe the behaviours of functional equations in modular spaces. The proof of the theorem presented here utilizes many concepts which are particular to modular spaces. Further, modular spaces without Δ_2- conditions have been considered here which warrants additional features in the proof itself. It is perceived that similar studies for other types of equations in modular spaces in the absence of Δ_2-conditions will be mathematically interesting. This can be taken up for future research.

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REFERENCES

[1] ABDOU, A. A. N.—KHAMSI, M. A.: *Fixed point theorems in modular vector spaces*, J. Nonlinear Sci. Appl. **10** (2017), 4046–4057.

[2] GRABIEC, A.: *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen **48** (1996), 217–235.

[3] MIRMOSTAFAEE, V.—MOSLEHIAN, M. S.: *Stability of additive mappings in non-Archimedean fuzzy normed spaces*, Fuzzy Sets and 5ystems **160** (2009), 1643–1652.

[4] KIM, C.—PARK, S. W.: *A Fixed point approach to the stability of additive-quadratic functional equations in modular spaces*, J. of The Chungcheong Math. Soc. **18** (2015), 321–330.

[5] HYERS, D. H.: *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[6] HYERS, D. H.: *Stability of functional equations in several variables*. Birkhauser, 1998.

[7] ELQORACHI, E.—RASSIAS, M. TH.: *Generalized Hyers-Ulam stability of trigonometric functional equations*, Mathematics, **6** (2018).

[8] CRIVELLI, F.: *Absolute values, valuations and completion*. Algebra in Positive Characteristic, 2008.

[9] SADEGHI, G.: *A fixed point approach to stability of functional equations in modular spaces*. Bull. Malays. Math. Sci. Soc. **37** (2014), no.2., 333–344.

[10] H. CHODA, H.—MIURA, F.—TAKAHASI, S.: *On the Hyers–Ulam stability of real continuous function valued differentiable map*, Tokyo J. Math. **24** (2001), 467–476.

[11] KIM, H.—HONG, Y.: *Approximate Cauchy-Jensen type mappings in modular spaces*, Far East J. Math. Sci. **7** (2017), 1319–1336.

[12] NAKANO, H.: *Modular on Semi-Ordered Spaces*. Tokyo, Japan, 1959.

[13] ACZEL, J.—DHOMBRES, J.: *Functional Equations in Several Variables. With Applications to Mathematics, Information Theory and to the Natural and Social Sciences*. Encyclopedia of Mathematics and Its Applications Vol. 31. Cambridge University Press, Cambridge, 1989.

[14] CHMIELINSKI, J.—TABOR, J.: On approximate solutons of the Pexider equation, Aequationes Math. **46** (1993), 143–163.

[15] WANG, J.: *Some further generalization of the Ulam-Hyers-Rassias stability of functional equations*, J. Math. Anal. Appl. **263** (2001), 406–423.

[16] PARNAMI, J. C.—VASUDEVA, H. L.: *On Jensen's functional equation*, Aequationes Math. **43** (1992), 211–218.

[17] MUSIELAK, J.: *Orlicz Spaces and Modular Spaces*. In: *Lecture Notes in Math. Vol. 1034*, Springer-Verlag, Berlin, 1983.

[18] WONGKUM, K.—CHAIPUNYA, P.—KUMAM P.: *On the generalized Ulam-Hyers-Rassias stability of quadratic mappings in modular spaces without Δ_2-conditions*, J. Funct. Spaces (2015), Article ID 461719, 6 p.

[19] WONGKUM, K.—KUMAMA, P.—CHOB, Y.—THOUNTHONGE, P.—CHAIPUNYAA, P.: *On the generalized Ulam-Hyers-Rassias stability for quartic functional equation in modular spaces*, J. Nonlinear Sci. Appl. **10** (2017), 1399–1406.

[20] CADARIU, L.—RADU, V.: *Fixed points and the stability of Jensen's functional equation*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, 1–15.

[21] CADARIU, L.—RADU, V.: *Fixed points and stability for functional equations in probabilistic metric and random normed spaces*, Fixed Point Theory and Appl. (2009), Article ID 589143, 18 p.

[22] KHAMSI, M. A.: *Quasicontraction mappings in modular spaces without Δ_2-condition*, Fixed Point Theory Appl. (2008), Article ID 916187, 6 p.

[23] KHAMSI, M. A.—KOZLOWSKI, W. M.: *Fixed Point Theory in Modular Function Spaces*. Birkhäuser/Springer, Cham, Heidelberg, New York, 2015.

[24] GHAEMI, M. B.—CHOUBIN, M.—SADEGHI, G.—GORDJI, M.: *A Fixed point approach to stability of quintic functional equations in modular spaces*, Kyungpook Math. J. **55** (2015), 313–326.

[25] ABDOLLAHPOUR, M. R.—RASSIAS, M. TH.: *Hyers-Ulam Stability of hypergeometric differential equations*, Aequationes Math. **93** (2019), 691–698.

[26] RASSIAS, M. TH.: *Solution of a functional equation problem of Steven Butler*, Octogon Math. Mag. **12** (2004), 152–153.

[27] KAYAL, N. C.—SAMANTA, T. K.—SAHA, P.—CHOUDHURY, B. S.: *A Hyers-Ulam-Rassias stability result for functional equations in intuitionistic fuzzy Banach spaces*, Iranian J. Fuzzy Systems **13**, (2016), 87–96.

[28] KANNAPPAN, PL.: *Functional Equations and Inequalities with Applications*. In: *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2009.

[29] SAHOO, P. K.—KANNAPPAN, PL.: *Introduction to Functional Equations*. CRC Press, Boca Raton, FL, 2011.

[30] SAHA, P. —SAMANTA, T. K. —MONDAL, P.—CHOUDHURY, B. S.: *Stability of two variable pexiderized quadratic functional equation in intuitionistic fuzzy Banach spaces*, Proyecciones **38**, no. 3, (2019), 447–467.

[31] JUNG, S. M.: *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*. Springer-Verlag, Berlin, 2011.

[32] JUNG, S. M.: *Hyers-Ulam stability of linear differential equations of first order II*, App. Math. Lett. **19**, (2006), 854–858.

[33] JUNG, S. M.—RASSIAS, M. TH.—MORTICI, C.: *On a functional equation of trigonometric type*, Appl. Math. Comp. **252** (2015), 294–303.

[34] JUNG, S. M.—RASSIAS, M. TH.: *A linear functional equation of third order associated to the Fibonacci numbers*, Abstr. Appl. Anal. **2014** (2014), Article ID 137468.

[35] JUNG, S. M.—POPA, D.—RASSIAS, M. TH.: *On the stability of the linear functional equation in a single variable complete metric group*, J. Global Optim. **59** (2014), 165–171.

[36] ULAM, S. M.: *Problems in Modern Mathematics*. Chapter VI, Science Editions John Wiley & Sons, Inc., New York, 1964.

[37] MIURA, T.—MIYAJIMA, S.—TIKAHASI, S.: *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Appl. **286** (2003), 136–146.

[38] SAMANTA, T. K.—MONDAL, P.—KAYAL, N. C.: *The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation in fuzzy Banach spaces*, Ann. Fuzzy Math. Inform. **6** (2013), 59–68.

[39] RASSIAS, M. TH.: *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soci. **72** (1978) no. 2, 297–300.

[40] RASSIAS, M. TH.: *Functional Equations and Inequalities*. Kluwer Academic Publishers, 2000.

[41] KOZLOWSKI, W. M.: *Modular Function Spaces*. In: *Series of Monographs and Textbooks in Pure and Applied Mathematics Vol. 122*, Marcel Dekker, New York, 1988.
[42] DONG, Y.: *On approximate isometries and application to stability of a function*, J. Math. Anal. Appl. **426** (2015), 125–137.
[43] LEE, Y. H.—JUNG, S. M.—RASSIAS, M. TH.: *On an n-dimensional mixed type additive and uadratic functional equation*, Applied Mathematics and Computation, **228** (2014), 13–16.
[44] LEE, Y. H.—JUNG, S. M.—RASSIAS, M. TH.: *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), no. 1, 43–61.
[45] GAJDA, Z.: *On stability of additive mapping*, J. Math. Math. Sci. **14** (1991), 431–434.

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