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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2021-0002
Tatra Mt. Math. Publ. **78** (2021), 9–24

ON O'MALLEY POROUSCONTINUOUS FUNCTIONS

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ABSTRACT. In 2014, J. Borsík and J. Holos defined porouscontinuous functions. Using the notion of density in O'Malley sense, we introduce new definitions of porouscontinuity, namely $\mathcal{MO}_r$ and $\mathcal{SO}_r$-continuity. Some relevant properties of these classes of functions are discussed.

1. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all positive integers, the set of all integers and the set of all real numbers, respectively. In the whole paper we will consider real functions defined on $\mathbb{R}$. The symbol $|I|$ stands for the length of an interval $I \subset \mathbb{R}$. By $f|_A$ we denote the restriction of f to $A \subset \mathbb{R}$. For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A. Then according to [1,6], the right porosity of the set A at $x \in \mathbb{R}$ is defined as

$$p^+(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the porosity of A at x is defined as

$$p(A, x) = \max\left\{p^-(A, x), p^+(A, x)\right\}.$$

2010 Mathematics Subject Classification: 54C30, 26A15, 54C08.
Keywords: porosity, porouscontinuity, porouscontinuity in O'Malley sense.

Definition 1.1 ([1], Definition 1)**.** Let $r \in [0,1)$. A point $x \in \mathbb{R}$ will be called a point of π_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) > r$.

Definition 1.2 ([1], Definition 1)**.** Let $r \in (0,1]$. A point $x \in \mathbb{R}$ will be called a point of μ_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) \geq r$.

Definition 1.3 ([1], Definition 2)**.** Let $r \in [0,1)$ and $x \in \mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ will be called:

- $\mathcal{P}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f|_A$ is continuous at x;
- $\mathcal{S}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f(A) \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$.

Let $r \in (0,1]$ and $x \in \mathbb{R}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ will be called:

- $\mathcal{M}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f|_A$ is continuous at x;
- $\mathcal{N}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f(A) \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$.

All these functions are called porouscontinuous functions.

Symbols $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ denote the set of all points at which f is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous and $\mathcal{N}_r$-continuous, respectively. In [1], the equality $\mathcal{M}_r(f) = \mathcal{N}_r(f)$ for every f and every $r \in (0,1]$ was proved. Observe that if f is continuous from the right or from the left at some x, then f is porouscontinuous at x.

Theorem 1.4 ([4], Theorem 2.1)**.** *Let $r \in [0,1)$, $x \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then, $x \in \mathcal{P}_r(f)$ if and only if $\lim_{\varepsilon \to 0^+} p\big(\mathbb{R} \setminus \{t\colon |f(x) - f(t)| < \varepsilon\}, x\big) > r$.*

For further consideration we recall main results of [1].

Theorem 1.5 ([1], Theorem 6)**.** *Let $0 < r < s < 1$ and $f\colon \mathbb{R} \to \mathbb{R}$. Then*

$$\mathcal{C}(f) \subset \mathcal{M}_1(f) \subset \mathcal{P}_s(f) \subset \mathcal{S}_s(f) \subset \mathcal{M}_s(f) \subset \mathcal{P}_r(f) \subset \mathcal{P}_0(f) \subset \mathcal{S}_0(f) \subset \mathcal{Q}(f)$$

($\mathcal{C}(f)$ and $\mathcal{Q}(f)$ denote the set of all points at which $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and quasicontinuous, respectively).

Following [1], we introduce the following notations:

- $\mathcal{C} = \{f\colon \mathcal{C}(f) = \mathbb{R}\}$, $\mathcal{Q} = \{f\colon \mathcal{Q}(f) = \mathbb{R}\}$ and $\mathcal{C}^{\pm}$ is the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that at every $x \in \mathbb{R}$, f is continuous from the left or from the right (obviously $\mathcal{C} \subsetneqq \mathcal{C}^{\pm}$),
- for $r \in (0,1]$, let $\mathcal{M}_r = \{f\colon \mathcal{M}_r(f) = \mathbb{R}\}$,
- for $r \in [0,1)$, let $\mathcal{P}_r = \{f\colon \mathcal{P}_r(f) = \mathbb{R}\}$ and $\mathcal{S}_r = \{f\colon \mathcal{S}_r(f) = \mathbb{R}\}$.

Theorem 1.6 ([1], Theorem 7). *Let $0 < r < s < 1$. Then*

$$\mathcal{C} \subset \mathcal{M}_1 \subset \mathcal{P}_s \subset \mathcal{S}_s \subset \mathcal{M}_s \subset \mathcal{P}_r \subset \mathcal{M}_r \subset \mathcal{P}_0 \subset \mathcal{S}_0 \subset \mathcal{Q}.$$

All inclusions are proper.

In [4], we can find comparison of presented classes of functions in the topology of uniform convergence which is generated by the metric

$$\varrho(f, g) = \min\big\{1, \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}\big\}$$

in the space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 1.7 ([4], Theorem 3.3).

1) $\mathcal{C}^{\pm}$ *is nowhere dense and closed in* $\mathcal{M}_1$.
2) *For* $r \in [0, 1)$, $\mathcal{M}_1$ *is nowhere dense and closed in* $\mathcal{P}_r$.
3) *For* $r \in (0, 1)$, $\mathcal{S}_r$ *is nowhere dense and closed in* $\mathcal{M}_r$.
4) *For* $0 \le r < s \le 1$, $\mathcal{M}_s$ *is nowhere dense and closed in* $\mathcal{P}_r$.
5) *For* $r \in (0, 1]$, $\mathcal{M}_r$ *is nowhere dense and closed in* $\mathcal{P}_0$.
6) $\mathcal{S}_0$ *is nowhere dense and closed in* $\mathcal{Q}$.

Theorem 1.8 ([4], Theorem 3.4). $\mathcal{P}_r$ *is a first category subset of* $\mathcal{S}_r$ *for every* $r \in [0, 1)$.

Theorem 1.9 ([4], Theorem 3.9). *There exists* $f : \mathbb{R} \to \mathbb{R}$ *from* $\mathcal{M}_1$ *which does not belong to Baire class one.*

If we take lower limit instead of upper limit and if we take maximum of lower porosities in definitions of the right porosity and the left porosity, then we obtain so-called v-porosity. Some other properties of different kinds of porosity are described in [6, 7].

2. O'Malley porouscontinuous function

In [5], R. J. O'Malley modified the notion of preponderant continuity. He showed how one can replace density of a set by another condition involving the Lebesgue measure, [2, 5]. Combining the notion of porouscontinuity defined by J. Borsík and J. Holos and using the concept of R. J. O'Malley, we define other types of porouscontinuity.

Definition 2.1. Let $r \in [0, 1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. A point x will be called a point of πO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0, \eta)$ and an open interval $(a, b) \subset A \cap \big((x - \delta, x + \delta) \setminus \{x\}\big)$ such that $\frac{b-a}{\delta} > r$.

Definition 2.2. Let $r \in (0,1]$, $x \in \mathbb{R}$, $A \subset \mathbb{R}$. A point x will be called a point of μO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0,\eta)$ and $(a,b) \subset A \cap \big((x-\delta, x+\delta) \setminus \{x\}\big)$ such that $\frac{b-a}{\delta} \geq r$.

Directly from the above definitions and from Definitions 1.1 and 1.2 we obtain the following remarks.

Remark 2.3. Let $r \in (0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of πO_r-density of A, then x is a point of μO_r-density of A.

Remark 2.4. Let $r \in [0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of π_r-density of A, then x is a point of πO_r-density of A.

Remark 2.5. Let $r \in (0,1]$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of μO_r-density of A, then x is a point of μ_r-density of A.

Definition 2.6. Let $r \in [0,1)$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{SO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of πO_r-density of a set $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$.

Definition 2.7. Let $r \in (0,1]$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{MO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of μO_r-density of a set $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$.

Symbols $\mathcal{SO}_r(f)$ and $\mathcal{MO}_r(f)$ denote the set of all points at which f is $\mathcal{SO}_r$-continuous and $\mathcal{MO}_r$-continuous, respectively, for corresponding r.

In a simple manner we may define $\mathcal{SO}_r$-continuity from the right, $\mathcal{SO}_r$--continuity from the left, $\mathcal{MO}_r$-continuity from the right, $\mathcal{MO}_r$-continuity from the left at some point for corresponding r.

Symbols $\mathcal{SO}_r^+(f)$, $\mathcal{SO}_r^-(f)$, $\mathcal{MO}_r^+(f)$ and $\mathcal{MO}_r^-(f)$ denote the set of all points at which f is $\mathcal{SO}_r$-continuous from the right, $\mathcal{SO}_r$-continuous from the left, $\mathcal{MO}_r$-continuous from the right and $\mathcal{MO}_r$-continuous from the left, respectively, for corresponding r.

It is easy to see that f is $\mathcal{SO}_r$-continuous at x if and only if f is $\mathcal{SO}_r$-continuous from the right at x or f is $\mathcal{SO}_r$-continuous from the left at x. Obviously, similar condition holds for $\mathcal{MO}_r$-continuity.

Remark 2.8. Let $r \in [0,1)$, $x \in \mathbb{R}$ and $f, g \colon \mathbb{R} \to \mathbb{R}$. If f is continuous at x and g is $\mathcal{SO}_r$-continuous at x, then $f+g$ is $\mathcal{SO}_r$-continuous at x. More generally, if f is continuous from the right at x and g is $\mathcal{SO}_r$-continuous from the right at x, then $f+g$ is $\mathcal{SO}_r$-continuous from the right at x. And similarly, if f is continuous from the left at x and g is $\mathcal{SO}_r$-continuous from the left at x, then $f+g$ is $\mathcal{SO}_r$-continuous from the left at x.

Remark 2.9. Let $r \in (0,1]$, $x \in \mathbb{R}$ and $f, g\colon \mathbb{R} \to \mathbb{R}$. If f is continuous at x and g is $\mathcal{MO}_r$-continuous at x, then $f+g$ is $\mathcal{MO}_r$-continuous at x. More generally, if f is continuous from the right (left) at x and g is $\mathcal{MO}_r$-continuous from the right (left) at x, then $f+g$ is $\mathcal{MO}_r$-continuous from the right (left) at x.

From definitions of $\mathcal{SO}_r$-continuity and $\mathcal{MO}_r$-continuity we obtain the following corollaries.

Corollary 2.10. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{MO}_1$-continuous at $x \in \mathbb{R}$ if and only if f is continuous from the right or from the left at x. In particular, $\mathcal{C}^{\pm} = \mathcal{MO}_1$.*

Corollary 2.11. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{SO}_0$-continuous at $x \in \mathbb{R}$ if and only if f is quasicontinuous at x. In particular, $\mathcal{Q} = \mathcal{SO}_0$.*

Theorem 2.12. *Let $r \in [0,1)$, $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{SO}_r$-continuous at x if and only if there exists a set $E \subset \mathbb{R}$ such that $x \in E$, x is a point of πO_r-density of E and $f|_E$ is continuous at x.*

Proof. Assume that f is $\mathcal{SO}_r$-continuous at x. Then, for $\varepsilon = 1$ and $\eta = 1$ there exist

$$\delta_1 \in (0,\eta) \quad \text{and} \quad (a_1,b_1) \subset f^{-1}\big((f(x)-1, f(x)+1)\big) \cap \big((x-\delta_1, x+\delta_1) \setminus \{x\}\big)$$

such that $\frac{b_1-a_1}{\delta_1} > r$. Next, for $\varepsilon = \frac{1}{2}$ and $\eta = \frac{1}{2}$ there exist

$$\delta_2 \in \left(0, \min\left\{\frac{1}{2}, |a_1 - x|, |x - b_1|\right\}\right)$$

and

$$(a_2,b_2) \subset f^{-1}\left(\left(f(x)-\frac{1}{2}, f(x)+\frac{1}{2}\right)\right) \cap \Big((x-\delta_2, x+\delta_2) \setminus \{x\}\Big)$$

such that $\frac{b_2-a_2}{\delta_2} > r$. Generally, for $\varepsilon = \frac{1}{n}$ and $\eta = \frac{1}{n}$ there exist

$$\delta_n \in \left(0, \min\left\{\frac{1}{n}, |a_{n-1} - x|, |x - b_{n-1}|\right\}\right)$$

and

$$(a_n,b_n) \subset f^{-1}\left(\left(f(x)-\frac{1}{n}, f(x)+\frac{1}{n}\right)\right) \cap \big((x-\delta_n, x+\delta_n) \setminus \{x\}\big)$$

such that $\frac{b_n-a_n}{\delta_n} > r$ for each $n \geq 2$. Put

$$E = \bigcup_{n=1}^{\infty} (a_n,b_n) \cup \{x\}.$$

Obviously, $f|_E$ is continuous at x. Take $\eta > 0$. Since $\lim_{n\to\infty} \delta_n = 0$, there exists n_0 such that $\delta_{n_0} < \eta$. Therefore, $(a_{n_0}, b_{n_0}) \subset E \cap \big((x-\delta_{n_0}, x+\delta_{n_0}) \setminus \{x\}\big)$ and $\frac{b_{n_0}-a_{n_0}}{\delta_{n_0}} > r$. This means that x is a point of πO_r-density of E.

Now, assume that there exists $E \subset \mathbb{R}$ such that $x \in E$, $f|_E$ is continuous at x and x is a point of πO_r-density of E. Take $\varepsilon > 0$ and $\eta > 0$. By continuity of $f|_E$ at x, there exists $\delta_\varepsilon > 0$ such that $|f(t) - f(x)| < \varepsilon$ for each $t \in E \cap (x - \delta_\varepsilon, x + \delta_\varepsilon)$. Since x is a point of πO_r-density of E, there exist $\delta \in (0, \min\{\eta, \delta_\varepsilon\})$ and an interval (a, b) such that $(a, b) \subset E \cap \big((x - \delta, x + \delta) \setminus \{x\}\big)$ and $\frac{b-a}{\delta} > r$. Therefore, f is $\mathcal{SO}_r$-continuous at x. □

In a similar way we can proof the following theorem.

Theorem 2.13. *Let $r \in (0, 1]$, $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{MO}_r$-continuous at x if and only if there exists a set $E \subset \mathbb{R}$ such that $x \in E$, x is a point of μO_r--density of E and $f|_E$ is continuous at x.*

Let

$$\mathcal{SO}_r = \{f : \mathcal{SO}_r(f) = \mathbb{R}\} \qquad \text{for } r \in [0, 1)$$

and

$$\mathcal{MO}_r = \{f : \mathcal{MO}_r(f) = \mathbb{R}\} \qquad \text{for } r \in (0, 1].$$

Directly from Remarks 2.3, 2.4 and 2.5 we obtain the following theorem.

Theorem 2.14. *Let $r \in (0, 1)$. Then, $\mathcal{S}_r \subset \mathcal{SO}_r \subset \mathcal{MO}_r \subset \mathcal{M}_r$.*

All inclusions presented in Theorem 2.14 are proper which the following examples show.

Example 2.15. Let $r \in [0, 1)$. We will construct $f : \mathbb{R} \to \mathbb{R}$ from $\mathcal{SO}_r \setminus \mathcal{S}_r$. Let $E = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where $0 < \cdots < b_{n+1} < a_n < b_n < \cdots$, $\frac{b_n - a_n}{b_n} = r + \frac{1-r}{n+1}$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by letting

$$f(x) = \begin{cases} 0, & x \in E \cup \{0\}, \\ 1, & x \notin E \cup \{0\}. \end{cases}$$

Then, f is continuous from the right or from the left at each point except 0. Therefore,

$$\mathbb{R} \setminus \{0\} \subset \mathcal{S}_r(f) \quad \text{and} \quad \mathbb{R} \setminus \{0\} \subset \mathcal{SO}_r(f).$$

Let $\eta > 0$. Since $\lim_{n\to\infty} a_n = 0$, there exists n_0 such that $b_{n_0} < \eta$. Taking $\delta = b_{n_0}$ we obtain $\frac{b_{n_0} - a_{n_0}}{\delta} = \frac{b_{n_0} - a_{n_0}}{b_{n_0}} = r + \frac{1-r}{n_0+1} > r$. Thus, $0 \in \mathcal{SO}_r(f)$. Moreover,

$$\{x : |f(x) - f(0)| < 1\} = E \cup \{0\}$$

and

$$p\,(\mathbb{R} \setminus E, 0) = \limsup_{n\to\infty} \frac{b_n - a_n}{b_n} = \limsup_{n\to\infty} \left(r + \frac{1-r}{n+1}\right) = r.$$

Therefore,

$$0 \notin \mathcal{S}_r(f) \quad \text{and} \quad f \in \mathcal{SO}_r \setminus \mathcal{S}_r.$$

Example 2.16. Let $r \in (0,1)$. We will construct $f\colon \mathbb{R} \to \mathbb{R}$ from $\mathcal{MO}_r \setminus \mathcal{SO}_r$. Let $E = \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $0 < \cdots < b_{n+1} < a_n < b_n < \cdots$, $\lim_{n\to\infty} a_n = 0$ and $\frac{b_n - a_n}{b_n} = r$ for each $n \in \mathbb{N}$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in E \cup \{0\}, \\ 1, & x \notin E \cup \{0\}. \end{cases}$$

Observe that f is continuous from the right or from the left at each point different from 0. Therefore, $\mathbb{R} \setminus \{0\} \subset \mathcal{SO}_r(f)$. Moreover, $f|_{E\cup\{0\}}$ is constant and for each $\eta > 0$ we can find n_0 such that $b_{n_0} < \eta$. Taking $\delta = b_{n_0}$, we get

$$\frac{b_{n_0} - a_{n_0}}{\delta} = \frac{b_{n_0} - a_{n_0}}{b_{n_0}} = r.$$

Hence, $0 \in \mathcal{MO}_r(f)$. Let $\varepsilon = \frac{1}{2}$ and $\eta = b_1$. Then, $\{x \in \mathbb{R}\colon |f(x) - f(0)| < \varepsilon\} = E \cup \{0\}$ and if $(a,b) \subset \{x \in \mathbb{R}\colon |f(x) - f(0)| < \varepsilon\} \cap \big((-\delta,\delta) \setminus \{0\}\big)$ for some $\delta \in (0,\eta)$, then $(a,b) \subset (a_{n_0}, b_{n_0})$ for some $n_0 \geq 1$. Thus, $\frac{b-a}{\delta} \leq \frac{b_{n_0} - a_{n_0}}{b_{n_0}} = r$. Therefore, $0 \notin \mathcal{SO}_r(f)$ and $f \in \mathcal{MO}_r \setminus \mathcal{SO}_r$.

Example 2.17. Let $r \in (0,1]$. We shall show that there exists $f\colon \mathbb{R} \to \mathbb{R}$ from $\mathcal{M}_r \setminus \mathcal{MO}_r$. Let $E = \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $0 < \cdots < b_{n+1} < a_n < b_n < \cdots$ and $\frac{b_n - a_n}{b_n} = r - \frac{r}{n+1}$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in E \cup \{0\}, \\ 1, & x \notin E \cup \{0\}. \end{cases}$$

Obviously, f is continuous from the right or from the left at each point different from 0. Hence, $\mathbb{R} \setminus \{0\} \subset \mathcal{MO}_r(f)$. Moreover, $f|_E$ is continuous at 0 and

$$p\,(\mathbb{R} \setminus E, 0) = \limsup_{n\to\infty} \frac{b_n - a_n}{b_n} = \limsup_{n\to\infty} \left(r - \frac{r}{n+1}\right) = r.$$

So, $0 \in \mathcal{M}_r(f)$. Let $\varepsilon = \frac{1}{2}$ and $\eta = b_1$. Then, $\{x \in \mathbb{R}\colon |f(x) - f(0)| < \varepsilon\} = E \cup \{0\}$ and if $(a,b) \subset \{x \in \mathbb{R}\colon |f(x) - f(0)| < \varepsilon\} \cap \big((-\delta,\delta) \setminus \{0\}\big)$ for some $\delta \in (0,\eta)$, then $(a,b) \subset (a_{n_0}, b_{n_0})$ for some $n_0 \geq 1$. Thus, $\frac{b-a}{\delta} \leq \frac{b_{n_0} - a_{n_0}}{b_{n_0}} = r - \frac{r}{1+n_0} < r$. Therefore,

$$0 \notin \mathcal{MO}_r(f) \quad \text{and} \quad f \in \mathcal{M}_r \setminus \mathcal{MO}_r.$$

By Corollary 2.10 and Corollary 2.11 we obtain the following inclusions.

Corollary 2.18. $\mathcal{C}^{\pm} = \mathcal{MO}_1 \subset \mathcal{M}_1$ *and* $\mathcal{S}_0 \subset \mathcal{SO}_0 = \mathcal{Q}$.

Again, all inclusions presented in the previous corollary are proper which follows from Example 2.15 and Example 2.17.

Combining Theorem 1.6, Theorem 2.14 and Corollary 2.18, we obtain full chain of inclusions between different kinds of porouscontinuity.

Remark 2.19. Let $0 < r < t < 1$. Then

$$\mathcal{C}^{\pm} = \mathcal{MO}_1 \subset \mathcal{M}_1 \subset \mathcal{P}_t \subset \mathcal{S}_t \subset \mathcal{SO}_t \subset \mathcal{MO}_t \subset \mathcal{M}_t \subset \mathcal{P}_r \subset \mathcal{S}_r \subset \mathcal{SO}_r \subset \mathcal{MO}_r \subset \mathcal{M}_r \subset \mathcal{P}_0 \subset \mathcal{S}_0 \subset \mathcal{SO}_0 = \mathcal{Q} \quad (2.1)$$

and all inclusions are proper.

Theorem 2.20. *Fix $r \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$, $f_n\colon \mathbb{R} \to \mathbb{R}$, be a sequence of functions such that each of them is $\mathcal{SO}_r$-continuous at x. If $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f\colon \mathbb{R} \to \mathbb{R}$, then f is $\mathcal{SO}_r$-continuous.*

Proof. Fix $r \in [0,1)$, $x \in \mathbb{R}$. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{3}$$

for each $n \geq n_0$ and $t \in \mathbb{R}$. Since f_{n_0} is $\mathcal{SO}_r$-continuous at x, there exists $E \subset \mathbb{R}$ such that $x \in E$, x is a point of πO_r-density of E and $f_{n_0}|_E$ is continuous at x. Therefore, we can find $\delta_{n_0} > 0$ such that

$$|f_{n_0}(t) - f_{n_0}(x)| < \frac{\varepsilon}{3}$$

for each $t \in E \cap (x - \delta_{n_0}, x + \delta_{n_0})$. Hence

$$|f(t) - f(x)| \leq |f(t) - f_{n_0}(t)| + |f_{n_0}(t) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for each $t \in E \cap (x - \delta_{n_0}, x + \delta_{n_0})$. This means that f is $\mathcal{SO}_r$-continuous at x. □

Theorem 2.21. *Fix $r \in (0,1]$. Let $(f_n)_{n\in\mathbb{N}}$, $f_n\colon \mathbb{R} \to \mathbb{R}$, be a sequence of functions such that each of them is $\mathcal{MO}_r$-continuous at x. If $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f\colon \mathbb{R} \to \mathbb{R}$, then f is $\mathcal{MO}_r$-continuous.*

The proof of this theorem is very similar to that of the previous theorem and we omit it.

3. Comparison of families of porouscontinuous functions in terms of porosity

The aim of the section is to describe the size of the presented families of functions, one into another, in terms of porosity.

Let us equip the space $\mathcal{X}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the metric

$$\varrho(f,g) = \min\bigl\{1, \sup\{|f(x) - g(x)|\colon x \in \mathbb{R}\}\bigr\}.$$

If $\mathcal{F} \subset \mathcal{X}$ consists of bounded functions, then $\|f\| = \sup\{|f(x)| \colon x \in \mathbb{R}\}$ is a norm in $\mathcal{F}$ such that the metric generated by $\| \ \|$ is equivalent to ϱ. If $\mathcal{F} \subset \mathcal{X}$, then we consider in $\mathcal{F}$ the metric ϱ restricted to $\mathcal{F}$, and the open ball in $\mathcal{F}$ with the center f and radius $R > 0$ will be denoted by $B_{\mathcal{F}}(f, R)$.

First, we recall the usual definition of lower porosity in a metric space X. The open ball with the center $x \in X$ and radius R will be denoted by $B(x, R)$. Let $M \subset X$, $x \in X$ and $R > 0$. Then, according to [6], we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \backslash M$ by $\gamma(x, R, M)$. The number

$$\underline{p}(M, x) = 2 \liminf_{R \to 0^+} \frac{\gamma(x, R, M)}{R}$$

is called the lower porosity of M at x. We say that the set M is lower porous at x if $\underline{p}(M, x) > 0$. The set M is called lower porous if M is lower porous at each point $x \in M$, and M is called σ-lower porous if M is the countable union of lower porous sets.

We will transfer the notion of porosity into function spaces. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{X}$, $f \in \mathcal{F}_2$ and $R > 0$. Then, $\gamma(f, R, \mathcal{F}_1, \mathcal{F}_2)$ will denote the supremum of the set of all $r > 0$ for which there exists $g \in \mathcal{F}_2$ such that $B_{\mathcal{F}_2}(g, r) \subset B_{\mathcal{F}_2}(f, R) \setminus \mathcal{F}_1$. The number

$$\underline{p}(\mathcal{F}_1, \mathcal{F}_2, f) = 2 \liminf_{R \to 0^+} \frac{\gamma(f, R, \mathcal{F}_1, \mathcal{F}_2)}{R}$$

is called the lower porosity of $\mathcal{F}_1$ in $\mathcal{F}_2$ at f. We will say that $\mathcal{F}_1$ is lower porous in $\mathcal{F}_2$ at f if $\underline{p}(\mathcal{F}_1, \mathcal{F}_2, f) > 0$ and we will say that $\mathcal{F}_1$ is strongly lower porous in $\mathcal{F}_2$ at f if $\underline{p}(\mathcal{F}_1, \mathcal{F}_2, f) \geq 1$. The family $\mathcal{F}_1$ is called lower porous (strongly lower porous) in $\mathcal{F}_2$ if it is lower porous (strongly lower porous) in $\mathcal{F}_2$ at each of its points. The family $\mathcal{F}_1$ is called σ-lower porous (σ-strongly lower porous) in $\mathcal{F}_2$ if it is a countable union of lower porous (strongly lower porous) sets in $\mathcal{F}_2$.

Obviously, every strongly lower porous set is lower porous, every lower porous set is nowhere dense and every σ-lower porous is meager. Moreover, none of the reverse inclusions is true.

In the sequel, we will need some results from [3].

Definition 3.1 ([3], Definition 2.2). Let $I \subset \mathbb{R}$ be any interval. We will say that the family $\mathcal{F}$ of real functions $f \colon I \to \mathbb{R}$ is admissible if:

- $\mathcal{C}(f) \cap \operatorname{int} I \neq \emptyset$ for each $f \in \mathcal{F}$ ($\operatorname{int} I$ denotes the interior of the interval I);
- if $f \in \mathcal{F}$, then $af + b \in \mathcal{F}$ for all real numbers a, b.

Remark 3.2. For every $r \in [0, 1)$, families $\mathcal{S}_r$ and $\mathcal{SO}_r$ are admissible.

Remark 3.3. For every $r \in (0, 1]$, families $\mathcal{M}_r$ and $\mathcal{MO}_r$ are admissible.

Theorem 3.4 ([3], Corollary 2.4). *Let $I \subset \mathbb{R}$ be any interval. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be admissible families of functions, $\mathcal{F}_1 \subset \mathcal{F}_2$. If for all $x_0 \in \operatorname{int} I$ there exists $h_{x_0} \colon I \to \mathbb{R}$ with the properties:*

1) $h_{x_0} \in \mathcal{F}_2$,
2) $\|h_{x_0}\| = \frac{1}{2}$,
3) $\forall_{f \in \mathcal{F}_2}\ (x_0 \in \mathcal{C}(f) \Rightarrow h_{x_0} + f \in \mathcal{F}_2)$,
4) *for $f \colon I \to \mathbb{R}$, if there exists $\delta > 0$ such that* $\sup\{|h_{x_0}(x) - f(x)| \colon x \in (x_0 - \delta, x_0 + \delta) \cap I\} < \frac{1}{2}$, *then $f \notin \mathcal{F}_1$,*

then $\mathcal{F}_1$ is strongly lower porous in $\mathcal{F}_2$.

Theorem 3.5. *Let $r \in [0,1)$. Then, $\mathcal{S}_r$ is strongly lower porous in $\mathcal{SO}_r$.*

Proof. Let us take any $x_0 \in \mathbb{R}$. Let $(a_n)_{n \geq 1}$, $(b_n)_{n \geq 1}$, $(c_n)_{n \geq 1}$, $(d_n)_{n \geq 1}$ be four sequences of reals with properties $x_0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots$, $\lim_{n\to\infty} a_n = x_0$, $\frac{b_n - a_n}{b_n - x_0} = r + \frac{1-r}{2(n+1)}$ and $\frac{d_n - c_n}{d_n - x_0} = r + \frac{1-r}{n+1}$ for each $n \geq 1$. Denote $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$. Define $h_{x_0} \colon \mathbb{R} \to \mathbb{R}$ by

$$h_{x_0}(x) = \begin{cases} \frac{1}{2}, & x \in (-\infty, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] \cup [d_1, \infty), \\ -\frac{1}{2}, & x \in \{x_0\} \cup A, \\ \text{linear} & \text{on } [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

1) Observe that h_{x_0} is continuous at each point different from x_0, so $\mathbb{R} \setminus \{x_0\} \in \mathcal{SO}_r(h_{x_0})$. Obviously, $h_{x_0}|_A$ is continuous at x_0. Let us take any $\varepsilon > 0$ and $\eta > 0$. Since $\lim_{n\to\infty} b_n = x_0$, we can find n_0 such that $b_{n_0} < x_0 + \eta$. Taking $\delta = b_{n_0} - x_0$ and $(a,b) = (a_{n_0}, b_{n_0})$, we get $\frac{b_{n_0} - a_{n_0}}{\delta} = \frac{b_{n_0} - a_{n_0}}{b_{n_0} - x_0} = r + \frac{1-r}{2(n+1)} > r$. Thus, x_0 is πO_r-density point of A. Finally, $h_{x_0} \in \mathcal{SO}_r$.
2) Obviously, $\|h_{x_0}\| = \frac{1}{2}$.
3) Take $f \colon \mathbb{R} \to \mathbb{R}$ from $\mathcal{SO}_r$, which is continuous at x_0. By Remark 2.8,
$$f + h_{x_0} \in \mathcal{SO}_r.$$
4) Now, take any $f \colon \mathbb{R} \to \mathbb{R}$ such that there exists $\delta > 0$ for which
$$\sup\{|h_{x_0}(x) - f(x)| \colon x \in (x_0 - \delta, x_0 + \delta)\} = \alpha < \frac{1}{2}.$$
Without loss of generality we may assume that $d_1 < x_0 + \delta$. Then,
$$-\frac{1}{2} - \alpha \leq f(x_0) \leq -\frac{1}{2} + \alpha \quad \text{and} \quad 0 < \frac{1}{2} - \alpha \leq f(x) \leq \frac{1}{2} + \alpha$$
$$\text{for } x \in (x_0 - \delta, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n].$$

Thus,

$$|f(x) - f(x_0)| \geq 1 - 2\alpha > 0 \quad \text{for} \quad x \in (x_0 - \delta, x_0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n].$$

Denote $\varepsilon = 1 - 2\alpha$. Then

$$\{x \in (x_0 - \delta, d_1) \colon |f(x) - f(x_0)| < \varepsilon\} \subset \{x_0\} \cup \bigcup_{n=1}^{\infty} [c_n, d_n].$$

Since

$$p\left(\mathbb{R} \setminus \left(\{x_0\} \cup \bigcup_{n=1}^{\infty} [c_n, d_n]\right), x_0\right) = \lim_{n\to\infty} \frac{d_n - c_n}{d_n - x_0} = \lim_{n\to\infty} \left(r + \frac{1-r}{n+1}\right) = r,$$

we conclude $x_0 \notin \mathcal{S}_r(f)$. And finally, $f \notin \mathcal{S}_r$.

All assumptions of Theorem 3.4 are satisfied, so $\mathcal{S}_r$ is strongly lower porous in $\mathcal{SO}_r$. □

Theorem 3.6. *Let $r \in (0, 1]$. Then, $\mathcal{MO}_r$ is strongly lower porous in $\mathcal{M}_r$.*

Proof. Let us take any $x_0 \in \mathbb{R}$. Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$, $(d_n)_{n\geq 1}$ be four sequences of reals with properties:

$$x_0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots,$$

$$\lim_{n\to\infty} a_n = x_0,$$

$$\frac{b_n - a_n}{b_n - x_0} = r - \frac{r}{2(n+1)} \quad \text{and} \quad \frac{d_n - c_n}{d_n - x_0} = r - \frac{r}{n+1} \quad \text{for each} \quad n \geq 1.$$

Observe that

$$p\left(\mathbb{R} \setminus \left(\{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]\right), x_0\right) = \lim_{n\to\infty} \frac{b_n - a_n}{b_n - x_0} = \lim_{n\to\infty} \left(r - \frac{r}{2(n+1)}\right) = r.$$

Therefore, x_0 is μ_r-density point of the set $\{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$. It is easy to see that x_0 is not μO_r-density point of $\{x_0\} \cup \bigcup_{n=1}^{\infty} [c_n, d_n]$. Defining $f \colon \mathbb{R} \to \mathbb{R}$ in the same way as in the proof of Theorem 3.5 and repeating arguments from the mentioned proof, we can show that $\mathcal{MO}_r$ is strongly lower porous in $\mathcal{M}_r$. □

We would like to prove that for every $r \in (0, 1)$ the family $\mathcal{SO}_r$ is strongly lower porous in $\mathcal{MO}_r$. It turns out that Theorem 3.4 is not strong enough and we need to prove a small generalization of Theorem 3.4.

Theorem 3.7. *Let I be any interval. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be admissible families of functions, $\mathcal{F}_1 \subset \mathcal{F}_2$. If for each $s \in \left(0, \frac{1}{2}\right)$ and for each $f \in \mathcal{F}_1$ there exist $x_0 \in \mathcal{C}(f)$ and a function $h \colon I \to \mathbb{R}$ with properties:*

1) $h \in \mathcal{F}_2$,
2) $\|h\| = \frac{1}{2}$,

3) $\forall_{c\in\mathbb{R}}\ (ch + f \in \mathcal{F}_2)$,

4) *for* $g\colon I \to \mathbb{R}$, *if there exists* $\delta > 0$ *such that* $\sup\{|h(x) - g(x)|\colon x \in (x_0 - \delta, x_0 + \delta) \cap I\} < s$, *then* $g \notin \mathcal{F}_1$,

then $\mathcal{F}_1$ *is strongly lower porous in* $\mathcal{F}_2$.

P r o o f. Choose any $f \in \mathcal{F}_1$ and $s \in (0, \frac{1}{2})$. Let $x_0 \in \mathcal{C}(f)$ and $h\colon I \to \mathbb{R}$ satisfy conditions 1) – 4). Choose any $r \in (0, 1)$. Since $\mathcal{F}_2$ is admissible, $rh \in \mathcal{F}_2$. Clearly,

$$\varrho(f, f + rh) = \frac{r}{2}.$$

By 3), $f+rh \in \mathcal{F}_2$. Since $x_0 \in \mathcal{C}(f)$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{r^2 s}{2}$ for $x \in (x_0 - \delta, x_0 + \delta) \cap I$. We shall show that

$$B_{\mathcal{F}_2}\left(f + rh, rs - r^2 s\right) \cap \mathcal{F}_1 = \emptyset.$$

Pick $g \in B_{\mathcal{F}_2}\left(f + rh, rs - r^2 s\right)$. Let $d = \varrho(g, f + rh)$. Then

$$|g(x) - f(x_0) - rh(x)| \le |g(x) - f(x) - rh(x)| + |f(x) - f(x_0)| < \\ d + \frac{r^2 s}{2} < rs - r^2 s + \frac{r^2 s}{2} = rs - \frac{r^2 s}{2} = rs\left(1 - \frac{r}{2}\right)$$

for $x \in (x_0 - \delta, x_0 + \delta) \cap I$. Therefore

$$\sup\left\{\left|\left(g(x) - f(x_0)\right) - rh(x)\right|\colon x \in (x_0 - \delta, x_0 + \delta) \cap I\right\} \le rs(1 - \tfrac{r}{2}) < rs.$$

Thus

$$\sup\left\{\left|\frac{g(x) - f(x_0)}{r} - h(x)\right| : x \in (x_0 - \delta, x_0 + \delta) \cap I\right\} < s.$$

By 4), $\frac{g - f(x_0)}{r} \notin \mathcal{F}_1$. Since $\mathcal{F}_1$ is admissible, $g \notin \mathcal{F}_1$. Moreover,

$$\varrho(f, g) \le \varrho(f, f + rh) + \varrho(f + rh, g) < \frac{r}{2} + rs - r^2 s < r.$$

Thus, we have shown

$$B_{\mathcal{F}_2}\left(f + rh, rs - r^2 s\right) \cap \mathcal{F}_1 = \emptyset \quad \text{and} \quad B_{\mathcal{F}_2}\left(f + rh, rs(1 - r)\right) \subset B_{\mathcal{F}_2}(f, r).$$

Since $r > 0$ was arbitrary,

$$\underline{p}(\mathcal{F}_1, \mathcal{F}_2, f) = \liminf_{r\to 0} \frac{2\gamma(f, r, \mathcal{F}_1, \mathcal{F}_2)}{r} \ge \liminf_{r\to 0} \frac{2(rs - r^2 s)}{r} = \liminf_{r\to 0} 2s(1 - r) = 2s$$

for every $s \in (0, \frac{1}{2})$. Therefore

$$p(\mathcal{F}_1, \mathcal{F}_2, f) \ge \lim_{s\to \frac{1}{2}^-} 2s = 1.$$

This means that $\mathcal{F}_1$ is strongly lower porous in $\mathcal{F}_2$. □

Corollary 3.8. *Let $I \subset \mathbb{R}$ be any interval. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be admissible families of functions, $\mathcal{F}_1 \subset \mathcal{F}_2$. If for each $f \in \mathcal{F}_1$ there exist $x_0 \in \mathcal{C}(f)$ and a function $h\colon I \to \mathbb{R}$ with properties:*

1) $h \in \mathcal{F}_2$,
2) $\|h\| = \frac{1}{2}$,
3) $\forall_{c\in\mathbb{R}}\ (ch + f \in \mathcal{F}_2)$,
4) *for $g\colon I \to \mathbb{R}$, if there exists $\delta > 0$ such that* $\sup\{|h(x) - g(x)|\colon x \in (x_0 - \delta, x_0 + \delta) \cap I\} < \frac{1}{2}$, *then $g \notin \mathcal{F}_1$,*

then $\mathcal{F}_1$ is strongly lower porous in $\mathcal{F}_2$.

Theorem 3.9. *For every $r \in (0,1)$, the family $\mathcal{SO}_r$ is strongly lower porous in $\mathcal{MO}_r$.*

Proof. Take any $f \in \mathcal{SO}_r$. Since $\mathcal{SO}_r \subset \mathcal{Q}$, there exists a point $x_0 \in \mathbb{R}$ at which f is continuous. Without loss of generality we may assume that $x_0 = 0$. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences of reals with the properties $\lim_{n\to\infty} a_n = 0$, $0 < \cdots < b_{n+1} < a_n < b_n$ and $\frac{b_n - a_n}{b_n} = r$ for each $n \geq 1$. Since $f \in \mathcal{SO}_r$, f is $\mathcal{SO}_r$-continuous from the right or $\mathcal{SO}_r$-continuous from the left at each point from the set $\bigcup_{n=1}^{\infty}\{a_n, b_n\}$. Denote

$$A = \{a_n\colon a_n \in \mathcal{SO}_r^+(f)\} \cup \{b_n\colon b_n \in \mathcal{SO}_r^-(f)\} \quad \text{and} \quad B = \bigcup_{n=1}^{\infty}\{a_n, b_n\} \setminus A.$$

Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{1}{2}, & x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty}(b_{n+1}, a_n) \cup (b_1, \infty) \cup B, \\ -\frac{1}{2}, & x \in \{0\} \cup \bigcup_{n=1}^{\infty}(a_n, b_n) \cup A. \end{cases}$$

1) Observe that h is continuous from the left or from the right at each point different from 0. Thus, $\mathbb{R} \setminus \{0\} \subset \mathcal{SO}_r(h)$. Moreover, $h|_{\bigcup_{n=1}^{\infty}(a_n, b_n)\cup A}$ is continuous at 0. Take $\eta > 0$. There exists $n_0 \in \mathbb{N}$ such that $b_{n_0} < \eta$ and $\frac{b_{n_0} - a_{n_0}}{b_{n_0}} = r$. Hence, 0 is a point of μO_r-density of $\bigcup_{n=1}^{\infty}(a_n, b_n) \cup A$ and $h \in \mathcal{MO}_r$.
2) Obviously, $\|h\| = \frac{1}{2}$.
3) Let $c \in \mathbb{R}$, $x \in \mathbb{R}$. We will consider the following cases.
   - Let $x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty}\big((b_{n+1}, a_n) \cup (a_n, b_n)\big) \cup (b_1, \infty)$. Then, the function ch is continuous at x and f is $\mathcal{SO}_r$-continuous at x. Therefore, $f + ch$ is $\mathcal{SO}_r$-continuous at x, by Remark 2.8.

- Let $x \in \bigcup_{n=1}^{\infty}\{a_n\}$.
  If $x \in A$, then f is $\mathcal{SO}_r$-continuous from the right at x and ch is continuous from the right at x. Thus, $f + ch$ is $\mathcal{SO}_r$-continuous from the right at x.
  If $x \in B$, then f is $\mathcal{SO}_r$-continuous from the left at x and ch is continuous from the left at x. Thus, $f + ch$ is $\mathcal{SO}_r$-continuous from the left at x.
- Let $x \in \bigcup_{n=1}^{\infty}\{b_n\}$.
  If $x \in A$, then f is $\mathcal{SO}_r$-continuous from the left at x and ch is continuous from the left at x. Thus, $f + ch$ is $\mathcal{SO}_r$-continuous from the left at x.
  If $x \in B$, then f is $\mathcal{SO}_r$-continuous from the right at x and ch is continuous from the right at x. Thus, $f + ch$ is $\mathcal{SO}_r$-continuous from the right at x.
- If $x = 0$, then ch is $\mathcal{MO}_r$-continuous at x and f is continuous at 0. Therefore, $f + ch$ is $\mathcal{MO}_r$-continuous at x.

Finally, $f + ch \in \mathcal{MO}_r$.

4) Assume that for $g\colon \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that

$$\sup\{|h(x) - g(x)|\colon x \in (-\delta, \delta)\} = \alpha < \frac{1}{2}.$$

We shall show that

$$g \notin \mathcal{SO}_r.$$

Without loss of generality we may assume $b_1 < \delta$. Observe that

$$g(0) \le -\frac{1}{2} + \alpha \quad \text{and} \quad g(x) \ge \frac{1}{2} - \alpha$$

if $x \in (-\delta, 0) \cup \bigcup_{n=1}^{\infty}(b_{n+1}, a_n) \cup (b_1, \delta) \cup B$. Thus

$$|g(x) - g(0)| \ge 1 - 2\alpha \;\text{ if }\; x \in (-\delta, 0) \cup \bigcup_{n=1}^{\infty}(b_{n+1}, a_n) \cup (b_1, \delta) \cup B.$$

Take any $\varepsilon \in (0, 1 - 2\alpha)$ and $\eta \in (0, \delta)$. Then

$$\{x \in (-\delta, b_1)\colon |g(x) - g(0)| < \varepsilon\} \subset \bigcup_{n=1}^{\infty}(a_n, b_n) \cup A.$$

Hence, for every $\delta_\eta \in (0, \eta)$ if

$$(a, b) \subset g^{-1}\big((g(0) - \varepsilon, g(0) + \varepsilon)\big) \cap \big((-\delta_\eta, \delta_\eta) \setminus \{0\}\big),$$

then

$$\frac{b - a}{\delta_\eta} \le \sup_{n \ge 1} \frac{b_n - a_n}{b_n} = r.$$

This means that $0 \notin \mathcal{SO}_r(g)$. Therefore, $g \notin \mathcal{SO}_r$.

By Corollary 3.8, we conclude that $\mathcal{SO}_r$ is strongly lower porous in $\mathcal{MO}_r$. □

Theorem 3.10. *Let $0 \le r < t \le 1$. Then, $\mathcal{M}_t$ is strongly lower porous in $\mathcal{P}_r$.*

Proof. Let $x_0 \in \mathbb{R}$. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ be four sequences of reals with the properties:

$$x_0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots, \quad \lim_{n\to\infty} a_n = x_0,$$

$$\frac{b_n - a_n}{b_n - x_0} = r + \frac{t-r}{3} \quad \text{and} \quad \frac{d_n - c_n}{d_n - x_0} = r + \frac{2(t-r)}{3} \quad \text{for each } n \in \mathbb{N}.$$

Then

$$p\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (a_n, b_n), x_0\right) = \lim_{n\to\infty} \frac{b_n - a_n}{b_n - x_0} = \lim_{n\to\infty}\left(r + \frac{t-r}{3}\right) > r$$

and

$$p\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (c_n, d_n), x_0\right) = \lim_{n\to\infty} \frac{d_n - c_n}{d_n - x_0} = \lim_{n\to\infty}\left(r + \frac{2(t-r)}{3}\right) < t,$$

which means that x_0 is a point of π_r-density of $\bigcup_{n=1}^{\infty}(a_n, b_n)$ and x_0 is not a point of μ_t-density of $\bigcup_{n=1}^{\infty}(c_n, d_n)$. Define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{1}{2}, & x \in (-\infty, x_0) \cup \bigcup_{n=1}^{\infty}[d_{n+1}, c_n] \cup (d_1, \infty), \\ -\frac{1}{2}, & x \in \{x_0\} \cup \bigcup_{n=1}^{\infty}[a_n, b_n], \\ \text{linear} & \text{on } [b_n, d_n], [c_n, a_n], n \ge 1. \end{cases}$$

In a similar way as in proof of Theorem 3.5, we can show that $\mathcal{M}_t$ is strongly lower porous in $\mathcal{P}_r$. □

Let us recall the other results from [3].

Theorem 3.11 ([3], Theorem 3.31)**.**

- *$\mathcal{P}_t$ is strongly lower porous in $\mathcal{P}_r$ for $0 \le r < t < 1$;*
- *$\mathcal{S}_t$ is strongly lower porous in $\mathcal{S}_r$ for $0 \le r < t < 1$;*
- *$\mathcal{M}_t$ is strongly lower porous in $\mathcal{M}_r$ for $0 < r < t \le 1$.*

Theorem 3.12 ([3], Theorem 3.36)**.** *Let $r \in (0,1)$. Then, $\mathcal{S}_r$ is strongly lower porous in $\mathcal{M}_r$.*

Remark 3.13. Let us rewrite and expand (2.1). This enlargement contains majority of results obtained in the paper. Let $0 < r < t < 1$. Then

$$\mathcal{C}^{\pm} = \mathcal{MO}_1 \overset{3.6}{\subset} \mathcal{M}_1 \overset{3.10}{\subset} \mathcal{P}_t \subset \mathcal{S}_t \overset{3.5}{\subset} \mathcal{SO}_t \overset{3.9}{\subset}$$

$$\mathcal{MO}_t \overset{3.6}{\subset} \mathcal{M}_t \overset{3.10}{\subset} \mathcal{P}_r \subset \mathcal{S}_r \overset{3.5}{\subset} \mathcal{SO}_r \overset{3.9}{\subset}$$

$$\mathcal{MO}_r \overset{3.6}{\subset} \mathcal{M}_r \overset{3.10}{\subset} \mathcal{P}_0 \subset \mathcal{S}_0 \overset{3.5}{\subset} \mathcal{SO}_0 = \mathcal{Q}.$$

The number located above the inclusion mark denotes the number of theorem which says about porosity of the smaller family into the greater one.

Question 3.14 ([3])**.** Let $r \in [0,1)$. What can we say about the porosity of $\mathcal{P}_r$ in $\mathcal{S}_r$?

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Received October 12, 2020

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