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THE FAMILY OF CENTRAL CANTOR SETS WITH PACKING DIMENSION ZERO

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ABSTRACT. As in the recent article of M. Balcerzak, T. Filipczak and P. Nowakowski, we identify the family $\mathcal{CS}$ of central Cantor subsets of $[0,1]$ with the Polish space $X := (0,1)^{\mathbb{N}}$ equipped with the probability product measure μ. We investigate the size of the family $\mathcal{P}_0$ of sets in $\mathcal{CS}$ with packing dimension zero. We show that $\mathcal{P}_0$ is meager and of μ measure zero while it is treated as the corresponding subset of X. We also check possible inclusions between $\mathcal{P}_0$ and other subfamilies of $\mathcal{CS}$ consisting of small sets.

1. Introduction

The paper is a continuation of the studies presented in [2] where the family $\mathcal{CS}$ of central Cantor sets was considered and the sizes (with respect to measure and category) of several subfamilies of $\mathcal{CS}$ consisting of small sets were investigated.

Recall the standard construction of a central Cantor subset of $[0,1]$ (see [2], cf. also [11], [10] and [7]). An interval $I \subset \mathbb{R}$ is called *concentric with* an interval J if they have a common centre. By $|I|$ we will denote the length of the interval I. Let $a=(a_n)\in(0,1)^{\mathbb{N}}$. The set $C(a)$ is defined similarly as the classical Cantor ternary set, but in the n-th step of the construction, from every interval I obtained in the previous step, we remove the open interval concentric with interval I of a length equal to $a_n|I|$ (for the classical Cantor set, it is $\frac{1}{3}|I|$ for any n). Specifically, first, we define by induction the intervals I_t and P_t indexed by finite binary sequences. By $I_\emptyset$ we denote the interval $[0,1]$ and by $P_\emptyset$ the open interval with a length equal to $a_1|I_\emptyset| = a_1$, concentric with $I_\emptyset$. If we have already defined the intervals I_t and P_t, where $t\in\{0,1\}^n$, $n\in N\cup\{0\}$, then the left and the right component

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of the set $I_t \setminus P_t$ will be denoted by $I_{t\hat{}0}$ and $I_{t\hat{}1}$, respectively. The open intervals concentric with them, with a length equal to $a_{n+1}\,|I_{t\hat{}0}| = a_{n+1}\,|I_{t\hat{}1}|$, are denoted by $P_{t\hat{}0}$ and $P_{t\hat{}1}$, respectively.

For all $n \in \mathbb{N}$, set

$$\mathcal{I}_n := \left\{ I_{t_1,\ldots,t_n} \colon (t_1,\ldots,t_n) \in \{0,1\}^n \right\} \quad \text{and} \quad C_n(a) := \bigcup \mathcal{I}_n.$$

Let $C(a) := \bigcap_{n\in\mathbb{N}} C_n(a)$. Then, $C(a)$ is called a *central Cantor set*. Of course, every family $\mathcal{I}_n$ covers the set $C(a)$. Each interval of the family $\mathcal{I}_n$ is called basic of rank n. From the construction, it follows that the length of this interval is equal to

$$d_n = \frac{1}{2^n}(1-a_1)\cdots(1-a_n). \tag{1}$$

We will consider d_n as a function of a variable a, that is $a \mapsto d_n(a)$. However, for the simplicity of notation, we will keep writing d_n.

Consider $X := (0,1)^{\mathbb{N}}$ and equip it with the product topology generated by the natural topology in $(0,1)$. Then, X is a Polish space. Also, having Lebesgue measure on the σ-algebra of measurable subsets of $(0,1)$, we equip X with the product σ-algebra $\mathcal{S}$ generated by the σ-algebra of measurable subsets of $(0,1)$ and the product measure μ on $\mathcal{S}$ generated by the Lebesgue measure on $(0,1)$. Then, $(X,\mathcal{S},\mu)$ is a probability space.

Since every central Cantor subset of $[0,1]$ is uniquely determined by the respective sequence $(a_n) \in X$, we can identify the family $\mathcal{CS}$ of all central Cantor subsets of $[0,1]$ with the set X. Thanks to this idea, we can identify subsets of $\mathcal{CS}$ with the corresponding subsets of X, and then we can use topological and measure structure of X. Namely, for a subset $\mathcal{A}$ of $\mathcal{CS}$ we denote

$$\mathcal{A}^* := \{a \in X \colon C(a) \in \mathcal{A}\},$$

and, identifying $\mathcal{A}$ with $\mathcal{A}^*$, we may interpret $\mathcal{A}$ as a subset of X.

In [2], the following families of small sets were considered: microscopic sets, sets of Hausdorff dimension zero, strongly porous sets, porous sets, and Lebesgue null sets. It was proved that the intersections of these families with $\mathcal{CS}$ ordered as above, form an increasing sequence with respect to inclusion, and all inclusions between these families are proper. It also turned out that all the considered intersections, treated as subsets of X (via the identification $\mathcal{A} \mapsto \mathcal{A}^*$), are residual. Moreover, the families of microscopic sets and sets of Hausdorff dimension zero are of μ measure zero, and the remaining ones are of full measure.

In this paper, we will consider the family $\mathcal{P}_0$ of central Cantor sets with packing dimension zero. We will show that this family is meager and of μ--measure zero. We will also examine possible inclusions between $\mathcal{P}_0$ and the families considered in [2]. The obtained results answer the question posed by Professor Mariusz Urbański during the conference “Dynamics, measures and dimensions” which took place in Będlewo in 2019.

2. Main results

In [2], a useful class of small central Cantor sets was introduced. Namely, let $f\colon \mathbb{N} \to [1,\infty)$, and define the family $\mathcal{M}(f) \subset \mathcal{CS}$ as follows:

$$\mathcal{M}(f) := \left\{ C(a) \in \mathcal{CS}\colon \forall_{\varepsilon>0}\ \exists_{n\in\mathbb{N}}\ \ d_n < \varepsilon^{f(n)} \right\}, \tag{2}$$

where the numbers d_n depend on $a = (a_n)$ according to formula (1).

The family $\mathcal{M}(f)$ may be characterized as below.

Proposition 2.1 ([2]). *For a sequence $f\colon \mathbb{N} \to [1,\infty)$ and a set $C(a) \in \mathcal{CS}$, the following conditions are equivalent:*

(i) $C(a) \in \mathcal{M}(f)$;

(ii) $\forall_{\varepsilon>0}\ \forall_{m\in\mathbb{N}}\ \exists_{n\geq m}\ \ d_n < \varepsilon^{f(n)}$;

(iii) $\liminf\limits_{n\to\infty} (d_n)^{1/f(n)} = 0$;

(iv) $\liminf\limits_{n\to\infty} \frac{f(n)}{-\ln d_n} = 0$.

Theorem 2.2 ([2]). *Let $f\colon \mathbb{N} \to [1,\infty)$. Then, the set $\mathcal{M}(f)$ is residual of type G_δ while it is treated as the corresponding subset $\mathcal{M}(f)^*$ of X.*

For the simiplicity, when f is given by an explicit formula, we will write

$$\mathcal{M}\big(f(n)\big) \quad \text{instead of} \quad \mathcal{M}(f).$$

Now, let us recall the notions of Hausdorff and packing measures and dimensions (compare [4], [5], [7], [8], [9],). For $s > 0$ we define the s-dimensional Hausdorff measure of a set $E \subset \mathbb{R}$ by the formula

$$H^s(E) := \lim_{\delta\to 0^+} H^s_\delta(E),$$

where

$$H^s_\delta(E) := \inf\left\{ \sum_{i=1}^{\infty} |I_i|^s \colon I_i - \text{open intervals}, E \subset \bigcup_{i=1}^{\infty} I_i, |I_i| \leq \delta \right\}.$$

The Hausdorff dimension of a set E is given by

$$\dim_H(E) := \sup\{s > 0\colon H^s(E) > 0\} = \inf\{s > 0\colon H^s(E) < \infty\} \in [0,1].$$

For $s > 0$ we define the s-dimensional packing measure of a set $E \subset \mathbb{R}$ by the formula $P^s(E) := \lim_{\delta\to 0^+} P^s_\delta(E)$, where

$$P^s_\delta(E) := \sup\left\{ \sum_{i=1}^{\infty} |I_i|^s \colon I_i - \text{pairwise disjoint open intervals with centres in E}, |I_i| \leq \delta \right\}.$$

The packing dimension of a set E is given by the formula

$$\dim_P(E) := \sup\{s>0\colon P^s(E)>0\} = \inf\{s>0\colon P^s(E)<\infty\} \in [0,1].$$

The subfamily of the family $\mathcal{CS}$ of all sets with Hausdorff dimension equal to s will be denoted by $\mathcal{H}_s$, and of all sets with packing dimension equal to s, by $\mathcal{P}_s$.

It is known that (see [4]) for any set $E \subset \mathbb{R}$ we have $\dim_H(E) \leq \dim_P(E)$. Therefore, $\mathcal{P}_0 \subset \mathcal{H}_0$.

We will need the following result.

Theorem 2.3 ([7], [5]). *The Hausdorff dimension of a central Cantor set $C(a)$ is equal to*

$$\liminf_{n\to\infty} \frac{n\ln 2}{-\ln d_n},$$

and the packing dimension of the set $C(a)$ is equal to

$$\limsup_{n\to\infty} \frac{n\ln 2}{-\ln d_n}.$$

Corollary 2.4. $\mathcal{H}_0 = \left\{C(a)\colon \liminf\limits_{n\to\infty} \frac{n}{-\ln d_n} = 0\right\} = \mathcal{M}(n).$

Using Theorem 2.3, we can also prove the following theorem. Its proof is based on ideas of [2, Theorem 2.5].

Theorem 2.5.

$$\mu(\mathcal{H}_s^*) = \mu(\mathcal{P}_s^*) = \begin{cases} 1 & \text{if } s = \frac{\ln 2}{\ln 2 + 1}, \\ 0, & \text{otherwise.} \end{cases}$$

P r o o f. Observe that $Y_j(a) := \ln(1-a_j)$, for $j \in \mathbb{N}$, are independent random variables on X with the same distribution and with the expected value equal to

$$\mathbb{E}Y_j = \int\limits_{(0,1)} \ln(1-x)\, d\lambda(x) = \int\limits_{(0,1)} \ln t\, d\lambda(t) = -1.$$

By the strong law of large numbers [3, Theorem 2.25], we have $\mu(A) = 1$, where

$$A := \left\{a \in X\colon \frac{1}{n}\sum_{j=1}^{n} Y_j(a) \to -1\right\}.$$

Hence, for any $a \in A$ we have

$$\frac{\ln 2}{\ln 2 + 1} = \frac{\ln 2}{\ln 2 - \lim\limits_{n\to\infty} \frac{1}{n}\cdot \sum_{i=1}^{n} \ln(1-a_i)} = \lim_{n\to\infty} \frac{n\ln 2}{n\ln 2 - \sum_{i=1}^{n}\ln(1-a_i)}$$

$$= \lim_{n\to\infty} \frac{n\ln 2}{-\ln\left(\prod_{i=1}^{n} \frac{1-a_i}{2}\right)} = \lim_{n\to\infty} \frac{n\ln 2}{-\ln d_n}.$$

Therefore, by Theorem 2.3, we get that for a random central Cantor set $C(a)$

$$\dim_H\big(C(a)\big) = \dim_P\big(C(a)\big) \overset{a.s.}{=} \frac{\ln 2}{\ln 2 + 1},$$

where 'a.s.' means 'almost surely'. □

Corollary 2.6. *The expected value of the Hausdorff dimension and of the packing dimension of a central Cantor set is equal to* $\frac{\ln 2}{\ln 2+1}$.

From Theorem 2.5, Theorem 2.2, and Corollary 2.4 we get the following result.

Corollary 2.7 ([2]). *The family $\mathcal{H}_0$ is residual of type G_δ and of measure μ zero while it is treated as the corresponding subset of X.*

The properties of the family of sets with packing dimension zero are different from those for the family $\mathcal{H}_0$. Specifically, we have the following theorem.

Theorem 2.8. *The family of sets in $\mathcal{CS}$ with packing dimension zero is of μ measure zero, meager, and of type $F_{\sigma\delta}$, while it is treated as the corresponding family $\mathcal{P}_0^*$ in X.*

P r o o f. Directly from Theorem 2.5 we have $\mu(\mathcal{P}_0^*) = 0$. By Theorem 2.3, we obtain the following equivalences

$$C(a) \in \mathcal{P}_0 \Leftrightarrow \limsup_{n\to\infty} \frac{n}{-\ln d_n} = 0 \Leftrightarrow \lim_{n\to\infty} \frac{\ln d_n}{n} = -\infty \Leftrightarrow \lim_{n\to\infty} (d_n)^{\frac{1}{n}} = 0$$

$$\Leftrightarrow \forall_{k\in\mathbb{N}} \exists_{m\in\mathbb{N}} \forall_{n\geq m}\, d_n \leq \frac{1}{k^n} \Leftrightarrow a \in \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{a \in X \colon d_n \leq \frac{1}{k^n}\right\}.$$

Thus,

$$\mathcal{P}_0^* = \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \left\{a \in X \colon (1-a_1)(1-a_2)\dots(1-a_n) \leq \left(\frac{2}{k}\right)^n\right\}.$$

The sets $\{a \in X \colon (1-a_1)(1-a_2)\cdots(1-a_n) \leq \big(\frac{2}{k}\big)^n\}$ are closed, so $\mathcal{P}_0^*$ is of type $F_{\sigma\delta}$.

Observe that the set $A := \big\{a \in X \colon \dim_P\big(C(a)\big) = 1\big\}$ is included in $X \setminus \mathcal{P}_0^*$. To complete the proof, it suffices to show that the set A is residual. Since $\dim_P\big(C(a)\big) \leq 1$ for any set $C(a)$, we have the following equivalences

$$a \in A \Leftrightarrow \limsup_{n\to\infty} \frac{n \ln 2}{-\ln d_n} = 1 \Leftrightarrow \forall_{k\in\mathbb{N}} \forall_{m\in\mathbb{N}} \exists_{n\geq m} \frac{n\ln 2}{-\ln d_n} > 1 - \frac{1}{k}.$$

Observe that the inequality $\frac{n\ln 2}{-\ln d_n} > 1 - \frac{1}{k}$ is equivalent to the inequality

$$d_n > \left(\frac{1}{2}\right)^{\frac{nk}{k-1}}.$$

Hence,

$$A = \bigcap_{k\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{ a \in X \colon d_n > \left(\frac{1}{2}\right)^{\frac{nk}{k-1}} \right\}$$
$$= \bigcap_{k\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{ a \in X \colon (1-a_1)(1-a_2)\cdots(1-a_n) > \left(\frac{1}{2}\right)^{\frac{n}{k-1}} \right\}.$$

Let

$$A_{km} := \bigcup_{n\geq m} \left\{ a \in X \colon (1-a_1)(1-a_2)\cdots(1-a_n) > \left(\frac{1}{2}\right)^{\frac{n}{k-1}} \right\}$$

for $m, k \in \mathbb{N}$. Of course, the sets A_{km} are open in X, so A is of type G_δ. Let $m, k \in \mathbb{N}$. We will show that A_{km} is dense. Consider a basic open set in X of the form

$$U := U_1 \times U_2 \times \cdots \times U_j \times (0,1) \times (0,1) \times \cdots,$$

where $j \in \mathbb{N}$, and U_i are nonempty and open sets in $(0,1)$ for $i = 1,\ldots,j$. We will find $a \in U \cap A_{km}$. Choose $a_i \in U_i$ for $i = 1,\ldots,j$. Let

$$M = (1-a_1)(1-a_2)\cdots(1-a_j).$$

There exists

$$n \geq \max\{m, j+1\} \quad \text{such that} \quad M > \left(\frac{1}{2}\right)^{\frac{n}{k-1}}.$$

For $i \in \{j+1,\ldots,n\}$ let $a_i < 1 - \sqrt[n-j]{\frac{1}{M\cdot 2^{\frac{n}{k-1}}}} \in (0,1)$, and let a_i be an arbitrary element from $(0,1)$ for $i > n$. Observe that $a = (a_i) \in U \cap A_{km}$. Indeed,

$$(1-a_1)(1-a_2)\cdots(1-a_n) = M\cdot(1-a_{j+1})\cdots(1-a_n) > M\cdot\left(\sqrt[n-j]{\frac{1}{M\cdot 2^{\frac{n}{k-1}}}}\right)^{n-j} = \frac{1}{2^{\frac{n}{k-1}}}.$$

Hence, $a \in A_{km}$. Therefore, the sets A_{km} are open and dense in the Polish space X, and thus, they are residual. So, $\mathcal{A}$ is also residual in X, and consequently, the set $\mathcal{P}_0^*$ is meager in X. □

Corollary 2.9. $\mathcal{P}_0 \subsetneq \mathcal{H}_0$.

Now, let us consider the family of microscopic sets in $\mathcal{CS}$ which is denoted by $\mathcal{M}$. Recall that we say that a set $E \subset \mathbb{R}$ is microscopic if for any $\varepsilon > 0$ there exists a sequence of intervals (J_n) such that $E \subset \bigcup_{n\in\mathbb{N}} J_n$ and $|J_n| \leq \varepsilon^n$. For more information about microscopic sets, see [1] or [6]. We have the following inclusions.

Theorem 2.10 ([2]). $\mathcal{M}(2^n) \subset \mathcal{M} \subset \bigcap_{p\in(1,2)} \mathcal{M}(p^n)$.

We will now compare the families $\mathcal{M}$ and $\mathcal{P}_0$.

EXAMPLE 1. We give an example of a set $C(a)$ such that $C(a) \in \mathcal{M}(2^n) \setminus \mathcal{P}_0 \subset \mathcal{M} \setminus \mathcal{P}_0$. Let (k_n) and (l_n) be sequences of natural numbers such that $k_1 = 1$ and

$$k_n < l_n < 2 \cdot 3^{l_n} < k_{n+1} < 3 \cdot 3^{l_n}$$

for $n \in \mathbb{N}$. Define a sequence (x_n). Put $x_{k_n} = \left(\frac{1}{16}\right)^{k_n}$ and $x_{l_n} = \left(\frac{1}{16}\right)^{3^{l_n}}$ for $n \in \mathbb{N}$. We have

$$\frac{x_{l_n}}{x_{k_n}} = \left(\frac{1}{16}\right)^{3^{l_n} - k_n} < \left(\frac{1}{2}\right)^{l_n - k_n}$$

and

$$\frac{x_{k_{n+1}}}{x_{l_n}} = \left(\frac{1}{16}\right)^{k_{n+1} - 3^{l_n}} < \left(\frac{1}{16}\right)^{3^{l_n}} = \left(\frac{1}{2}\right)^{4 \cdot 3^{l_n}} < \left(\frac{1}{2}\right)^{k_{n+1} - l_n}.$$

So, we can define the rest of terms of the sequence (x_i) in such a way that $\frac{x_{i+1}}{x_i} < \frac{1}{2}$ for $i \in \mathbb{N}$. Putting $a_1 = 1 - 2x_1$ and $a_{i+1} = 1 - 2\frac{x_{i+1}}{x_i}$ for $i \in \mathbb{N}$, we receive the sequence $a \in (0,1)^{\mathbb{N}}$, for which $d_i = x_i$. In particular,

$$d_{k_n} = \left(\frac{1}{16}\right)^{k_n} \quad \text{and} \quad d_{l_n} = \left(\frac{1}{16}\right)^{3^{l_n}}.$$

We have

$$\limsup_{n\to\infty} \frac{n \ln 2}{-\ln d_n} \geq \lim_{n\to\infty} \frac{k_n \ln 2}{-\ln d_{k_n}} = \lim_{n\to\infty} \frac{k_n \ln 2}{k_n \ln 16} = \frac{1}{4} > 0,$$

and hence, $C(a) \notin \mathcal{P}_0$. However,

$$\liminf_{n\to\infty} \frac{2^n}{-\ln d_n} \leq \lim_{n\to\infty} \frac{2^{l_n}}{-\ln d_{l_n}} = \lim_{n\to\infty} \frac{2^{l_n}}{3^{l_n} \ln 16} = 0.$$

Thus, by Proposition 2.1 and Theorem 2.10, we have $C(a) \in \mathcal{M}(2^n) \subset \mathcal{M}$.

EXAMPLE 2. We give an example of a set $C(a)$ such that $C(a) \in \mathcal{P}_0 \setminus \mathcal{M}$. Let $a_n = 1 - \frac{2}{e^{2n-1}} \in (0,1)$ for $n \in \mathbb{N}$ and $a = (a_n)$. Then,

$$d_n = \prod_{i=1}^{n} \frac{1 - a_i}{2} = \prod_{i=1}^{n} \frac{1}{e^{2i-1}} = \frac{1}{e^{\sum_{i=1}^{n}(2i-1)}} = \frac{1}{e^{n^2}}.$$

Therefore,

$$\limsup_{n\to\infty} \frac{n \ln 2}{-\ln d_n} = \lim_{n\to\infty} \frac{n \ln 2}{n^2} = 0,$$

so, $C(a) \in \mathcal{P}_0$. However, for any $p \in (1,2)$ we have

$$\liminf_{n\to\infty} \frac{p^n}{-\ln d_n} = \lim_{n\to\infty} \frac{p^n}{n^2} = \infty,$$

and thus, $C(a) \in \mathcal{P}_0 \setminus \mathcal{M}(p^n) \subset \mathcal{P}_0 \setminus \mathcal{M}$.

COROLLARY 2.11. *$\mathcal{M} \not\subset \mathcal{P}_0$ and $\mathcal{P}_0 \not\subset \mathcal{M}$.*

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