sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2024-0015
Tatra Mt. Math. Publ. **86** (2024), 147–174

NEUTROSOPHIC FUZZY TRIBONACCI $\mathcal{I}$-LACUNARY STATISTICAL CONVERGENT SEQUENCE SPACES

Vakeel A. Khan[1] — Ömer Kişi[2] — Chiranjib Choudhury[3]

[1]Department of Mathematics, Aligarh Muslim University, Aligarh, INDIA

[2]Department of Mathematics, Bartın University, Bartın, TURKEY

[3]Department of Mathematics, University, Tripura University, Agartala, INDIA

ABSTRACT. The aim of this paper is to introduce and investigate some neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistical convergent sequence spaces by utilizing the domain of regular tribonacci matrix $A = \big(a_{jk}\big)$. Morever, we also put forward various algebraic and topological features of these convergent sequence spaces and establish several interesting inclusion relations.

1. Introduction

Statistical convergence and ideal convergence were put forward by Fast [3] and Kostyrko et al. [18], respectively. Various works on their corresponding sequence spaces can be examined in [5–8, 22, 26, 28, 29, 32, 39, 40].

After the original study of Zadeh [43], a huge number of research works have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy sets cannot always overcome the absence of knowledge of membership degrees. Atanassov [1] proposed intuitionistic fuzzy set (IFS). Kramosil and Michalek [19] investigated fuzzy metric space (FMS) by help of the notions of fuzzy and probabilistic metric space. Park [27] presented intuitionistic fuzzy metric space (IFMS). In [20], influenced by Park's definition of an IFM, Lael and Nourouzi first worked an IF-normed space. Statistical convergence in IFNS

2020 Mathematics Subject Classification: 46S40, 40A35, 03E72.
Keywords: Neutrosophic normed spaces, ideal convergence, tribonacci sequence, regular tribonacci matrix.

was studied by Karakuş et al. [10]. For extensive study in this topic, one may refer to the works of [2, 9, 11–14, 23–25, 30, 31].

Smarandache [33, 34] investigated the concept of 'Neutrosophic set' (NS), a generalisation of FS, IFS, interval-valued IFS etc. to overcome the complexity arising from uncertainty in solving many practical problems in our routine life activity more precisely. Several times, decision makers face some hesitations beside going to direct approaches (i.e., yes or no) in a decision making. In addition, we can obtain a tri-component outcomes in some real events like sports, procedure for voting etc. Smarandache applied the IFS theory by defining a new component namely indeterminacy membership function. Thus, an element belonging to an NS consists of a triplet namely truth-membership function (T), indeterminacy-membership function (I) and falsity-membership function (F). A neutrosophic set (NS) is identified as a set where every component of the universe has a degree of T, F and I. These three functions are independent in nature in NS. So, the concept of neutrosophy implies impartial knowledge of idea and then neutral describes the fundamental difference between neutral, fuzzy, intuitive fuzzy set and logic.

The uncertainty is based on the degree of membership in the IFSs, whereas the uncertainty in NS is thought independently from T and F values. Since there are no limitations among the degree of T, F, I, NSs are actually more general than IFS.

FS provide a framework for handling uncertainty by assigning degrees of membership to elements, allowing for partial membership. IFS extends this concept by introducing pairs of membership and non-membership degrees, separately capturing uncertainty and hesitation. NS further generalizes this idea by incorporating a third degree of membership, representing indeterminacy, thus distinguishing between absolute and relative membership.

Kirişçi and Şimşek [15] proposed new notion known as neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. Neutrosophic normed space (NNS) and statistical convergence in NNS were investigated by Kirişci and Şimşek [16]. Ideal convergence in NNS was put forward by Kişi and khan [17, 35–37].

In [42], the authors put forward the matrix corresponding to the tribonacci sequence in [4, 38].

The intention of this paper is to survey the new kind of convergence for sequences in NNS. The following is how the paper is structured. The literature review is covered in Section 1 of the introduction. The key findings are then given in Section 2. Namely, we intend to investigate neutrosophic fuzzy tribonacci $\mathcal{I}$--lacunary statistical convergent sequence spaces and to develop essential features of these convergent sequence spaces.

2. Preliminaries

Assume T and Q are two sequence spaces and $A = \big(b_{js}\big)$ is an infinite matrix of real entries. A_j is utilized to indicate the sequence in the nth row of matrix A. Recalling that A determines a matrix mapping from sequence space P to Q provided that for all sequence $\phi = (\phi_s)$, the A transform of ϕ is identified as $A\phi = \{A_j(\phi)\}_{j=1}^{\infty} \in Q$, where

$$A_j(\phi) = \sum_{s=1} b_{js}\phi_s, \quad j \in \mathbb{N}.$$

For any sequence space R, the sequence space R_A determined by

$$R_A = \{\phi = (\phi_s) \in l_\infty : A\phi \in R\}$$

is recognized as domain of the matrix A.

Lemma 2.1 (Wilansky [41], Yaying and Hazarika [42]). *An infinite matrix $A = (b_{js})_{j,s\in\mathbb{N}}$ is regular if and only if*

(i) *for all $j \in \mathbb{N}$, there is a $K > 0$ so that $\sum_s \|b_{js}\| \leq K$,*

(ii) $\lim_{j\to\infty} b_{js} = 0$ *for all $s \in \mathbb{N}$,*

(iii) $\lim_{j\to\infty} \sum_s b_{js} = 1$.

The sequence $b_t)_{t\in\mathbb{N}}$ of tribonacci numbers was investigated by Mark Feinberg [4] as follows:

$$b_t = b_{t-1} + b_{t-2} + b_{t-3}, \quad t \geq 3 \quad \text{with} \quad b_1 = b_2 = 1 \quad \text{and} \quad b_3 = 2.$$

The first numbers of tribonacci sequence are $1, 1, 2, 4, 7, 13, 24, \ldots$ Several fundamental features of tribonacci sequence are

$$\lim_{s\to\infty} \tfrac{b_s}{b_{s+1}} = 0.54368901\ldots$$

and

$$\sum_{s=1}^{j} b_s = \tfrac{b_{j+2}+b_j-1}{2}.$$

In this paper, we utilize the lower triangular tribonacci matrix $A = \big(b_{js}\big)$, given in [42], as follows:

$$b_{js} = \begin{cases} \frac{2b_s}{b_{j+2}+b_j-1}, & \text{if } 1 \leq s \leq j, \\ 0, & \text{if not}, . \end{cases}$$

According to the lemma, we can obviously see that A is regular matrix. For any sequence $\phi = (\phi_s)$, the A-transformation of (ϕ_s) is given by

$$A_j\phi) = \sum_{s=1}^{j} \tfrac{2b_s}{b_{j+2}+b_j-1}\phi_s, \quad j \in \mathbb{N}.$$

Definition 2.2 (Kostyrko et al. [18]). Take $K \neq \emptyset$. $\mathcal{I} \subset 2^K$ is named an ideal on K provided that

(i) for all $C, D \in \mathcal{I}$ means $C \cup D \in \mathcal{I}$;
(ii) for all $C \in \mathcal{I}$ and $D \subset C$ means $D \in \mathcal{I}$.

Definition 2.3 (Kostyrko et al. [18]). Take $K \neq \emptyset$. $\mathcal{F} \subset 2^K$ is named a filter on K provided that

(i) for all $C, D \in \mathcal{F}$ means $C \cap D \in \mathcal{F}$;
(ii) for all $C \in \mathcal{F}$ and $D \supset C$ means $D \in \mathcal{F}$.

An ideal $\mathcal{I}$ is identified as non-trivial when $K \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non-trivial ideal $\mathcal{I} \subset P(K)$ is named an admissible ideal in K if and only if $\mathcal{I} \supset \{\{w\} : w \in Y\}$. Also, the filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{K - L : L \in \mathcal{I}\}$ is known as the filter connected with the ideal.

By utilizing the notion of ideals, Kostyrko et al. [18] presented the notion of $\mathcal{I}$-convergence.

Definition 2.4 (Kostyrko et al. [18]). A sequence $\phi = (\phi_s)$ is named to be $\mathcal{I}$-convergent to α provided that for all $\gamma > 0$,

$$\{s \in \mathbb{N} : |\phi_s - \alpha| \geq \gamma\} \in \mathcal{I}.$$

In this situation, we indicate as $\mathcal{I}\text{-}\lim \phi_s = \alpha$.

Definition 2.5 (Menger [21]). Suppose $\triangle : [0,1] \times [0,1] \to [0,1]$ is an operation. When $\triangle$ holds subsequent statements, it is named continuous t-norm (TN). Assume $p, q, r, s \in [0,1]$,

(a) $p \triangle 1 = p$,
(b) when $p \leq r$ and $q \leq s$, then $p \triangle q \leq r \triangle s$,
(c) $\triangle$ *is continuous,*
(d) $\triangle$ *commutative and associative.*

Definition 2.6 (Menger [21]). Suppose $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is an operation. When $\diamondsuit$ supplies subsequent statements, it is named to be continuous $t-$conorm (TC).

(a) $p \diamondsuit 0 = p$,
(b) when $p \leq r$ and $q \leq s$, then $p \diamondsuit q \leq r \diamondsuit s$,
(c) $c \diamondsuit$ *is continuous,*
(d) $\diamondsuit$ *commutative and associative.*

Definition 2.7 (Kirişçi and Şimşek [16]). Assume that F is a vector space, $\mathcal{N}=\{\langle u, \Phi(u), \Psi(u)\rangle \Theta(u) u \in F\}$ is a normed space (NS) so that $\mathcal{N}: F \times \mathbb{R}^{+} \to [0,1]$. While following statements supply, $V = (F, \mathcal{N}, \triangle, \diamondsuit)$ is named to be NNS. For every $u, v \in F$ and $\lambda, \mu > 0$ and for all $\sigma \neq 0$,

(a) $0 \leq \Phi(u,\lambda) \leq 1$, $0 \leq \Psi(u,\lambda) \leq 1$, $0 \leq \Theta(u,\lambda) \leq 1$ $\forall \lambda \in \mathbb{R}^{+}$,

(b) $\Phi(u,\lambda) + \Psi(u,\lambda) + \Theta(u,\lambda) \leq 3$ (for $\lambda \in \mathbb{R}^{+}$),

(c) $\Phi(u,\lambda) = 1$ (for $\lambda > 0$) if an only if $u = 0$,

(d) $\Phi(\sigma u, \lambda = \Phi\big(u, \frac{\lambda}{|\sigma|}\big)$,

(e) $\Phi(u,\mu) \triangle \Phi(v,\lambda) \leq \Phi(u+v, \mu+\lambda)$,

(f) $\Phi(u,.)$ *is non-decreasing continuous function,*

(g) $\lim_{\lambda\to\infty} \Phi(u,\lambda) = 1$,

(h) $\Psi(u,\lambda) = 0$ (for $\lambda > 0$) if and only if $u = 0$,

(i) $\Psi(\sigma u, \lambda) = \Psi\big(u, \frac{\lambda}{bl|\sigma|}\big)$,

(j) $\Psi(u,\mu) \diamondsuit \Psi(v,\lambda) \geq \Psi(u+v, \mu+\lambda)$,

(k) $\Psi(u,.)$ *is non-decreasing continuous function,*

(l) $\lim_{\lambda\to\infty} \Psi(u,\lambda) = 0$,

(m) $\Theta(u,\lambda) = 0$ (for $\lambda > 0$) if an only if $u = 0$,

(n) $\Theta(\sigma u, \lambda) = \Theta\big(u, \frac{\lambda}{|\sigma|}\big)$,

(o) $\Theta(u,\mu) \diamondsuit \Theta(v,\lambda) \geq \Theta(u+v, \mu+\lambda)$,

(p) $\Theta(u,.)$ *is non-decreasing continuous function,*

(r) $\lim_{\lambda\to\infty} \Theta(u,\lambda) = 0$,

(s) If $\lambda \leq 0$, then $\Phi(u,\lambda) = 0, \Psi(u,\lambda) = 1$ and $\Theta(u,\lambda) = 1$.

Then, $\mathcal{N} = (\Phi, \Psi, \Theta)$ (is named neutrosophic norm (NN)).

Definition 2.8 (Kirişçi and Şimşek [16]). Assume V is an NNS, the sequence (ϕ_s) in V, $\varepsilon \in (0,1)$ and $\lambda > 0$. Then, the sequence (ϕ_s) is Cauchy in NNS V if there is a $N \in N$ such that $\Phi(\phi_s - \phi_r, \lambda) > 1 - \varepsilon$, $\Psi(\phi_s - \phi_r, \lambda) < \varepsilon$, $\Theta(\phi_s - \phi_r, \lambda) < \varepsilon$ for $s, r \geq N$.

Definition 2.9 (Kirişçi and Şimşek [16]). A sequence $\phi = (\phi_s)$ is named to be statistically convergent to $\alpha \in F$ w.r.t NN (Φ, Ψ, Θ), provided that for all $\sigma > 0$ and $\mu \in (0,1)$ the set

$$T_\mu := \big\{ s \leq p : \Phi(\phi_s - \alpha, \sigma) \leq 1 - \mu \ \text{ or } \ \Psi(\phi_s - \alpha, \sigma) \geq \mu,\ \Theta(\phi_s - \alpha, \sigma) \geq \mu \big\}$$

has natural density zero. Namely, $d(T_\mu) = 0$ or

$$\lim_{p\to\infty} \frac{1}{p} \Big| \big\{ s \leq p : \Phi(\phi_s - \alpha, \sigma) \leq 1 - \mu \ \text{ or } \ \Psi(\phi_s - \alpha, \sigma) \geq \mu,\ \Theta(\phi_s - \alpha, \sigma) \geq \mu \big\} \Big| = 0.$$

It is indicated by $S_{\mathcal{N}}\text{-}\lim \phi_s = \alpha$ or $\phi s \to \alpha(S_{\mathcal{N}})$.

3. Main results

In this section, we present our findings. We begin with the following definitions which play a key role throughout the paper.

Definition 3.1. A sequence $\phi = (\phi_s) \in l_\infty$ is named to be tribonacci $\mathcal{I}$-statistical convergent to $\alpha \in \mathbb{R}$, provided that for all γ, $\mu > 0$

$$T_1 := \left\{ p \in \mathbb{N} : \tfrac{1}{p} \big| \{ s \le p, j \in \mathbb{N} : |A_j(\phi_s) - \alpha| \ge \mu \} \big| \ge \gamma \right\} \in \mathcal{I}.$$

Definition 3.2. A sequence $\phi = (\phi_s) \in l_\infty$ is named to be tribonacci $\mathcal{I}$-lacunary statistical convergent to $\alpha \in \mathbb{R}$, provided that for all γ, $\mu > 0$

$$T_1 := \left\{ p \in \mathbb{N} : \tfrac{1}{h_p} \big| \{ s \in I_p, j \in \mathbb{N} : |A_j(\phi_s) - \alpha| \ge \mu \} \big| \ge \gamma \right\} \in \mathcal{I}.$$

Definition 3.3. A sequence $\phi = (\phi_s) \in l_\infty$ is named to be neutrosophic tribonacci $\mathcal{I}$-lacunary statistical convergent to $\alpha \in \mathbb{R}$ w.r.t. NN (Φ, Ψ, Θ), provided that for all σ, $\gamma > 0$ and $\mu \in (0, 1)$

$$K_1 := \Big\{ p \in \mathbb{N} : \tfrac{1}{h_p} \Big| \Big\{ s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha, \sigma\big) \le 1 - \mu \ \ or$$
$$\Psi\big(A_j(\phi_s) - \alpha, \sigma\big) \ge \mu, \Theta\big(A_j(\phi_s) - \alpha, \sigma\big) \ge \mu \Big\} \Big| \ge \gamma \Big\} \in \mathcal{I}.$$

We express this as

$$S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A){-}\lim \phi_s = \alpha \quad \text{or} \quad \phi_s \to \alpha \left(S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) \right).$$

Now, we present the following sequence spaces:

$$\begin{aligned} T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A) \quad := \quad & \Big\{ \phi = (\phi_s) \in l_\infty : \Big\{ p \in \mathbb{N} : \tfrac{1}{h_p} \Big| \Big\{ s \in I_p, j \in \mathbb{N} : \\ & \Phi\big(A_j(\phi_s) - \alpha, \sigma\big) \le 1 - \mu \\ & \text{or} \\ & \Psi\big(A_j(\phi_s) - \alpha, \sigma\big) \ge \mu, \\ & \Theta\big(A_j(\phi_s) - \alpha, \sigma\big) \ge mu \} \big| \ge \gamma \Big\} \in \mathcal{I} \Big\}, \end{aligned}$$

$$\begin{aligned} T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A) \quad := \quad & \Big\{ \phi = (\phi_s) \in l_\infty : \Big\{ p \in \mathbb{N} : \tfrac{1}{h_p} \Big| \Big\{ s \in I_p, j \in \mathbb{N} : \\ & \Phi\big(A_j(\phi_s), \sigma\big) \le 1 - \mu \\ & \text{or} \\ & \Psi\big(A_j(\phi_s), \sigma\big) \ge \mu, \\ & \Theta\big(A_j(\phi_s), \sigma\big) \ge \mu \} \big| \ge \gamma \Big\} \in \mathcal{I} \Big\}, \end{aligned}$$

$$
\begin{aligned}
T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A) \quad := \quad & \Big\{\Theta = (\Theta_m) \in l_\infty : \Big\{p \in \mathbb{N} : \exists\zeta \in (0,1), \tfrac{1}{h_p}\big|\big\{s \in I_p, \\
& j \in \mathbb{N} : \Phi\big(A_j(\phi_s), \sigma\big) \leq 1-\zeta \\
& \text{or} \\
& \Psi\big(A_j(\phi_s), \sigma\big) \geq \zeta, \Theta\big(A_j(\phi_s), \sigma\big) \geq \zeta\big\}\big| \geq \gamma\Big\} \in \mathcal{I}\Big\}.
\end{aligned}
$$

We put forward an open ball and a closed ball with center at ϖ and radius $\sigma > 0$ w.r.t. neutrosophic $\mu \in (0,1)$ and $\gamma > 0$ parameters, indicated by

$$
\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) \quad \text{and} \quad \mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)
$$

as follows:

$$
\begin{aligned}
\mathcal{B}_{\varpi}^{S_\theta^{(\Phi\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A) \quad := \quad & \Big\{q = (q_s) \in l_\infty : \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big|\big\{s \in I_p, \\
& j \in \mathbb{N} : \Phi\big(A_j(\varpi) - A_j(q), \sigma\big) \leq 1-\mu \\
& \text{or} \\
& \Psi\big(A_j(\varpi) - A_j(q), \sigma\big) \geq \mu, \\
& \Theta\big(A_j(\varpi) - A_j(q), \sigma\big) \geq \mu\big\}\big| \geq \gamma\Big\} \in \mathcal{I}\Big\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A) \quad := \quad & \Big\{q = (q_s) \in l_\infty : \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big|\big\{s \in I_p, \\
& j \in \mathbb{N} : \Phi\big(A_j(\varpi) - A_j(q), \ \sigma\big) < 1-\mu \\
& \text{or} \\
& \Psi\big(A_j(\varpi) - A_j(q), \ \sigma\big) > \mu, \\
& \Theta\big(A_j(\varpi) - A_j(q), \sigma\big) > \mu\big\}\big| > \gamma\Big\} \in \mathcal{I}\Big\}.
\end{aligned}
$$

If $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$, then $\phi = (\phi_s)$ converges to some $\alpha \in \mathbb{R}$, indicated by $(\phi_s) \overset{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)}{\longrightarrow} \alpha$ and in that case we denote $S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) - \lim \phi_s = \alpha$.

Lemma 3.4. *Consider the space $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Let $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Then for all $\eta \in (0,1)$ and $\sigma, \gamma > 0$, the following statements are equivalent:*

(i) $$S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) - \lim \phi_s = \alpha,$$

(ii) $$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\varpi) - \alpha, \ \sigma\big) \leq 1-\mu \quad \textit{or} \quad \Psi\big(A_j(\varpi) - \alpha, \ \sigma\big) \geq \mu, \ \Theta\big(A_j(\varpi) - \alpha, \ \sigma\big) \geq \mu\big\}\big| \geq \gamma\Big\} \in \mathcal{I},$$

(iii) $$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\varpi) - \alpha, \ \sigma\big) > 1-\mu \quad \textit{and} \quad \Psi\big(A_j(\varpi) - \alpha, \sigma\big) < \mu, \ \Theta\big(A_j(\varpi) - \alpha, \sigma\big) < \mu\big\}\big| < \gamma\Big\} \in \mathcal{F}(\mathcal{I})$$

$$\text{(iv)}\qquad \begin{aligned} S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)-\lim \Phi\big(A_j(\varpi)-\alpha,\ \sigma\big) &= 1 \\ \textit{and} \\ S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)-\lim \Psi\big(A_j(\varpi)-\alpha,\ \sigma\big) &= 0, \\ S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)-\lim \Theta\big(A_j(\varpi)-\alpha,\ \sigma\big) &= 0. \end{aligned}$$

Theorem 3.5. *The inclusion relation*

$$T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)\subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)\subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A)$$

supplies.

Proof. It is obvious that $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)\subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Then, we have to illustrate that $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)\subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A)$. Contemplate $\phi=(\phi_s)\in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Then, there is an $\alpha\in\mathbb{R}$ such that $S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)-\lim\phi_s=\alpha$. So, for all $\eta\in(0,1)$ and $\sigma,\gamma>0$ the set

$$K:=\Bigg\{p\in\mathbb{N}:\tfrac{1}{h_p}\Bigg\|\Bigg\{s\in I_p, j\in\mathbb{N}:\Phi\Big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\Big)>1-\eta\quad\text{and}$$
$$\Psi\Big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\Big)<\mu,\Theta\Big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\Big)<\mu\Bigg\}\Bigg\|<\gamma\Bigg\}\in\mathcal{F}(\mathcal{I}).$$

Assume

$$\Phi\Big(\alpha,\frac{\sigma}{2}\Big)=p,\quad \Psi\Big(\alpha,\frac{\sigma}{2}\Big)=q\quad\text{and}\quad \Theta\Big(\alpha,\frac{\sigma}{2}\Big)=r\quad\text{for all }\ \sigma>0.$$

Since $p,q,r\in(0,1)$ and $\eta\in(0,1)$, there are $s_1,s_2,s_3\in(0,1)$ such that $(1-\eta)\triangle p>1-s_1$, $\eta\diamondsuit q<s_2$ and $\eta\diamondsuit r<s_3$. So, we obtain

$$\begin{aligned}\Phi\big(A_j(\phi_s),\sigma\big) &= \Phi\big(A_j(\phi_s)-\alpha+\alpha,\ \sigma\big)\\ &\geq \Phi\big(A_j(\phi_s)-\alpha\tfrac{\sigma}{2}\big)\triangle\Phi\big(\alpha,\tfrac{\sigma}{2}\big)\\ &\geq (1-\eta)\triangle p\\ &> 1-s_1,\end{aligned}$$

$$\begin{aligned}\Psi\big(A_j(\phi_s),\ \sigma\big) &= \Psi\big(A_j(\phi_s)-\alpha+\alpha,\ \sigma\big)\\ &\leq \Psi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\diamondsuit\Psi\big(\alpha,\tfrac{\sigma}{2}\big)\\ &< \eta\diamondsuit q\\ &< s_2,\end{aligned}$$

$$\begin{aligned}\Theta\big(A_j(\phi_s),\ \sigma\big) &= \Theta\big(A_j(\phi_s)-\alpha+\alpha,\ \sigma\big)\\ &\leq \Theta\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\diamondsuit\Theta\big(\alpha,\tfrac{\sigma}{2}\big)\\ &< \eta\diamondsuit r\\ &< s_3.\end{aligned}$$

Getting $s = \max\{s_1, s_2, s_3\}$, then we obtain

$$\Big\{p \in \mathbb{N} : \exists\, s \in (0,1), \tfrac{1}{h_p}\big\|\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s), \sigma\big) > 1 - s \quad \text{and} \\ \Psi\big(A_j(\phi_s), \sigma\big) < s, \Theta\big(A_j(\phi_s), \sigma\big) < \eta\}\big\| < s\Big\} \in \mathcal{F}(\mathcal{I}).$$

Hence, $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A)$. This gives that

$$T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A) \subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A). \qquad \square$$

The converse of the inclusion relation does not supply. We establish the following example in support of our claim.

Example. Assume $(\mathbb{R}, \|\cdot\|)$ is a normed space such that $\|\phi\| = \sup_s \|\phi_s\|$, $u\triangle v = \min\{u, v\}$ and $u\diamondsuit v = \max\{u, v\}$, $\forall u, v \in (0,1)$. Now, determine norms $\mathcal{H} = (\Phi, \Psi, \Theta)$ on $\mathbb{R}^2 \times (0, \infty)$ as follows:

$$\Phi(\phi, \sigma) = \frac{\sigma}{\sigma + \|\phi\|}, \quad \Psi(\phi, \sigma) = \frac{\|\phi\|}{\sigma + \|\phi\|} \quad \text{and} \quad \Theta(\phi, \sigma) = \frac{\|\phi\|}{\sigma}.$$

Then, $\big(\mathbb{R}, \mathcal{H}, \triangle, \diamondsuit\big)$ is an NNS. Contemplate the sequence $(\phi_s) = \{1\}$. It can be easily examined that $(\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ and $S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) - \lim \phi_s = 1$, however $(\phi_s) \notin T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$.

Example. Assume $(\mathbb{R}, \|\cdot\|)$ is a normed space and (Φ, Ψ, Θ) are the neutrosophic fuzzy norms as defined in the above example. Contemplate the sequence $(\phi_s) = (-1)^s$. Then, $(\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}^\infty)}(A)$, however, $(\phi_s) \notin T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$.

Now, we provide a detailed examination of the algebraic and topological properties of the spaces $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$ and $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$.

Theorem 3.6. *The spaces $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$ and $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ are linear spaces, respectively.*

Proof. It is clear that

$$T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A) \subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A).$$

So, we have to illustrate the result for

$$T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A).$$

The proof of linearity of the space $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$ follows in the same way. Presume sequences $\phi = (\phi_s)$, $q = (q_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Also, suppose

$$S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) - \lim \phi_s = \alpha_1 \quad \text{and} \quad S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A) - \lim q_s = \alpha_2.$$

We should denote that for any scalars λ and ρ the sequence $(\lambda\phi_s + \rho q_s)$ neutrosophic tribonacci $\mathcal{I}$-lacunary statistically converges to $\lambda\alpha_1 + \rho\alpha_2$. For $\sigma, \gamma > 0$ and $\eta \in (0,1)$, take the subsequent sets;

$$T_1 := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) \le 1 - \eta \quad \text{or} \\ \Psi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) \ge \eta,\ \ \Theta\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) \ge \eta\big\}\big\| \ge \gamma\Big\} \in \mathcal{I};$$

$$T_2 := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) \le 1 - \eta. \quad \text{or} \\ \Psi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) \ge \eta,\ \Theta\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) \ge \eta\big\}\big\| \ge \gamma\Big\} \in \mathcal{I}.$$

So, we can write

$$T_1^c := \Big\{p \in \mathbb{N} : \frac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) > 1 - \eta \quad \text{and} \\ \Psi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) < \eta,\ \ \Theta\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) < \eta\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I});$$

$$T_2^c := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p,\ \ j \in \mathbb{N} : \Phi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) > 1 - \eta \quad \text{and} \\ \Psi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) < \eta,\ \ \Theta\big(A_j(q_s) - \alpha_2, \frac{\sigma}{2\|\rho\|}\big) < \eta\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

Therefore, the set $T = T_1^c \cap T_2^c$ is non-empty and $T \in \mathcal{F}(\mathcal{I})$. Take $j \in T$, then

$$\Phi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) > 1 - \eta \quad \text{and} \quad \Psi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) < \eta, \\ \Theta\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) < \eta,$$

and

$$\Phi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) > 1 - \eta \quad \text{and} \quad \Psi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) < \eta, \\ \Theta\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) < \eta.$$

So, we can write

$$\begin{aligned} &\Phi\big(A_j(\lambda\phi_s + \rho q_s) - (\lambda\alpha_1 + \rho\alpha_2),\ \sigma\big) \\ &\ge \Phi\big(\lambda A_j(\phi_s) - \lambda\alpha_1, \tfrac{\sigma}{2}\big) \triangle \Phi\big(\rho A_j(q_s) - \rho\alpha_2, \tfrac{\sigma}{2}\big) \\ &= \Phi\big(A_j(\phi_s) - \alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big) \triangle \Phi\big(A_j(q_s) - \alpha_2, \tfrac{\sigma}{2\|\rho\|}\big) \\ &> (1 - \eta) \triangle (1 - \eta) \\ &> 1 - \eta. \end{aligned}$$

So, we get

$$\Phi\big(A_j(\lambda\phi_s + \rho q_s) - (\lambda\alpha_1 + \rho\alpha_2),\ \sigma\big) > 1 - \eta.$$

Also,

$$\begin{aligned}
&\Psi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\ \sigma\big)\\
&\le \Psi\big(\lambda A_j(\phi_s)-\lambda\alpha_1, \tfrac{\sigma}{2}\big)\diamondsuit\Psi\big(\rho A_j(q_s)-\rho\alpha_2, \tfrac{\sigma}{2}\big)\\
&=\Psi\big(A_j(\phi_s)-\alpha_1, \tfrac{\sigma}{2\|\lambda\|}\big)\diamondsuit\Psi\big(A_j(q_s)-\alpha_2, \tfrac{\sigma}{2\|\rho\|}\big)\\
&<\eta\diamondsuit\eta<\eta.
\end{aligned}$$

At that time, we get

$$\Psi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta.$$

In addition, we have

$$\Theta\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta.$$

Hence,

$$\begin{aligned}
T\ \subset\ \Big\{&p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\\
&\Phi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)>1-\eta\quad\text{and}\\
&\Psi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta,\\
&\Theta\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta\big\}\big\|<\gamma\Big\}.
\end{aligned}$$

Since $T\in\mathcal{F}(\mathcal{I})$. Hence,

$$\begin{aligned}
\Big\{&p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\\
&\Phi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)>1-\eta\quad\text{and}\\
&\Psi\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta,\\
&\Theta\big(A_j(\lambda\phi_s+\rho q_s)-(\lambda\alpha_1+\rho\alpha_2),\sigma\big)<\eta\big\}\big\|<\gamma\Big\}\in\mathcal{F}(\mathcal{I}),
\end{aligned}$$

which yields that the sequence $(\lambda\phi_s+\rho q_s)$ neutrosophic tribonacci $\mathcal{I}$-lacunary statistically converges to $\lambda\alpha_1+\rho\alpha_2$, i.e.,

$$S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})(A)-\lim(\lambda\phi_s+\rho q_s)=\lambda\alpha_1+\rho\alpha_2.$$

So,

$$(\lambda\phi_s+\rho q_s)\in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A).$$

As a result, we get $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ is a linear space. □

Theorem 3.7.

Every open ball $\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(r,\sigma,\gamma)(A)$ is an open set in $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ with respect to the norms (Φ,Ψ,Θ).

P r o o f. Assume $\mathcal{B}_{\varpi}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(r,\sigma,\gamma)(A)$ is an open ball with center at ϖ and radius $\sigma>0$ w.r.t. neutrosophic $\mu\in(0,1)$ and $\gamma>0$ parameters,

$$\begin{aligned}\mathcal{B}_{\varpi}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A):=\Big\{&q=(q_s)\in l_\infty:\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\\&\Phi\big(A_j(\varpi)-A_j(q),\sigma\big)\le 1-\mu \quad \text{or}\\&\Psi\big(A_j(\varpi)-A_j(q),\sigma\big)\ge\mu,\\&\Theta\big(A_j(\varpi)-A_j(q),\sigma\big)\ge\mu\big\}\big\|\ge\gamma\Big\}\in\mathcal{I}\Big\}.\end{aligned}$$

Then,

$$\begin{aligned}\Big(\mathcal{B}_{\varpi}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma,\mu,\gamma)(A):=\Big\{&q=(q_s)\in l_\infty:\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\\&\Phi\big(A_j(\varpi)-A_j(q),\sigma\big)> 1-\mu \quad \text{and}\\&\Psi\big(A_j(\varpi)-A_j(q),\sigma\big)<\mu,\\&\Theta\big(A_j(\varpi)-A_j(q),\sigma\big)<\mu\big\}\big\|<\gamma\Big\}\in\mathcal{F}(\mathcal{I})\Big\}.\end{aligned}$$

Let $q=(q_s)\in\Big(\mathcal{B}_{\varpi}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma,\mu,\gamma)(A)$. Then, for

$$\Phi\big(A_j(\varpi)-A_j(q),\sigma\big)>1-\mu \quad \text{and} \quad \Psi\big(A_j(\varpi)-A_j(q),\sigma\big)<\mu,$$
$$\Theta\big(A_j(\varpi)-A_j(q),\sigma\big)<\mu,$$

so there exists a $\sigma_0\in\big(0,\sigma\big)$ such that

$$\Phi\big(A_j(\varpi)-A_j(q),\sigma_0\big)>1-\mu \quad \text{and} \quad \Psi\big(A_j(\varpi)-A_j(q),\sigma_0\big)<\mu,$$
$$\Theta\big(A_j(\varpi)-A_j(q),\sigma_0\big)<\mu.$$

Putting $\mu_0=\Phi(A_j(\phi_s)-A_j(q_s\big),\sigma_0)$, implies $\mu_0>1-\mu$.

Then, $\exists t\in(0,1)$ so that $\mu_0>1-t>1-\mu$. For $\mu_0>1-t$, we get $\mu_1,\mu_2,\mu_3\in(0,1)$ such that $\mu_0\triangle\mu_1>1-t$, $(1-\mu_0)\diamondsuit\big(1-\mu_2\big)<t$ and $(1-\mu_0)\diamondsuit(1-\mu_3)<t$. Take $\mu_4=\max\{\mu_1,\mu_2,\mu_3\}$.

Now, consider the open ball

$$\Big(\mathcal{B}_{q}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma-\sigma_0,1-\mu_4,\gamma)(A).$$

We have to indicate that

$$\Big(\mathcal{B}_{q}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma-\sigma_0,1-\mu_4,\gamma)(A)\subset\Big(\mathcal{B}_{\varpi}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\mu,\sigma,\gamma)(A).$$

Take $x=(x_s)\in\Big(\mathcal{B}_{q}^{S_{\theta}^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma-\sigma_0,1-\mu_4,\gamma)(A)$. Then,

$$\Phi\big(A_j(q)-A_j(x),\sigma-\sigma_0\big)>\mu_4 \quad \text{and} \quad \Psi\big(A_j(q)-A_j(x),\sigma-\sigma_0\big)<1-\mu_4,$$

$$\Theta\big(A_j(q)-A_j(x),\sigma-\sigma_0\big)<-\mu_4.$$

So we get

$$\begin{aligned}\Phi\big(A_j(\varpi)-A_j(x),\,\sigma\big) &\geq \Phi\big(A_j(\varpi)-A_j(q),\,\sigma_0\big)\\ &\triangle\,\Phi\big(A_j(q)-A_j(x),\,\sigma-\sigma_0\big)\Phi\big(A_j(\varpi)-A_j(q),\,\sigma_0\big)\\ &\geq \mu_0\triangle\mu_4 \geq \mu_0\triangle\mu_1\Phi\big(A_j(\varpi)-A_j(q),\,\sigma_0\big)\\ &> 1-t > 1-\mu,\end{aligned}$$

$$\begin{aligned}\Psi\big(A_j(\varpi)-A_j(x),\,\sigma\big) &\leq \Psi\big(A_j(\varpi)-A_j(q),\,\sigma_0\big)\\ &\diamondsuit\,\Psi\big(A_j(q)-A_j(x),\,\sigma-\sigma_0\big)\\ &< (1-\mu_0)\diamondsuit(1-\mu_4) \leq \mu_0\diamondsuit\mu_2\\ &< t < \mu.\end{aligned}$$

and

$$\begin{aligned}\Theta\big(A_j(\varpi)-A_j(x),\,\sigma\big) &\leq \Theta\big(A_j(\varpi)-A_j(q),\,\sigma_0\big)\\ &\diamondsuit\,\Theta\big(A_j(q)-A_j(x),\,\sigma-\sigma_0\big)\\ &< \mu_0\triangle\mu_4 \leq \mu_0\diamondsuit\mu_3\\ &< t < \mu.\end{aligned}$$

So, we obtain

$$\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big|\big\{s\in I_p:\Phi\big(A_j(\varpi)-A_j(x),\,\sigma\big)>1-\mu \quad\text{and}$$
$$\Psi\big(A_j(\varpi)-A_j(x),\sigma\big)<\mu,\Theta\big(A_j(\varpi)-A_j(x),\,\sigma\big)<\mu\big\}\big|<\gamma\Big\}\in\mathcal{F}(\mathcal{I}).$$

Therefore,

$$x=(x_s)\in\Big(\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma,\mu,\gamma)(A).$$

So, we have

$$\Big(\mathcal{B}_{q}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\sigma-\sigma_0,1-\mu_4,\gamma)\subset\Big(\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c(\mu,\,\sigma,\gamma)(A).$$

As a result, every open ball

$$\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) \quad\text{is an open set in}\quad T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$$

w.r.t. NN (Φ,Ψ,Θ). □

Remark 1. The spaces $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ and $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$ are tribonacci $\mathcal{I}$--lacunary statistical convergent and tribonacci $\mathcal{I}$-null lacunary statistical convergent in NNS w.r.t. NN (Φ,Ψ,Θ).

Now, we identify a collection $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}$ of subsets of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ as follows:

$$\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A) = \Big\{P \subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A) : \text{ for all } \varpi \in P$$

$$\text{there are } \sigma, \gamma > 0 \text{ and } \eta \in (0,1) \text{ so that } \mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) \subset P\Big\}.$$

Clearly, $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ is a topology on $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. The collection signified by

$$\mathcal{B} = \Big\{\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma) : \varpi \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A),\ \sigma,\gamma > 0 \text{ and } \eta \in (0,1)\Big\}$$

is the basis for the topology $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ on $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$.

Theorem 3.8. *The topology $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ on the space of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ is first countable.*

Proof. For all $\varpi \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$, establish the set

$$\mathcal{B} = \left\{\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\left(\frac{1}{q},\frac{1}{q},\frac{1}{q}\right) : q = 1,2,3,4,\dots\right\},$$

which is a countable local base at ϖ. Hence, the topology

$$\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A) \text{ on } T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$$

is first countable. □

Theorem 3.9. *The spaces $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A)$ and $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ are Hausdorff spaces.*

Proof. It is clear that $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I}_0)}(A) \subset T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. We have to prove the result for only $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Let $\phi = (\phi_s)$, $q = (q_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ so that $\phi \neq q$. Then, for all $j \in \mathbb{N}$ and $\sigma > 0$, we get

$$0 < \Phi(A_j(\phi) - A_j(q), \sigma) < 1 \quad \text{and} \quad 0 < \Psi\big(A_j(\phi) - A_j(q), \sigma\big) < 1,$$
$$0 < \Theta\big(A_j(\phi) - A_j(q), \sigma\big) < 1.$$

Let

$$r_1 = \Phi\big(A_j(\phi) - A_j(q), \sigma\big), \qquad r_2 = \Psi\big(A_j(\phi) - A_j(q), \sigma\big),$$

$$r_3 = \Theta\big(A_j(\phi) - A_j(q), \sigma\big),$$

and

$$r = \max\{r_1, 1 - r_2, 1 - r_3\}.$$

At this time, for all $r_0 \in (r,1)$ there are $r_4, r_5, r_6 \in (0,1)$, so that

$$r_4 \triangle r_4 \geq r_0, (1-r_5)\diamondsuit(1-r_5) \leq 1-r_0 \quad and \quad (1-r_6)\diamondsuit(1-r_6) \leq 1-r_0.$$

Again, let $r_7 = \max\{r_4, 1-r_5, 1-r_6\}$ and consider the open balls

$$\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) \quad \text{and} \quad \mathcal{B}_{q}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A)$$

centered at ϖ and q, respectively. Then, it is clear that

$$\Big(\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) \cap \Big(\mathcal{B}_{q}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) = \emptyset.$$

If possible, assume

$$\begin{aligned} x = (x_s) \in & \Big(\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) \\ & \cap \Big(\mathcal{B}_{q}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A). \end{aligned}$$

Then, we obtain

$$\begin{aligned} r_1 &= \Phi\big(A_j(\varpi) - A_j(q), \sigma\big) \\ &\geq \Phi\big(A_j(\varpi) - A_j(x), \tfrac{\sigma}{2}\big) \triangle \Phi\big(A_j(x) - A_j(q), \tfrac{\sigma}{2}\big) \\ &> r_7 \triangle r_7 \\ &\geq r_4 \triangle r_4 \\ &\geq r_0 > r_1, \end{aligned}$$

$$\begin{aligned} r_2 &= \Psi\big(A_j(\varpi) - A_j(q), \sigma\big) \\ &\leq \Psi\big(A_j(\varpi) - A_j(x), \tfrac{\sigma}{2}\big) \diamondsuit \Psi\big(A_j(x) - A_j(q), \tfrac{\sigma}{2}\big) \\ &< (1-r_7)\diamondsuit(1-r_7) \\ &\leq (1-r_5)\diamondsuit(1-r_5) \\ &< (1-r_0) < r_2. \end{aligned}$$

and

$$\begin{aligned} r_3 &= \Psi\big(A_j(\varpi) - A_j(q), \sigma\big) \\ &\leq \Psi\big(A_j(\varpi) - A_j(x), \tfrac{\sigma}{2}\big) \diamondsuit \Psi\big(A_j(x) - A_j(q), \tfrac{\sigma}{2}\big) \\ &< (1-r_7)\diamondsuit(1-r_7) \\ &\leq (1-r_6)\diamondsuit(1-r_6) \\ &< (1-r_0) < r_3. \end{aligned}$$

From the above equations, we obtain a contradiction. So,

$$\Big(\mathcal{B}_{\varpi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) \cap \Big(\mathcal{B}_{q}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big)^c\Big(1-r_7, \frac{\sigma}{2}, \gamma\Big)(A) = \emptyset.$$

As a result, the space $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ is a Hausdorff space. $\square$

Theorem 3.10. *Let $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ be a topology on an NNS $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Assume $(\phi^s) = (\phi_k^s)_{s=1}^\infty$ is a sequence of points in $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Then, $\phi^s \to \phi$ as $s \to \infty$ if and only if $\Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 1$, $\Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0$, $\Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0$ as $j \to \infty$.*

Proof. Assume $\phi^s \to \phi$ as $s \to \infty$. Fix $\sigma > 0$. Then, for $\mu \in (0,1)$, there is a $k \in \mathbb{N}$ so that $(\phi^s) \in \mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$, $\forall s \geq k$. Then

$$T_1 := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \leq 1 - \eta \quad \text{or} \\ \Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \geq \eta,\ \ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \geq \eta\big\}\big\| \geq \gamma\Big\} \in \mathcal{I},$$

which is equivalent to

$$T_1^c := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) > 1 - \eta \quad \text{and} \\ \Psi\big(A_j(\phi^s) - A_j\big(\phi\big), \sigma\big) < \eta, \quad \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

For

$$\{p \in \mathbb{N}, j \in \mathbb{N}\} \subseteq T_1^c, \quad \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) > 1 - \eta.$$

Then, we can write

$$1 - \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta \quad \text{and} \quad \Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta, \\ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta.$$

So, we have

$$\Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 1, \quad \Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0, \\ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0, \ \text{as} \ j \to \infty.$$

Conversely, presume that for all $\sigma > 0$,

$$\Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 1, \quad \Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0, \\ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) \to 0, \ \text{as} \ j \to \infty.$$

Then, for all $\mu \in (0,1)$, there exists a $k \in \mathbb{N}$ such that

$$1 - \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta \quad \text{and} \quad \Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta, \\ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta \quad \text{for all } j \geq k.$$

Contemplate the ideal $\mathcal{I}$ generated by the set $\{j \in \mathbb{N} : j < k\}$ means that the collection of sets generated by the set $\{j \in \mathbb{N} : j \geq k\} \in \mathcal{F}(\mathcal{I})$. Hence, we obtain

$$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) > 1-\eta \quad \text{and}$$
$$\Psi\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta,\ \ \Theta\big(A_j(\phi^s) - A_j(\phi), \sigma\big) < \eta\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

As a result, $(\phi^s) \in \mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$ for every $s \geq k$ and so,

$$\phi^s \to \phi \quad \text{as} \quad s \to \infty. \qquad \square$$

Theorem 3.11. *Let $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. a sequence $\phi = (\phi_s)$ is neutrosophic tribonacci $\mathcal{I}$-lacunary statistically convergent w.r.t NN (Φ,Ψ,Θ), then $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}\big)(A) - \lim \phi_s$ is unique.*

Proof. Presume $\phi = (\phi_s)$ is neutrosophic tribonacci $\mathcal{I}$-lacunary statistically convergent w.r.t NN (Φ,Ψ,Θ). Conversely, let α_1 and α_2 be two different elements so that $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big) - \lim \phi_s = \alpha_1$ and $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big) - \lim \phi_s = \alpha_2$. For a given $\eta \in (0,1)$, select $r > 0$ so that $\big(1-r\big)\triangle\big(1-r\big) > 1-\eta$, $r\diamondsuit r < \eta$. For $\sigma > 0$, we have to denote $\alpha_1 = \alpha_2$. We establish the subsequent sets:

$$T_1 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha_1, \sigma\big) \leq 1-\eta\big\}\big\|\big\},$$
$$T_2 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Psi\big(A_j(\phi_s) - \alpha_1, \sigma\big) \geq \eta\big\}\big\|\big\},$$
$$T_3 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Theta\big(A_j(\phi_s) - \alpha_1, \sigma\big) \geq \eta\big\}\big\|\big\},$$
$$R_1 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha_2, \sigma\big) \leq 1-\eta\big\}\big\|\big\},$$
$$R_2 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Psi\big(A_j(\phi_s) - \alpha_2, \sigma\big) \geq \eta\big\}\big\|\big\},$$
$$R_3 := \big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Theta\big(A_j(\phi_s) - \alpha_2, \sigma\big) \geq \eta\big\}\big\|\big\}.$$

Let
$$T = \big(T_1 \cup R_1\big) \cap \big(T_2 \cup R_2\big) \cap \big(T_3 \cup R_3\big).$$

It is clear that $T_1, T_2, T_3, R_1, R_2, R_3$ and T have to belong to $\mathcal{I}$, as $\phi = (\phi_s)$ has two different neutrosophic tribonacci $\mathcal{I}$-lacunary statistically limits w.r.t. NN $\big(\Phi,\Psi,\Theta\big)$, i.e., α_1 and α_2. Hence, $T^c \in \mathcal{F}(\mathcal{I})$. It is clear, T^c is non-empty. So, $j_0 \in T^c$ and also $j_0 \in T_1^c \cap R_1^c$, $j_0 \in T_2^c \cap R_2^c$ and $j_0 \in T_3^c \cap R_3^c$. Getting $j_0 \in T_1^c \cap R_1^c$, gives that

$$\Phi\Big(A_{j_0}(\phi_s) - \alpha_1, \frac{\sigma}{2}\Big) > 1-r \quad \text{and} \quad \Phi\Big(A_{j_0}(\phi_s) - \alpha_2, \frac{\sigma}{2}\Big) > 1-r.$$

Therefore, we get

$$\begin{aligned}\Phi(\alpha_1-\alpha_2,\sigma) &\geq \Phi\big(A_{j_0}(\phi_s)-\alpha_1,\tfrac{\sigma}{2}\big)\triangle\Phi\big(A_{j_0}(\phi_s)-\alpha_2,\tfrac{\sigma}{2}\big)\\ &> (1-r)\triangle(1-r)\\ &> 1-\eta.\end{aligned}$$

Since $\eta>0$ was arbitrary, so $\Phi\big(\alpha_1-\alpha_2,\sigma\big)=1$ for all $\eta>0$. So, we have $\alpha_1=\alpha_2$, which is a contradiction.

If $j_0\in T_2^c\cap R_2^c$, then we get

$$\Psi\Big(A_{j_0}(\phi_s)-\alpha_1,\frac{\sigma}{2}\Big)<\eta \quad\text{and}\quad \Psi\Big(A_{j_0}(\phi_s)-\alpha_2,\frac{\sigma}{2}\Big)<\eta.$$

Therefore, we have

$$\begin{aligned}\Psi(\alpha_1-\alpha_2,\sigma) &\leq \Psi\big(A_{j_0}(\phi_s)-\alpha_1,\tfrac{\sigma}{2}\big)\Diamond\Psi\big(A_{j_0}(\phi_s)-\alpha_2,\tfrac{\sigma}{2}\big)\\ &< r\Diamond r\\ &< \eta.\end{aligned}$$

Since $\eta>0$ was arbitrary, so $\Psi(\alpha_1-\alpha_2,\sigma)=0$ for all $\eta>0$. Then, we get $\alpha_1=\alpha_2$, which is a contradiction. For all cases, we get $\alpha_1=\alpha_2$. We demonstrate that $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big)-\lim\phi_s$ is unique. $\square$

Definition 3.12. A sequence $\phi=(\phi_s)\in l_\infty$ is said to be neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistically Cauchy w.r.t. NN (Φ,Ψ,Θ) provided that for each $\sigma,\gamma>0$ and $\eta\in(0,1)$, there is a $r\in\mathbb{N}$ such that the set K_2 belongs to $\mathcal{I}$, where

$$K_2:=\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\Phi\big(A_j(\phi_s)-A_r(\phi_s),\sigma\big)\leq 1-\eta \quad\text{or}$$
$$\Psi\big(A_j(\phi_s)-A_r(\phi_s),\sigma\big)\geq\eta,\ \ \Theta\big(A_j(\phi_s)-A_r(\phi_s),\sigma\big)\geq\eta\big\}\big\|\geq\gamma\Big\}\in\mathcal{I}.$$

Theorem 3.13. *A sequence $\phi=(\phi_s)$ is neutrosophic fuzzy Tribonacci $\mathcal{I}$--lacunary statistically convergent with respect to the NN (Φ,Ψ,Θ) if and only if it is neutrosophic fuzzy Tribonacci $\mathcal{I}$-lacunary statistically Cauchy with respect to the same norms.*

Proof. Assume $\phi=(\phi_s)$ is neutrosophic fuzzy tribonacci $\mathcal{I}$-statistically convergent w.r.t. NN (Φ,Ψ,Θ) so that $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big)-\lim\phi_s=\alpha$. For a given $\eta\in(0,1)$, there exists $r_1\in(0,1)$ such that $(1-r_1)\triangle(1-r_1)>1-\eta$, $r_1\Diamond r_1<\eta$. Since $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big)-\lim\phi_s=\alpha$, therefore for all $\sigma,\gamma>0$

$$K_1:=\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\Phi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\leq 1-r_1 \quad\text{or}$$
$$\Psi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\geq r_1,\Theta\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\geq r_1\big\}\big\|\geq\gamma\Big\}\in\mathcal{I};$$

that implies

$$K_1^c := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) > 1 - r_1 \quad \text{and}$$
$$\Psi\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < r_1, \Theta\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < r_1\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

For $r \in K_1^c$, we get

$$\Phi\big(A_r(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) > 1 - r_1 \quad \text{and} \quad \Psi\big(A_r(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < r_1,$$
$$\Theta\big(A_r(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < r_1.$$

Now, we denote that for $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ $\exists$ a natural number $r(\phi, \eta, \sigma)$ such that the set

$$K_2 := \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - A_r(\phi_s), \sigma\big) \leq 1 - \eta \quad \text{or}$$
$$\Psi\big(A_j(\phi_s) - A_r(\phi_s), \sigma\big) \geq \eta, \Theta\big(A_j(\phi_s) - A_r(\phi_s), \sigma\big) \geq \eta\big\}\big\| \geq \gamma\Big\} \in \mathcal{I}.$$

We have to demonstrate that $K_2 \subset K_1$. On the contrary, let $K_2 \nsubseteq K_1$. Take $j \in K_2$, but $j \notin K_1$, then we get

$$\Phi\big(A_j(\phi_s) - A_r(\phi_r), \sigma\big) \leq 1 - \eta.$$

Then $\Phi\big(A_j(\phi_s) - \alpha, \frac{\sigma}{2}\big) \geq 1 - r_1$. In particular, $\Phi\big(A_r(\phi_s) - \alpha, \frac{\sigma}{2}\big) \geq \eta$. Then, we can write

$$\begin{aligned} 1 - \eta &\geq \Phi\big(A_j(\phi_s) - A_r(\phi_r), \sigma\big) \\ &\geq \Phi\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) \triangle \Phi\big(A_r(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) \\ &> (1 - r_1) \triangle (1 - r_1) \\ &> 1 - \eta, \end{aligned}$$

which is a contradiction. So, $\Phi\big(A_j(\phi_s) - \alpha, \frac{\sigma}{2}\big) \leq 1 - r_1$ supplies. In the same way, consider $\Psi\big(A_j(\phi_s) - A_r(\phi_r), \sigma\big) \geq \eta$. Then, $\Psi\big(A_j(\phi_s) - \alpha, \frac{\sigma}{2}\big) < r_1$.

In particular, $\Psi\big(A_r(\phi_s) - \alpha, \frac{\sigma}{2}\big) < r_1$. Then, we can write

$$\begin{aligned} \eta &\leq \Psi\big(A_j(\phi_s) - A_r(\phi_r), \sigma\big) \\ &\leq \Psi\big(A_j(\phi_s) - \beta, \tfrac{\sigma}{2} \diamondsuit \Psi\big(A_r(\phi_s) - \beta, \tfrac{\sigma}{2}\big) \\ &< r_1 \diamondsuit r_1 \\ &< \eta, \end{aligned}$$

which is a contradiction. So, $\Psi\big(A_j(\phi_s) - \alpha, \frac{\sigma}{2}\big) \geq r_1$ supplies.

Similarly, consider $\Theta\big(A_j(\phi_s)-A_r(\phi_r),\,\sigma\big)\geq\eta$. Then, $\Theta\big(A_j(\phi_s)-\beta,\frac{\sigma}{2}\big)<r_1$. If possible, let $\Theta\big(A_r(\phi_s)-\alpha,\frac{\sigma}{2}\big)<r_1$. Then, we can write

$$\begin{aligned}\eta &\leq \Theta\big(A_j(\phi_s)-A_r(\phi_r),\,\sigma\big)\\ &\leq \Theta\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\Diamond\Theta\big(A_r(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\\ &< r_1\Diamond r_1\\ &< \eta,\end{aligned}$$

which is a contradiction. So, $\Theta\big(A_j(\phi_s)-\alpha,\frac{\sigma}{2}\big)\geq r_1$ holds. So, for $j\in K_2$, we obtain

$$\Psi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\leq 1-r_1 \quad \text{or} \quad \Psi\big(A_j(\phi_s)-\beta,\tfrac{\sigma}{2}\big)\geq r_1,$$

$$\Theta\big(A_j(\phi_s)-\beta,\tfrac{\sigma}{2}\big)\geq r_1.$$

Therefore, $j\in K_1$. Hence, $K_2\subset K_1$. Since $K_1\in\mathcal{I}$, so we get $K_2\in\mathcal{I}$. As a result, $\phi=\big(\phi_s\big)$ is neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistically Cauchy w.r.t. NN (Φ,Ψ,Θ). Conversely, assume the sequence $\phi=(\phi_s)$ is neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistically Cauchy w.r.t. NN $\big(\Phi,\Psi,\Theta\big)$. On the contrary, let the sequence $\phi=(\phi_s)$ be not neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistically convergent indicated by $\mathcal{S}_2$Then, there exists $s\in\mathbb{N}$ so that

$$\mathcal{S}_1:=\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\Phi\big(A_j(\phi_s)-A_r(\phi_s),\,\sigma\big)\leq 1-\eta \quad \text{or}$$
$$\Psi\big(A_j(\phi_s)-A_r\big(\phi_s\big),\,\sigma\big)\geq\eta,\Theta\big(A_j(\phi_s)-A_r(\phi_s),\,\sigma\big)\geq\eta\big\}\big\|\geq\gamma\Big\}\in\mathcal{I}.$$

Conversely, let

$$\mathcal{S}_2:=\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p, j\in\mathbb{N}:\Phi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)>1-r_1 \quad \text{and}$$
$$\Psi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)<r_1,\Theta\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)<r_1\big\}\big\|<\gamma\Big\}\in\mathcal{I}.$$

So, we have

$$\begin{aligned}1-\eta &\geq \Phi\big(A_j(\phi_s)-A_r(\phi_s),\sigma\big)\\ &\geq \Phi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\triangle\Phi\big(A_r(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\\ &> (1-r_1)\triangle(1-r_1)\\ &> 1-\eta\end{aligned}$$

which is a contradiction. Now,

$$\begin{aligned}\eta &\leq \Psi\big(A_j(\phi_s)-A_r(\phi_s),\,\sigma\big)\\ &\leq \Psi\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\Diamond\Psi\big(A_r(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\\ &< r_1\Diamond r_1\\ &< \eta,\end{aligned}$$

which is a contradiction. Also,

$$\begin{aligned} \eta &\le \Theta\big(A_j(\phi_s)-A_r(\phi_s),\,\sigma\big)\\ &\le \Theta\big(A_j(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\Diamond\Theta\big(A_r(\phi_s)-\alpha,\tfrac{\sigma}{2}\big)\\ &< r_1\Diamond r_1\\ &< \eta, \end{aligned}$$

which is a contradiction. So, $\mathcal{S}_2\in\mathcal{F}\big(\mathcal{I}\big)$ and, as a result, $\phi=(\phi_s)$ is neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistically convergent w.r.t. $\mathrm{NN}\big(\Phi,\Psi,\Theta\big)$. $\square$

Definition 3.14. A subset P of l^{∞} is named to be neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistical bounded w.r.t. NN $\big(\Phi,\Psi,\Theta\big)$, if $\forall\phi=(\phi_s)\in P$, $\eta\in(0,1)$ and $\sigma,\gamma>0$ such that

$$\Big\{\phi=(\phi_s)\in l_\infty:\Big\{p\in\mathbb{N}:\exists\eta\in(0,1):\tfrac{1}{h_p}\big\|\big\{s\in I_p,j\in\mathbb{N}:\Phi\big(A_j(\phi_s),\,\sigma\big)\le 1-\eta\ \text{ or }\\ \Psi\big(A_j(\phi_s),\,\sigma\big)\ge\eta,\Theta\big(A_j(\phi_s),\,\sigma\big)\ge\eta\big\}\big\|\ge\gamma\Big\}\in\mathcal{I}\Big\}.$$

Now, $T^{f}_{l^{\infty}_{(\Phi,\Psi,\Theta)}}\big(S_\theta(\mathcal{I})\big)$ shows the space of all neutrosophic fuzzy tribonacci $\mathcal{I}$-lacunary statistical bounded sequences.

Theorem 3.15. *Every closed ball with centre at ϖ and radius $\sigma>0$ w.r.t. parametres of fuzziness $\eta\in(0,1)$, i.e., $\mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)$, is a closed set in $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$.*

Proof. Assume $q=(q_s)\in l_\infty$ is such that

$$q\in\overline{\mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)}.$$

Then, there is a sequence $(q^n)=(q^n_s)\in\mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)$, so that $q^n\to q$ as $n\to\infty$. This yields the set

$$T:=\Big\{p\in\mathbb{N}:\tfrac{1}{h_p}\big\|\big\{s\in I_p,j\in\mathbb{N}:\Phi\big(A_j(q^n_s)-A_j(\phi),\,\sigma\big)>1-\mu\quad\text{and}\\ \Psi\big(A_j(q^n_s)-A_j(\phi),\,\sigma\big)<\mu,\Theta\big(A_j(q^n_s)-A_j(\phi),\,\sigma\big)<\mu\big\}\big\|<\gamma\Big\}\in\mathcal{F}(\mathcal{I}).$$

Since $q^n\to q$ as $n\to\infty$,

$$\Phi\big(A_j(q^n)-A_j(q),u\big)\to 1\quad\text{and}\quad\Psi\big(A_j(q^n)-A_j(q),u\big)\to 0,$$

$$\Theta\big(A_j(q^n)-A_j(q),u\big)\to 0,$$

for all $u>0$ as $j\to\infty$.

So, for $j \in T$

$$\begin{aligned}\Phi\big(A_j(\phi) - A_j(q), u+\sigma\big) &\geq \lim_{j\to\infty} \Phi\big(A_j(q^n) - A_j(q), u\big)\\ &\triangle\, \Phi\big(A_j(q^n) - A_j(\phi), \sigma\big)\\ &\geq 1\triangle\big(1-\mu\big) = 1-\mu,\\ \Psi\big(A_j(\phi) - A_j(q), u+\sigma\big) &\leq \lim_{j\to\infty} \Psi\big(A_j(q^n) - A_j(q), u\big)\\ &\diamondsuit\, \Psi\big(A_j(q^n) - A_j(\phi), \sigma\big)\\ &< 0\diamondsuit\mu = \mu,\end{aligned}$$

and

$$\begin{aligned}\Theta\big(A_j(\phi) - A_j(q), u+\sigma\big) &\leq \lim_{j\to\infty} \Theta\big({}_j(q^n) - A_j(q), u\big)\\ &\diamondsuit\, \Theta\big(A_j(q^n) - A_j(\phi), \sigma\big)\\ &< 0\diamondsuit\mu = \mu.\end{aligned}$$

Especially, for $n \in \mathbb{N}$, take $u = \frac{1}{n}$. Then,

$$\Phi\big(A_j(\phi) - A_j(q),\, \sigma\big) = \lim_{n\to\infty} \Phi\big(A_j(\phi) - A_j(q),\, \sigma + \tfrac{1}{n}\big) < 1-\mu,$$

$$\Psi\big(A_j(\phi) - A_j(q),\, \sigma\big) = \lim_{n\to\infty} \Phi\big(A_j(\phi) - A_j(q),\, \sigma + \tfrac{1}{n}\big) < \mu,$$

and

$$\Theta\big(A_j(\phi) - A_j(q),\, \sigma\big) = \lim_{n\to\infty} \Theta\big(A_j(\phi) - A_j(q),\, \sigma + \tfrac{1}{n}\big) < \mu.$$

So, the set

$$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi) - A_j(q),\, \sigma\big) > 1-\mu \quad \text{and}$$

$$\Psi\big(A_j(\phi) - A_j(q),\, \sigma\big) < \mu, \Theta\big(A_j(\phi) - A_j(q),\, \sigma\big) < \mu\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

Hence, we get $q \in \mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)$. As a result, $\mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}[\sigma,\mu,\gamma](A)$ is a closed set. $\square$

Theorem 3.16. *Suppose $\phi = (\phi_s) \in l_\infty$ is a sequence. If there is a sequence $q = (q_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ so that $A_j(\phi_s) = A_j(q_s)$ for almost every s relative to an admissible ideal $\mathcal{I}$, then $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$.*

Proof. Consider $A_j(\phi_s) = A_j(q_s)$ for almost every s relative to an admissible ideal $\mathcal{I}$. At that time,

$$\big\{s \in I_p, j \in \mathbb{N} : A_j(\phi_s) \neq A_j(q_s)\big\} \in \mathcal{I}.$$

This implies

$$\big\{s \in I_p, j \in \mathbb{N} : A_j(\phi_s) = A_j(q_s)\big\} \in \mathcal{F}(\mathcal{I}).$$

Hence, for all $j \in \mathcal{F}(\mathcal{I})$ for all $\sigma > 0$, we get

$$\Phi\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) = 1 \quad \text{and} \quad \Psi\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) = 0,$$
$$\Theta\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) = 0.$$

Since $q = (q_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$, suppose $S_\theta^{(\Phi,\Psi,\Theta)}\big(\mathcal{I}(A)\big) - \lim q_s = \alpha$. So, for all $\sigma, \gamma > 0$ and $\eta \in (0,1)$,

$$K_1 = \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big) > 1 - \eta \quad \text{and}$$
$$\Psi\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big) < \eta, \Theta\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big) < \eta\big\}\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

Establish the following set,

$$K_2 = \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) > 1 - \eta \quad \text{and}$$
$$\Psi\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < \eta, \Theta\big(A_j(\phi_s) - \alpha, \tfrac{\sigma}{2}\big) < \eta\big\}\big\| < \gamma\big\}.$$

We denote that $K_1 \subset K_2$. So, for all $j \in K_1$, we get

$$\begin{aligned}
\Phi\big(A_j(\phi_s) - \alpha, \sigma\big) &\geq \Phi\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) \triangle \Phi\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big), \\
&> 1 \triangle (1 - \eta), \\
&= 1 - \eta,
\end{aligned}$$

$$\begin{aligned}
\Psi\big(A_j(\phi_s) - \alpha, \sigma\big) &\leq \Psi\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) \Diamond \Psi\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big), \\
&< 0 \Diamond \eta, \\
&< \eta
\end{aligned}$$

and

$$\begin{aligned}
\Theta\big(A_j(\phi_s) - \alpha, \sigma\big) &\leq \Theta\big(A_j(\phi_s) - A_j(q_s), \tfrac{\sigma}{2}\big) \Diamond \Theta\big(A_j(q_s) - \alpha, \tfrac{\sigma}{2}\big), \\
&< 0 \Diamond \eta, \\
&< \eta\,.
\end{aligned}$$

This yields that $j \in K_2$ and hence, $K_1 \subset K_2$. Since $K_1 \in \mathcal{F}(\mathcal{I})$, we get $K_2 \in \mathcal{F}(\mathcal{I})$. Hence, $\phi = (\phi_s) \in T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. $\square$

Definition 3.17. Contemplate $S \subseteq T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. At that time, S is named to be compact if every open cover of S by the open set of $\tau^{\mathcal{N}}_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ has a finite subcover.

Theorem 3.18. *Every finite subset S of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$ is compact.*

P r o o f. Assume $S = \{\phi_1, \phi_2, \dots, \phi_n\}$ is the finite subset of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. For $\sigma > 0$ and $\mu \in (0,1)$, let us presume $\big\{\mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\big(\sigma,\mu,\gamma\big)(A) : \phi \in S\big\}$ is an open cover of S. At that time, $S \subseteq \bigcup_{\phi \in S} \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$. For all $\phi_i \in S$, $i = 1, 2, \dots, n$, we get $\phi_i \in \bigcup_{\phi_i \in S} \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$ This means,

$$\phi_i \in \mathcal{B}_{\phi_j}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) \quad \text{for some} \quad j \in \big\{1, 2, \dots, n\big\}.$$

Then,

$$\big\{\mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) : \phi \in S\big\}$$

is a finite subcover of S. As a result, we get S is compact. □

Theorem 3.19. *Take $S \subseteq T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Then, S is compact if and only if every sequence in S has a convergent subsequence.*

P r o o f. Assume S is compact subset of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Assume $(\phi_s^k) = (\phi^k)_{k=1}^{\infty}$ is a sequence in S. For a given $\sigma > 0$ and $\mu \in (0,1)$, assume

$$\left\{\mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(\frac{\sigma}{3},\mu,\gamma\Big)(A) : \phi = (\phi_s) \in S\right\}$$

is an open cover of S. This means

$$\big(\phi^k\big) \in \bigcup_{\phi \in S} \mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\big(\frac{\sigma}{3},\mu,\gamma\big)(A).$$

Then, there is some $\phi = \big(\phi_s\big) \in S$ so that $\big(\phi^k\big) \in \mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\big(\frac{\sigma}{3},\mu,\gamma\big)(A)$. So, the set

$$K_1 = \Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j\big(\phi_s^k\big) - A_j(\phi), \tfrac{\sigma}{3}\big) > 1 - \eta \quad \text{and}$$
$$\Psi\big(A_j(\phi_s^k) - A_j(\phi), \tfrac{\sigma}{3}\big) < \eta, \Theta\big(A_j(\phi_s^k) - A_j(\phi), \tfrac{\sigma}{3}\big) < \eta\big\}\big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

Since S is compact, there is a finite subcover

$$\left\{\mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(\frac{\sigma}{3},\mu,\gamma\Big)(A) : \phi_i \in S \quad \text{and} \quad i = 1, 2, \dots, m\right\}$$

of S so that $S \subseteq \bigcup_{i=1}^{m} \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\big(\frac{\sigma}{3},\mu,\gamma\big)(A)$. Let (ϕ^{k_n}) be a subsequence of (ϕ^k). Then,

$$(\phi^{k_n}) \in \bigcup_{i=1}^{m} \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(\frac{\sigma}{3},\mu,\gamma\Big)(A)$$

gives

$$(\phi^{k_n}) \in \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}\Big(\frac{\sigma}{3},\mu,\gamma\Big)(A) \quad \text{for some } \phi_i \in S\ .$$

So, the set

$$K_2 = Bl\{p \in \mathbb{N} : \tfrac{1}{h_p}\Big\|\Big\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi_i), \tfrac{\sigma}{3}\big) > 1-\eta \quad \text{and}$$

$$\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi_i), \tfrac{\sigma}{3}\big) < \eta, \Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi_i), \tfrac{\sigma}{3}\big) < \eta\Big\}\Big\| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

Now, for $j \in K_1 \cap K_2$,

$$\begin{aligned}\Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &\geq \Phi\big(A_j\big(\phi_s^{k_n}\big) - A_j(\phi_i), \tfrac{\sigma}{3}\big)\\ &\triangle\, \Phi\big(A_j\big(\phi_s^{k}\big) - A_j\big(\phi_i\big), \tfrac{\sigma}{3}\big)\\ &\triangle\, \Phi\big(A_j\big(\phi_s^{k}\big) - A_j\big(\phi\big), \tfrac{\sigma}{3}\big)\\ &> (1-\eta)\triangle(1-\eta)\triangle\big(1-\eta\big) = 1-\eta,\end{aligned}$$

thus, we get $\Phi\big(A_j(\phi_s^{k_n}) - A_j\big(\phi\big), \tfrac{\sigma}{3}\big) > 1-\eta$.

$$\begin{aligned}\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &\leq \Psi\big(A_j\big(\phi_s^{k_n}\big) - A_j(\phi_i), \tfrac{\sigma}{3}\big)\\ &\diamondsuit\, \Psi\big(A_j\big(\phi_s^{k}\big) - A_j\big(\phi_i\big), \tfrac{\sigma}{3}\big)\\ &\diamondsuit\, \Psi\big(A_j\big(\phi_s^{k}\big) - A_j\big(\phi\big), \tfrac{\sigma}{3}\big)\\ &< \eta \diamondsuit =, \eta \diamondsuit \eta = \eta,\end{aligned}$$

so, we obtain $\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) < \eta$.

$$\begin{aligned}\Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &\leq \Theta\big(A_j\big(\phi_s^{k_n}\big) - A_j(\phi_i), \tfrac{\sigma}{3}\big)\\ &\diamondsuit\, \Theta\big(A_j\big(\phi_s^{k}\big) - A_j\big(\phi_i\big), \tfrac{\sigma}{3}\big)\\ &\diamondsuit\, \Theta\big(A_j(\phi_s^{k}) - A_j(\phi), \tfrac{\sigma}{3}\big)\\ &< \eta \diamondsuit \eta \diamondsuit \eta = \eta,\end{aligned}$$

so, we receive $\Theta bl(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) < \eta$. Let $\eta = \frac{1}{j}$.

$$\begin{aligned}\lim_{j\to\infty} \Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &= \lim_{j\to\infty} 1 - \tfrac{1}{j} &&= 1,\\ \lim_{j\to\infty} \Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &= \lim_{j\to\infty} \tfrac{1}{j} &&= 0,\\ \lim_{j\to\infty} \Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi), \tfrac{\sigma}{3}\big) &= \lim_{j\to\infty} \tfrac{1}{j} &&= 0.\end{aligned}$$

Hence, we have $\phi^{k_n} \to \phi$, as $n \to \infty$. Conversely, assume (ϕ^{k_n}) is the subsequence of a sequence (ϕ^k) in S so that $(\phi^{k_n}) \to \phi$ in S. Let, on the contrary, S is not compact subset of $T_{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(A)$. Assume

$$\Big\{\mathcal{B}_\phi^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) : \phi \in S\Big\}$$

is an open cover of S, means $S \subseteq \bigcup_{\phi \in S} \mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$. So, we get

$$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big|\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, \sigma\big) > 1-\eta \quad \text{and}$$

$$\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, \sigma\big) < \eta, \Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, sigma\big) < \eta\}\big| < \gamma\Big\} \in \mathcal{F}(\mathcal{I}).$$

As S is not compact, there is a finite subcover

$$\Big\{\mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A) : \phi_i \in S \quad and \;\; i = 1,2,\ldots,m\Big\}$$

so that $S \nsubseteq \bigcup_{\phi_i \in S} \mathcal{B}_{\phi_i}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\frac{\sigma}{3},\mu\gamma)(A)$, which gives that the set

$$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi_i),\, \sigma\big) > 1-\eta \quad \text{and}$$

$$\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi_i),\, \sigma\big) < \eta, \Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi_i),\, \sigma\big) < \eta\}\big\| < \gamma\Big\} \notin \mathcal{F}(\mathcal{I}),$$

so,

$$\Big\{p \in \mathbb{N} : \tfrac{1}{h_p}\big\|\{s \in I_p, j \in \mathbb{N} : \Phi\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, \sigma\big) > 1-\eta \quad \text{and}$$

$$\Psi\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, \sigma\big) < \eta, \Theta\big(A_j(\phi_s^{k_n}) - A_j(\phi),\, \sigma\big) < \eta\}\big\| < \gamma\Big\} \notin \mathcal{F}(\mathcal{I}).$$

For any $\sigma > 0$ and $\mu \in (0,1)$, $(\phi^{k_n}) \notin \mathcal{B}_{\phi}^{S_\theta^{(\Phi,\Psi,\Theta)}(\mathcal{I})}(\sigma,\mu,\gamma)(A)$. $\phi^{k_n} \nrightarrow \phi$, which is a contradiction. As a result, we obtain S is compact. □

REFERENCES

[1] ATANASSOV, K. T.: *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), no. 1, 87–96.

[2] DEBNATH, P.: *Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces*, Comput. Math. Appl. **63** (2012), no. 3, 708–715.

[3] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1952), 241–244.

[4] FEINBERG, M.: *Fibonacci-tribonacci*, Fibonacci Quart. **1** (1963), 71–74.

[5] FRIDY, J. A.—ORHAN, C.: *Lacunary statistical convergence*, Pacific J. Math. **160** (1993), no. 1, 43–51.

[6] GEORGE, A.—VEERAMANI, P.: *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems. **90** (1997), no. 3, 365–368.

[7] GÜRDAL, M.—ŞAHINER, A.: *Extremal $\mathcal{I}$-limit points of double sequences*, Appl. Math. E-Notes **8** (2008), 131–137.

[8] KADAK, U.—MOHIUDDINE, S. A.: *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-Gamma function and related approximation theorems*, Results Math. **73** (2018), no. 1, Paper no. 9, 31 pp.

[9] KALEVA, O.—SEIKKALA, S.: *On fuzzy metric spaces*, Fuzzy Sets and Systems. **12** (1984), no. 3, 215–229.

[10] KARAKUS, S.—DEMIRCI, K.—DUMAN, O.: *Statistical convergence on intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals **35** (2008), no. 4, 215–229.

[11] KHAN, V. A.—AHMAD, M.—FATIMA, H.—KHAN, F. M.: *On some results in intuitionistic fuzzy ideal convergence double sequence spaces*, Adv. Difference Equ. **2019** (2019), no. 1, Paper No. 375, 10 pp.

[12] KHAN, V. A.—FATIMA, H.—ALTAF, H.—LOHANI, Q. M. D.—SRIVASTAVA, H. M.: *Intuitionistic fuzzy $\mathcal{I}$-convergent sequence spaces defined by compact operator*, Cogent Math. **3** (2016), no. 1, Article ID: 1267904.

[13] KHAN, V. A.—KARA, E.—ALTAF, H.—KHAN, N.—AHMAD, M.: *Intuitionistic fuzzy $\mathcal{I}$-convergent Fibonacci difference sequence spaces*, J. Inequal. Appl. **2019** (2019), Article no. 202, 7 pp.

[14] KHAN, V. A.—RAHAMAN, S. K. A.: *Intuitionistic fuzzy tribonacci convergent sequence spaces*, Math. Slovaca **72** (2022), no. 3, 693–708.

[15] KIRIŞCI, M.—ŞIMŞEK, N.: *Neutrosophic metric spaces*, Math. Sci. **14** (2022), no. 3, 241–248.

[16] KIRIŞCI M.—ŞIMŞEK, N.: *Neutrosophic normed spaces and statistical convergence*, J. Anal. **28** (2020), no. 4, 1059–1073.

[17] KIŞI, Ö.: *Ideal convergence of sequences in neutrosophic normed spaces*, J. Intell. Fuzzy Systems **41** (2021), no. 2, 2581–2590.

[18] KOSTYRKO, P.—ŠALÁT, T.—WILCZYNSSKI, W.: *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000), no. 2, 669–686.

[19] KRAMOSIL, I.—MICHALEK, J.: *Fuzzy metric and statistical metric spaces*, Kybernetika **11** (1975), no. 5, 336–344.

[20] LAEL, F.—NOUROUZI, K.: *Some results on the IF-normed spaces*, Chaos Solitons Fractals **37** (2008), no. 3, 931–939.

[21] MENGER, K.: *Statistical metrics*, Proc. Nat. Acad. Sci. U.S.A **28** (1942), 535–537.

[22] MOHIUDDINE, S. A.—ASIRI, A.—HAZARIKA, H.: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst. **48** (2019), no. 5, 492–506.

[23] MOHIUDDINE, S. A.—DANISH LOHANI, Q. M.: *On generalized statistical convergence in intuitionistic fuzzy normed space*, Chaos Solitons Fractals **42** (2009), no. 3, 1731–1737.

[24] MURSALEEN, M.—MOHIUDDINE, S. A.: *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Comput. Appl. Math. **233** (2009), no. 2, 142–149.

[25] MURSALEEN, M.—MOHIUDDINE, S. A.—EDELY, O. H. H.: *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces*, Comput. Math. Appl. **59** (2010), no. 2, 603–611.

[26] NABIEV, A. A.—PEHLIVAN, S.—GÜRDAL, M.: *On $\mathcal{I}-$Cauchy sequences*, Taiwanese J. Math. **11** (2007), no. 2, 569–566.

[27] PARK, J. H.: *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals **22** (2004), no. 5, 1039–1046.

[28] RATH, D.—TRIPATHY, B. C.: *Matrix maps on sequence spaces associated with sets of integers*, Indian J. Pure Appl. Math. **27** (1996), 197–206.

[29] SAVAŞ, E.—DAS, P.: *A generalized statistical convergence via ideals*, Appl. Math. Lett. **24** (2011), no. 6, 826–830.

[30] SAVAŞ, E.—GÜRDAL, M.: *Certain summability methods in intuitionistic fuzzy normed spaces*, J. Intell. Fuzzy Systems **27** (2014), no. 4, 1621–1629.

[31] SAVAŞ, E.—GÜRDAL, M.: *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Science Asia **41** (2015), nno. 4, 289–294.

[32] SAVAŞ, E.—GÜRDAL, M.: *Ideal convergent function sequences in random 2-normed spaces*, Filomat **30** (2016), no. 3, 557–567.
[33] SMARANDACHE, F.: *Neutrosophy. Neutrosophic Probability, Set, and Logic.* Ed. of ProQuest Information & Learning, Ann Arbor, Michigan, USA, 1998.
[34] SMARANDACHE, F.: *Neutrosophic set, a generalisation of the intuitionistic fuzzy sets*, Int. J. Pure Appl. Math. **24** (2005), no. 3, 287–297.
[35] KHAN, V. A.—ARSHAD, M.: *On some properties of Nörlund ideal convergence of sequence in neutrosophic normed spaces*, Ital. J. Pure Appl. Math. **40** (2023) 1–8.
[36] KHAN, V. A.—ARSHAD, M.—KHAN, M. D.: *Some results of neutrosophic normed space VIA Tribonacci convergent sequence spaces*, J. Inequal. Appl. **2022**, paper no. 42, 27 pp.
[37] KHAN, V. A.— ARSHAD, M.——ALAM, M.: *Riesz deal convergence in neutrosophic normed spaces.* J. Intell. Fuzzy Systems. **42** (2023). no. 4, 1–10.
[38] TAN, B.—WEN, Z. Y.: *Some properties of the tribonacci sequence*, European J. Combin. **28** (2007), no. 6, 1703–1719.
[39] TRIPATHY, B. C.—HAZARIKA, B.—CHOUDHARY, B.: *Lacunary $\mathcal{I}-$convergent sequences*, Kyungpook Math. J. **52** (2012), no. 4, 473–482.
[40] TRIPATHY, B. C.—SEN, M.: *On fuzzy $\mathcal{I}-$convergent difference sequence spaces*, J. Intell. Fuzzy Syst. **25** (2013), no. 3, 643–647.
[41] WILANSKY, A.: *Summability through functional analysis.* North-Holland Mathematics Stud. Vol. 85. Notas Mat. Vol. 91. [Mathematical Notes] North-Holland Publishing Co., Amsterdam-New York-Oxford, 1984.
[42] YAYING, T.—HAZARIKA, B.: *On sequence spaces defined by the domain of a regular tribonacci matrix*, Math. Slovaca **70** (2020), no. 3, 697–706.
[43] ZADEH, L. A.: *Fuzzy sets*, Inf. Control **8** (1965), no. 3, 338–353.

Received February 18, 2024
Revised May 26, 2024
Accepted June 18, 2024
Publ. online September 30, 2024

Vakeel A. Khan
Department of Mathematics
Aligarh Muslim University
Aligarh 202002
INDIA
E-mail: vakhanmaths@gmail.com

Ömer Kişi
Department of Mathematics
Bartın University
Bartın 74100
TURKEY
E-mail: okisi@bartin.edu.tr

Chiranjib Choudhury
Department of Mathematics
Tripura University (A Central University)
Suryamaninagar, Agartala 799022
INDIA
E-mail: chiranjibchoudhury123@gmail.com