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SOME INEQUALITIES INVOLVING WEIGHTED POWER MEAN

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ABSTRACT. In this paper, we first show some inequalities on weighted power mean. When $a, b > 0$, $p \geq 1$ and $0 < v \leq \tau < 1$, we have

$$\frac{v}{\tau} \leq \frac{a\sharp_{p,v}b - a\sharp_v b}{a\sharp_{p,\tau}b - a\sharp_\tau b} \leq \frac{1-v}{1-\tau}$$

and

$$\frac{v}{\tau} \leq \frac{a\sharp_{p,v}b - a!_v b}{a\sharp_{p,\tau}b - a!_\tau b} \leq \frac{1-v}{1-\tau}.$$

Further, we obtain the range of corresponding inequalities involving the m power form of weighted power mean in the same form as above for $m \in \mathbb{N}^+$ or $p \geq m > 0$, $m \leq p < 0$. As further applications, we provide some inequalities about matrices and determinants, respectively.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices, ${M_n}^+(\mathbb{C})$ stands for the set of positive semi-definite matrices in $M_n(\mathbb{C})$ and ${M_n}^{++}(\mathbb{C})$ is the set of positive definite matrices in $M_n(\mathbb{C})$. As usual, for the two Hermitian matrices A and B, we say $A > B$, when $A - B \in {M_n}^{++}(\mathbb{C})$. $B(\mathcal{H})$ stands for the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$. The singular value of A, that is, the eigenvalue of the positive semi-definite matrix $|A| = (A^*A)^{\frac{1}{2}}$, is denoted by $s_i(A)$.

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We usually define weighted AM-GM-HM (Arithmetic-Geometric-Harmonic means) and weighted power mean [4] as

$$a\nabla_v b = (1-v)a + vb,$$

$$a\sharp_v b = a^{1-v}b^v,$$

$$a!_v b = \big((1-v)a^{-1} + vb^{-1}\big)^{-1}$$

and

$$a\sharp_{p,v} b = \big((1-v)a^p + vb^p\big)^{\frac{1}{p}}$$

for $a, b > 0,\ 0 \le v \le 1$ and $p \ne 0$.

The value $p \to 0$ gives the weighted geometric mean, while the values $p = 1, -1$ give the weighted arithmetic and harmonic means, respectively.

Similarly, we define corresponding weighted operator AM-GM-HM and power mean as

$$A\nabla_v B = (1-v)A + vB,$$

$$A\sharp_v B = A^{\frac{1}{2}}\Big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\Big)^v A^{\frac{1}{2}},$$

$$A!_v B = \Big((1-v)A^{-1} + vB^{-1}\Big)^{-1}$$

and

$$A\sharp_{p,v} B = A^{\frac{1}{2}}\bigg((1-v)I + v\Big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\Big)^p\bigg)^{\frac{1}{p}} A^{\frac{1}{2}}$$

for $A, B > 0,\ 0 \le v \le 1$ and $p \ne 0$.

The well-known Young inequality says that for positive real numbers a, b and $0 \le v \le 1$, we have

$$a^{1-v}b^v \le (1-v)a + vb.$$

In professional discussions, the above inequality is receiving increasing attention.

In 2015, progress on Young-type inequality was proposed by Alzer, Fonseca and Kovačec in their work [1], which can be expressed as following:

$$\frac{v^m}{\tau^m} \le \frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \le \frac{(1-v)^m}{(1-\tau)^m} \tag{1.1}$$

for $a, b > 0,\ 0 < v \le \tau < 1$ and $m \ge 1$.

In the same year, Liao and Wu [6] changed the weighted geometric mean in (1.1) to the weighted harmonic mean:

$$\frac{v^m}{\tau^m} \le \frac{(a\nabla_v b)^m - (a!_v b)^m}{(a\nabla_\tau b)^m - (a!_\tau b)^m} \le \frac{(1-v)^m}{(1-\tau)^m} \tag{1.2}$$

for $a, b > 0,\ 0 < v \le \tau < 1$ and $m \ge 1$.

Similarly, S a b a b h e h [10] showed inequalities directly analogous to weighted geometric mean and harmonic mean in 2016:

$$\frac{(a\sharp_v b)^m - (a!_v b)^m}{(a\sharp_\tau b)^m - (a!_\tau b)^m} \geq \left(\frac{v}{\tau}\right)^m \tag{1.3}$$

for $a, b > 0$, $0 < v < \tau < 1$ and $m > 1$.

In 2016, M a r y a m K h o s r a v i [5] showed the following inequalities about weighted power mean:

$$\frac{v}{\tau} \leq \frac{a\nabla_v b - a\sharp_{p,v} b}{a\nabla_\tau b - a\sharp_{p,v} b} \leq \frac{1-v}{1-\tau} \tag{1.4}$$

for $a, b > 0$, $0 < v \leq \tau < 1$, $p \in R$ and $p \neq 1$.

Furthermore, Y a n g and W a n g [11] have recently refined inequality (1.1) with the following results:

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leq \frac{v(1-v)}{\tau(1-\tau)}, \quad a \leq b \tag{1.5}$$

and

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \geq \frac{v(1-v)}{\tau(1-\tau)}, \quad a \geq b \tag{1.6}$$

for $a, b > 0$, $0 < v \leq \tau < 1$ and $m \in \mathbb{N}^+$.

For more similar form of the above inequalities, we refer the readers to [2,8,9] and the reference therein.

2. Main results

In this section, we first show the inequalities that are new results for (1.4). Then, we present some corresponding inequalities involving the m power form of the weighted power mean.

Theorem 2.1. *Let*

$$a, b > 0, p \geq 1 \quad \textit{and} \quad 0 < v \leq \tau < 1,$$

then we have that

$$\frac{v}{\tau} \leq \frac{a\sharp_{p,v} b - a\sharp_v b}{a\sharp_{p,\tau} b - a\sharp_\tau b} \leq \frac{1-v}{1-\tau}. \tag{2.1}$$

P r o o f. Suppose that $f(v)=\frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{v}$. Then

$$
\begin{aligned}
f'(v) &= \frac{\left(\frac{1}{p}(1-v+vx^p)^{\frac{1}{p}-1}(x^p-1)-x^v\ln x\right)v-(1-v+vx^p)^{\frac{1}{p}}+x^v}{v^2}\\
&= \frac{h(x)}{v^2},\\
h(x) &= \frac{v}{p}(x^p-1)(1-v+vx^p)^{\frac{1}{p}-1}-vx^v\ln x-(1-v+vx^p)^{\frac{1}{p}}+x^v.
\end{aligned}
$$

$$
\begin{aligned}
&h'(x)\\
&= \frac{v}{p}px^{p-1}(1-v+vx^p)^{\frac{1}{p}-1}+\frac{v}{p}(x^p-1)\left(\frac{1}{p}-1\right)(1-v+vx^p)^{\frac{1}{p}-2}pvx^{p-1}\\
&\quad -v^2x^{v-1}\ln x-vx^{v-1}-\frac{1}{p}(1-v+vx^p)^{\frac{1}{p}-1}pvx^{p-1}+vx^{v-1}\\
&= v^2x^{p-1}(x^p-1)\left(\frac{1}{p}-1\right)(1-v+vx^p)^{\frac{1}{p}-2}-v^2x^{v-1}\ln x.
\end{aligned}
$$

When $p\geq 1$, if $0<x\leq 1$, $h'(x)\geq 0$, if $x\geq 1$, $h'(x)\leq 0$, thus $h(x)\leq h(1)=0$, it is clear that $f'(v)\leq 0$ when $p\geq 1$, which means that $f(v)\geq f(\tau)$. Therefore,

$$
\frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{v}\geq\frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau}{\tau}.
$$

Taking $x=\frac{b}{a}$, we can get the first inequality of (2.1) directly. The proof of the second inequality of (2.1) is similar to the one presented above. □

The following inequalities are similar to the above, where the weighted geometric mean becomes the weighted harmonic mean.

Theorem 2.2. *Let $a,b>0$, $p\geq 1$ and $0<v\leq\tau<1$, then we have that*

$$
\frac{v}{\tau}\leq\frac{a\sharp_{p,v}b-a!_v b}{a\sharp_{p,\tau}b-a!_\tau b}\leq\frac{1-v}{1-\tau}. \tag{2.2}
$$

P r o o f. Suppose that $f(v)=\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{v}$. Then

$$
\begin{aligned}
f'(v) &= \frac{1}{v^2}\left[\left(\frac{1}{p}(1-v+vx^p)^{\frac{1}{p}-1}(x^p-1)+\left(1-v+vx^{-1}\right)^{-2}\left(x^{-1}-1\right)\right)v\right.\\
&\quad\left.-(1-v+vx^p)^{\frac{1}{p}}+\left(1-v+vx^{-1}\right)^{-1}\right]\\
&= \frac{h(x)}{v^2}.
\end{aligned}
$$

$$\begin{aligned}
&h'(x)\\
&= \frac{v}{p}\left(\frac{1}{p}-1\right)(1-v+vx^p)^{\frac{1}{p}-2}pvx^{p-1}(x^p-1)+\frac{v}{p}(1-v+vx^p)^{\frac{1}{p}-1}px^{p-1}\\
&\quad +2v\left(1-v+vx^{-1}\right)^{-3}vx^{-2}\left(x^{-1}-1\right)-\left(1-v+vx^{-1}\right)^{-2}vx^{-2}\\
&\quad -\frac{1}{p}(1-v+vx^p)^{\frac{1}{p}-1}pvx^{p-1}+\left(1-v+vx^{-1}\right)^{-2}vx^{-2}\\
&= v^2x^{p-1}(x^p-1)\left(\frac{1}{p}-1\right)(1-v+vx^p)^{\frac{1}{p}-2}\\
&\quad +2v^2x^{-2}\left(x^{-1}-1\right)\left(1-v+vx^{-1}\right)^{-3}.
\end{aligned}$$

When $p \geq 1$, if $0 < x \leq 1$, $h'(x) \geq 0$, if $x \geq 1$, $h'(x) \leq 0$, then $h(x) \leq h(1) = 0$, it is clear that $f'(v) \leq 0$ when $p \geq 1$, which means that $f(v) \geq f(\tau)$. So

$$\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{v} \geq \frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}}{\tau}.$$

Taking $x = \frac{b}{a}$, we can get the first inequality of (2.2) directly. Similarly, we can obtain the other inequality by using the above method. □

The following remark is to show that Theorem 2.1 and Theorem 2.2 are new results compared to inequalities (1.4).

Remark 1.

(i) Here are the comparisons of these three inequalities.

When $a = 3,\ b = 2,\ v = \frac{1}{4},\ \tau = \frac{2}{3},\ p = 4$,

$$\frac{a\nabla_v b - a\sharp_{p,v}b}{a\nabla_\tau b - a\sharp_{p,v}b} \approx 0.602 \leq \frac{a\sharp_{p,v}b - a\sharp_v b}{a\sharp_{p,\tau}b - a\sharp_\tau b} \approx 0.670 \leq \frac{a\sharp_{p,v}b - a!_v b}{a\sharp_{p,\tau}b - a!_\tau b} \approx 0.748.$$

When $a = \frac{2}{5},\ b = 1,\ v = \frac{1}{4},\ \tau = \frac{2}{3},\ p = 4$,

$$\frac{a\nabla_v b - a\sharp_{p,v}b}{a\nabla_\tau b - a\sharp_{p,v}b} \approx 1.599 \geq \frac{a\sharp_{p,v}b - a\sharp_v b}{a\sharp_{p,\tau}b - a\sharp_\tau b} \approx 1.281 \geq \frac{a\sharp_{p,v}b - a!_v b}{a\sharp_{p,\tau}b - a!_\tau b} \approx 1.041.$$

Therefore, they cannot be substantively compared.

(ii) Let $p = 1$ in Theorem 2.1, we can get inequality (1.1) when $m = 1$.

On the basis of inequalities (1.4), (2.1) and (2.2), we generalize the inequalities to the m power, respectively.

Theorem 2.3. *Let $p \neq 1$, $0 < v \leq \tau < 1$, $m \in \mathbb{N}^+$, then for all real positive numbers a, b:*

(i) *If $a \geq b$, then*
$$\frac{(a\nabla_v b)^m - (a\sharp_{p,v} b)^m}{(a\nabla_\tau b)^m - (a\sharp_{p,\tau} b)^m} \geq \frac{v}{\tau}. \tag{2.3}$$

(ii) *If $a \leq b$, then*
$$\frac{(a\nabla_v b)^m - (a\sharp_{p,v} b)^m}{(a\nabla_\tau b)^m - (a\sharp_{p,\tau} b)^m} \leq \frac{1-v}{1-\tau}. \tag{2.4}$$

Proof. Clearly,
$$\begin{aligned}&(1-v+vx)^m - (1-v+vx^p)^{\frac{m}{p}}\\ &= \left((1-v+vx) - (1-v+vx^p)^{\frac{1}{p}}\right)\sum_{i=1}^{m}\left((1-v+vx)^{m-i}(1-v+vx^p)^{\frac{i-1}{p}}\right).\end{aligned}$$

Let
$$f(v) = \sum_{i=1}^{m}\left((1-v+vx)^{m-i}(1-v+vx^p)^{\frac{i-1}{p}}\right),$$
then
$$\begin{aligned}f'(v) &= (x-1)\left(\sum_{i=1}^{m}(m-i)(1-v+vx)^{m-i-1}(1-v+vx^p)^{\frac{i-1}{p}}\right)\\ &\quad + (x^p-1)\left(\sum_{i=1}^{m}\frac{i-1}{p}(1-v+vx)^{m-i}(1-v+vx^p)^{\frac{i-1}{p}-1}\right).\end{aligned}$$

(i) When $0 < x \leq 1$, $f'(v) \leq 0$, so $f(v)$ is decreasing, $f(v) \geq f(\tau)$, we have
$$\begin{aligned}\frac{(1-v+vx)^m - (1-v+vx^p)^{\frac{m}{p}}}{(1-\tau+\tau x)^m - (1-\tau+\tau x^p)^{\frac{m}{p}}} &= \frac{\left((1-v+vx) - (1-v+vx^p)^{\frac{1}{p}}\right)f(v)}{\left((1-\tau+\tau x) - (1-\tau+\tau x^p)^{\frac{1}{p}}\right)f(\tau)}\\ &\geq \frac{(1-v+vx) - (1-v+vx^p)^{\frac{1}{p}}}{(1-\tau+\tau x) - (1-\tau+\tau x^p)^{\frac{1}{p}}}\\ &\geq \frac{v}{\tau} \quad (\text{by } (1.4)).\end{aligned}$$

(ii) When $x \geq 1$, $f'(v) \geq 0$, so $f(v)$ is increasing, $f(v) \leq f(\tau)$, we have
$$\begin{aligned}\frac{(1-v+vx)^m - (1-v+vx^p)^{\frac{m}{p}}}{(1-\tau+\tau x)^m - (1-\tau+\tau x^p)^{\frac{m}{p}}} &= \frac{\left((1-v+vx) - (1-v+vx^p)^{\frac{1}{p}}\right)f(v)}{\left((1-\tau+\tau x) - (1-\tau+\tau x^p)^{\frac{1}{p}}\right)f(\tau)}\\ &\leq \frac{(1-v+vx) - (1-v+vx^p)^{\frac{1}{p}}}{(1-\tau+\tau x) - (1-\tau+\tau x^p)^{\frac{1}{p}}}\\ &\leq \frac{1-v}{1-\tau} \quad (\text{by } (1.4)).\end{aligned}$$

Taking $x = \frac{b}{a}$, we can get the desired results directly. □

Remark 2.

(i) When $m > 1$, Theorem 2.3 can be regarded as a generalization of inequality (1.4).

(ii) Let $a = b,\ b = a,\ v = 1 - \tau$ and $\tau = 1 - v$ in inequality (2.3), we can also get inequality (2.4) directly.

(iii) Let $0 < v \le \tau < 1$, $p \to 0$ in Theorem 2.3, we can get

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \ge \frac{v}{\tau} \ge \frac{v^m}{\tau^m}, \quad a \ge b$$

and

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \le \frac{1-v}{1-\tau} \le \frac{(1-v)^m}{(1-\tau)^m}, \quad a \le b.$$

Therefore, it is obvious that Theorem 2.3 is a new result associated with inequality (1.1) with the corresponding additional $m \in \mathbb{N}^+,\ a \ge b$ or $a \le b$ conditions, respectively.

Theorem 2.4. *Let $p \ge 1$, $0 < v \le \tau < 1$ and $m \in \mathbb{N}^+$, then for all real positive numbers a, b, we have that*

(i) *If $a \ge b$, then*

$$\frac{(a\sharp_{p,v} b)^m - (a\sharp_v b)^m}{(a\sharp_{p,\tau} b)^m - (a\sharp_\tau b)^m} \ge \frac{v}{\tau}. \tag{2.5}$$

(ii) *If $a \le b$, then*

$$\frac{(a\sharp_{p,v} b)^m - (a\sharp_v b)^m}{(a\sharp_{p,\tau} b)^m - (a\sharp_\tau b)^m} \le \frac{1-v}{1-\tau}. \tag{2.6}$$

P r o o f. Clearly,

$$(1 - v + vx^p)^{\frac{m}{p}} - x^{mv} = \left((1 - v + vx^p)^{\frac{1}{p}} - x^v\right)\left(\sum_{i=1}^{m} (1 - v + vx^p)^{\frac{m-i}{p}} x^{(i-1)v}\right)$$

Let

$$f(v) = \sum_{i=1}^{m} (1 - v + vx^p)^{\frac{m-i}{p}} x^{(i-1)v},$$

then

$$f'(v) = (x^p - 1)\left(\sum_{i=1}^{m} \frac{m-i}{p} (1 - v + vx^p)^{\frac{m-i}{p}-1} x^{(i-1)v}\right) + \ln x \left(\sum_{i=1}^{m} (i-1)(1 - v + vx^p)^{\frac{m-i}{p}} x^{(i-1)v}\right).$$

(i) When $0 < x \le 1$, $p \ge 1$, we have $(x^p - 1) \le 0$, $\frac{m-1}{p} \ge \frac{m-2}{p} \ge \cdots \ge \frac{1}{p} \ge 0$, $\ln x \le 0$, so it's obvious that $f'(v) \le 0$, which means $\frac{f(v)}{f(\tau)} \ge 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}} - x^{mv}}{(1-\tau+\tau x^p)^{\frac{m}{p}} - x^{m\tau}} &= \frac{\left((1-v+vx^p)^{\frac{1}{p}} - x^{v}\right) f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}} - x^{\tau}\right) f(\tau)} \\ &\ge \frac{(1-v+vx^p)^{\frac{1}{p}} - x^{v}}{(1-\tau+\tau x^p)^{\frac{1}{p}} - x^{\tau}} \\ &\ge \frac{v}{\tau} \quad (\text{by } (2.1)).\end{aligned}$$

(ii) When $x \ge 1$, $p \ge 1$, we have $(x^p - 1) \ge 0$, $\frac{m-1}{p} \ge \frac{m-2}{p} \ge \cdots \ge \frac{1}{p} \ge 0$, $\ln x \ge 0$, so it's obvious that $f'(v) \ge 0$, which means $\frac{f(v)}{f(\tau)} \le 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}} - x^{mv}}{(1-\tau+\tau x^p)^{\frac{m}{p}} - x^{m\tau}} &= \frac{\left((1-v+vx^p)^{\frac{1}{p}} - x^{v}\right) f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}} - x^{\tau}\right) f(\tau)} \\ &\le \frac{(1-v+vx^p)^{\frac{1}{p}} - x^{v}}{(1-\tau+\tau x^p)^{\frac{1}{p}} - x^{\tau}} \\ &\le \frac{1-v}{1-\tau} \quad (\text{by } (2.1)).\end{aligned}$$

Taking $x = \frac{b}{a}$, we can obtain the desired results directly. □

Theorem 2.5. *Let $p \ge 1$, $0 < v \le \tau < 1$ and $m \in \mathbb{N}^+$, then for all real positive numbers a, b, we have that*

(i) *If $a \ge b$, then*

$$\frac{(a\sharp_{p,v}b)^m - (a!_v b)^m}{(a\sharp_{p,\tau}b)^m - (a!_\tau b)^m} \ge \frac{v}{\tau}. \tag{2.7}$$

(ii) *If $a \le b$, then*

$$\frac{(a\sharp_{p,v}b)^m - (a!_v b)^m}{(a\sharp_{p,\tau}b)^m - (a!_\tau b)^m} \le \frac{1-v}{1-\tau}. \tag{2.8}$$

P r o o f. It can be proved by an argument similar to the one used in Theorem 2.4. □

It should be emphasized that in the following results, the m in the power of the inequalities is not a common positive integer but a real number.

Theorem 2.6. *Let $p \ge m > 0$ or $m \le p < 0$, $0 < v \le \tau < 1$ and m is a real number, then for all positive real numbers a, b, we have that*

$$\frac{(a\sharp_{p,v}b)^m - (a\sharp_v b)^m}{(a\sharp_{p,\tau}b)^m - (a\sharp_\tau b)^m} \ge \frac{v}{\tau}. \tag{2.9}$$

P r o o f. Suppose that $f(v)=\frac{(1-v+vx^p)^{\frac{m}{p}}-x^{mv}}{v}$. Then

$$f'(v) = \frac{\left(\frac{m}{p}(1-v+vx^p)^{\frac{m}{p}-1}(x^p-1)-mx^{mv}\ln x\right)v-(1-v+vx^p)^{\frac{m}{p}}+x^{mv}}{v^2} = \frac{h(x)}{v^2}.$$

$$\begin{aligned} h'(x) &= \frac{mv}{p}(x^p-1)\left(\frac{m}{p}-1\right)(1-v+vx^p)^{\frac{m}{p}-2}pvx^{p-1}+\frac{mv}{p}px^{p-1}(1-v+vx^p)^{\frac{m}{p}-1} \\ &\quad - m^2v^2x^{mv-1}\ln x - mvx^{mv-1} - \frac{m}{p}(1-v+vx^p)^{\frac{m}{p}-1}pvx^{p-1}+mvx^{mv-1} \\ &= mv^2x^{p-1}(x^p-1)\left(\frac{m}{p}-1\right)(1-v+vx^p)^{\frac{m}{p}-2}-m^2v^2x^{mv-1}\ln x. \end{aligned}$$

When $p\geq m>0$ or $m\leq p<0$, if $0<x<1$, $h'(x)\geq 0$, if $x>1$, $h'(x)\leq 0$. So $h(x)\leq h(1)=0$, it is obvious that $f'(v)\leq 0$, it means that $f(v)$ is decreasing and $f(v)\geq f(\tau)$. Therefore,

$$\frac{(1-v+vx^p)^{\frac{m}{p}}-x^{mv}}{v}\geq\frac{(1-\tau+\tau x^p)^{\frac{m}{p}}-x^{m\tau}}{\tau}.$$

Taking $x=\frac{b}{a}$, we can obtain the desired inequality directly. □

3. Applications to matrices and determinants

Lemma 3.1 ([7])**.** *Let $X\in B(\mathcal{H})$ be self-adjoint, f and g be continuous real functions such that $f(t)\geq g(t)$ for all $t\in Sp(X)$ (the spectrum of X). Then, $f(X)\geq g(X)$.*

Theorem 3.2. *Let $I,Q\in M_n(\mathbb{C})$ be positive definite, $0<v\leq\tau<1$ and $p\geq 1$, the following inequalities hold:*

$$\frac{v}{\tau}\left(I\sharp_{p,\tau}Q-I\sharp_\tau Q\right)\leq I\sharp_{p,v}Q-I\sharp_v Q\leq\frac{1-v}{1-\tau}\left(I\sharp_{p,\tau}Q-I\sharp_\tau Q\right) \tag{3.1}$$

and

$$\frac{v}{\tau}\left(I\sharp_{p,\tau}Q-I!_\tau Q\right)\leq I\sharp_{p,v}Q-I!_v Q\leq\frac{1-v}{1-\tau}\left(I\sharp_{p,\tau}Q-I!_\tau Q\right). \tag{3.2}$$

P r o o f. According to the above conditions, we can let the positive definite matrix $Q=U^*YU$, where U represents some unitary matrix and Y represents diagonal matrix. $Y=\operatorname{diag}(\lambda_1,\lambda_2,\ldots\lambda_n)$, λ_i is the eigenvalue of Q and

$$\lambda_i>0,\quad i=1,2,\ldots n,\quad I\sharp_{p,v}Y=\operatorname{diag}\left(1\sharp_{p,v}\lambda_1,1\sharp_{p,v}\lambda_2,\ldots,1\sharp_{p,v}\lambda_n\right).$$

Thus, we can apply the first inequality of (2.1) for $a = 1$ and $b = \lambda_i$, we have

$$I\sharp_{p,v}Y - I\sharp_v Y \geq \frac{v}{\tau}\left(I\sharp_{p,\tau}Y - I\sharp_\tau Y\right).$$

Then, multiply left by U and right by U^* on the inequality to get the desired results. This proves inequality (3.1), similarly, (3.2) holds by the second inequality of (2.1). □

Theorem 3.3. *Let $A, B \in M_n(\mathbb{C})$ be positive definite, $0 < v \leq \tau < 1$ and $p \geq 1$, the following inequalities hold:*

$$\frac{v}{\tau}\left(A\sharp_{p,\tau}B - A\sharp_\tau B\right) \leq A\sharp_{p,v}B - A\sharp_v B \leq \frac{1-v}{1-\tau}\left(A\sharp_{p,\tau}B - A\sharp_\tau B\right) \tag{3.3}$$

and

$$\frac{v}{\tau}\left(A\sharp_{p,\tau}B - A!_\tau B\right) \leq A\sharp_{p,v}B - A!_v B \leq \frac{1-v}{1-\tau}\left(A\sharp_{p,\tau}B - A!_\tau B\right). \tag{3.4}$$

Proof. Put $Q = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in inequalities (3.1) and (3.2), then multiply both sides of the inequalities by $A^{\frac{1}{2}}$ to get the results. □

Theorem 3.4. *Let $I, Q \in M_n(\mathbb{C})$ be positive definite, $0 < v \leq \tau < 1$, $p \neq 1$ and $m \in \mathbb{N}^+$, the following inequalities hold:*

(i) *If $I \geq Q$, then we have*

$$(I\nabla_v Q)^m - (I\sharp_{p,v}Q)^m \geq \frac{v}{\tau}\big((I\nabla_\tau Q)^m - (I\sharp_{p,\tau}Q)^m\big). \tag{3.5}$$

(ii) *If $I \leq Q$, then we have*

$$(I\nabla_v Q)^m - (I\sharp_{p,v}Q)^m \leq \frac{1-v}{1-\tau}\big((I\nabla_\tau Q)^m - (I\sharp_{p,\tau}Q)^m\big). \tag{3.6}$$

Proof. According to the above conditions, we can let the positive definite matrix $Q = U^*YU$, where U represents some unitary matrix and Y represents diagonal matrix. $Y = \operatorname{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$, λ_i is the eigenvalue of Q and $\lambda_i > 0$, $i = 1, 2, \ldots n$, then

$$(I\sharp_{p,v}Y)^m = \operatorname{diag}\big((1\sharp_{p,v}\lambda_1)^m, (1\sharp_{p,v}\lambda_2)^m, \ldots, (1\sharp_{p,v}\lambda_n)^m\big).$$

Thus, we can apply (2.3) for $a = 1$, $b = \lambda_i \leq 1$, we have

$$(I\nabla_v Y)^m - (I\sharp_{p,v}Y)^m \geq \frac{v}{\tau}\big((I\nabla_\tau Y)^m - (I\sharp_{p,\tau}Y)^m\big). \tag{3.7}$$

Then multiply left by U and right by U^* on inequality to get the desired results. This proves inequality (3.5), similarly, (3.6) holds by inequality (2.4). □

Theorem 3.5. *Let $I, Q \in M_n(\mathbb{C})$ be positive definite, $0 < v \leq \tau < 1$, $p \geq 1$ and $m \in \mathbb{N}^+$, the following inequalities hold:*

(i) *If $I \geq Q$, then we have that*

$$(I\sharp_{p,v}Q)^m - (I\sharp_v Q)^m \geq \frac{v}{\tau}\big((I\sharp_{p,\tau}Q)^m - (I\sharp_\tau Q)^m\big) \tag{3.8}$$

and

$$(I\sharp_{p,v}Q)^m - (I!_v Q)^m \geq \frac{v}{\tau}\big((I\sharp_{p,\tau}Q)^m - (I!_\tau Q)^m\big). \tag{3.9}$$

(ii) *If $I \leq Q$, then we have that*

$$(I\sharp_{p,v}Q)^m - (I\sharp_v Q)^m \leq \frac{1-v}{1-\tau}\big((I\sharp_{p,\tau}Q)^m - (I\sharp_\tau Q)^m\big) \tag{3.10}$$

and

$$(I\sharp_{p,v}Q)^m - (I!_v Q)^m \leq \frac{1-v}{1-\tau}\big((I\sharp_{p,\tau}Q)^m - (I!_\tau Q)^m\big). \tag{3.11}$$

P r o o f. We can prove it by the same method as that described in Theorem 3.4. □

Lemma 3.6 ([3]). *Let $a = [a_i], b = [b_i], i = 1, 2, \ldots, n$, such that a_i, b_i are positive real numbers. Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n} (a_i + b_i)\right)^{\frac{1}{n}}$$

Theorem 3.7. *Let $A, B \in {M_n}^{++}(\mathbb{C})$, $0 \leq v \leq \tau < 0$ and $p \geq 1$, then*

$$\det(A\sharp_{p,v}B)^{\frac{1}{n}} - \det(A\sharp_v B)^{\frac{1}{n}} \geq \frac{v}{\tau}\det(A\sharp_{p,\tau}B - A\sharp_\tau B)^{\frac{1}{n}} \tag{3.12}$$

and

$$\det(A\sharp_{p,v}B)^{\frac{1}{n}} - \det(A!_v B)^{\frac{1}{n}} \geq \frac{v}{\tau}\det(A\sharp_{p,\tau}B - A!_\tau B)^{\frac{1}{n}}. \tag{3.13}$$

P r o o f. Let $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, $a = 1$ and $b = s_i(T)$ in the first inequality of (2.1), we have

$$\frac{1\sharp_{p,v}s_i(T) - 1\sharp_v s_i(T)}{1\sharp_{p,\tau}s_i(T) - 1\sharp_\tau s_i(T)} \geq \frac{v}{\tau}, \quad s_i(T) \neq 1, \quad i = 1, 2, \ldots, n.$$

$$\begin{aligned}
\det\left(I\sharp_{p,v}T\right)^{\frac{1}{n}} &= \left(\prod_{i=1}^{n} 1\sharp_{p,v}s_i\left(T\right)\right)^{\frac{1}{n}} \\
&\geq \left(\prod_{i=1}^{n}\left[\frac{v}{\tau}\big(1\sharp_{p,\tau}s_i\left(T\right)-1\sharp_{\tau}s_i\left(T\right)\big)+1\sharp_{v}s_i\left(T\right)\right]\right)^{\frac{1}{n}} \\
&\geq \frac{v}{\tau}\prod_{i=1}^{n}\big(1\sharp_{p,\tau}s_i\left(T\right)-1\sharp_{\tau}s_i\left(T\right)\big)^{\frac{1}{n}}+\prod_{i=1}^{n}\big(1\sharp_{v}s_i\left(T\right)\big)^{\frac{1}{n}} \\
&\quad \text{(by Lemma 3.6)} \\
&= \frac{v}{\tau}\det\left(I\sharp_{p,\tau}T-I\sharp_{\tau}T\right)^{\frac{1}{n}}+\det\left(I\sharp_{v}T\right)^{\frac{1}{n}}.
\end{aligned}$$

Then, multiply the both sides of the above inequalities by $\big(\det A^{\frac{1}{2}}\big)^{\frac{1}{n}}$, we can get the desired results. Using the same method, we can obtain (3.13) easily. □

Theorem 3.8. *Let $A, B \in M_n{}^{++}(\mathbb{C})$, $A \leq B$, $0 \leq v \leq \tau < 0$, $p \neq 1$ and $m \in \mathbb{N}^+$, we have that*

$$\det\left(A\nabla_{\tau}B\right)^{\frac{m}{n}}-\det\left(A\sharp_{p,\tau}B\right)^{\frac{m}{n}} \geq \frac{1-\tau}{1-v}\det\left(A\nabla_{v}B-A\sharp_{p,v}B\right)^{\frac{m}{n}}. \qquad (3.14)$$

Proof. We have $s_i\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big) \geq 1$ for $0 < A \leq B$. Let $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, $a = 1$ and $b = s_i\left(T\right) \geq 1$ in inequality (2.4), then

$$\frac{\left(1\nabla_{v}s_i\left(T\right)\right)^m-\left(1\sharp_{p,v}s_i\left(T\right)\right)^m}{\left(1\nabla_{\tau}s_i\left(T\right)\right)^m-\left(1\sharp_{p,\tau}s_i\left(T\right)\right)^m} \leq \frac{1-v}{1-\tau}, \quad s_i\left(T\right) \neq 1, \quad i = 1, 2, \ldots, n.$$

$$\begin{aligned}
&\det\left(I\nabla_{\tau}T\right)^{\frac{m}{n}} = \big(\left(1\nabla_{\tau}s_i\left(T\right)\right)^m\big)^{\frac{1}{n}} \\
&\geq \prod_{i=1}^{n}\left[\frac{1-\tau}{1-v}\Big(\big(1\nabla_{v}s_i\left(T\right)\big)^m-\big(1\sharp_{p,v}s_i\left(T\right)\big)^m\Big)+\big(1\sharp_{p,\tau}s_i\left(T\right)\big)^m\right]^{\frac{1}{n}} \\
&\geq \prod_{i=1}^{n}\left[\frac{1-\tau}{1-v}\Big(\big(1\nabla_{v}s_i\left(T\right)\big)^m-\big(1\sharp_{p,v}s_i\left(T\right)\big)^m\Big)\right]^{\frac{1}{n}}+\prod_{i=1}^{n}\big(1\sharp_{p,\tau}s_i\left(T\right)\big)^{\frac{m}{n}} \\
&\quad \text{(by Lemma 3.6)} \\
&\geq \frac{1-\tau}{1-v}\prod_{i=1}^{n}\big(1\nabla_{v}s_i\left(T\right)-1\sharp_{p,v}s_i\left(T\right)\big)^{\frac{m}{n}}+\prod_{i=1}^{n}\big(1\sharp_{p,\tau}s_i\left(T\right)\big)^{\frac{m}{n}} \\
&= \frac{1-\tau}{1-v}\det\left(I\nabla_{v}T-I\sharp_{p,v}T\right)^{\frac{m}{n}}+\det\left(I\sharp_{p,\tau}T\right)^{\frac{m}{n}}.
\end{aligned}$$

Since $a \geq b > 0$ and $m \in \mathbb{N}^+$, $a^m - b^m \geq (a-b)^m$, we can get the last inequality. Then, multiply the both sides of the above inequalities by $\big(\det A^{\frac{1}{2}}\big)^{\frac{m}{n}}$, we can obtain the desired results. □

Theorem 3.9. *Let $A, B \in M_n^{++}(\mathbb{C})$, $A \geq B$, $0 \leq v \leq \tau < 0$, $p \geq 1$ and $m \in \mathbb{N}^+$, we have that*

$$\det\left(A\sharp_{p,v}B\right)^{\frac{m}{n}} - \det\left(A\sharp_v B\right)^{\frac{m}{n}} \geq \frac{v}{\tau}\det\left(A\sharp_{p,\tau}B - A\sharp_\tau B\right)^{\frac{m}{n}} \tag{3.15}$$

and

$$\det\left(A\sharp_{p,v}B\right)^{\frac{m}{n}} - \det\left(A!_v B\right)^{m} \geq \frac{v}{\tau}\det\left(A\sharp_{p,\tau}B - A!_\tau B\right)^{\frac{m}{n}}. \tag{3.16}$$

Proof. We have $s_i\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \leq 1$ for $A \geq B > 0$. Let $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, $a = 1$ and $b = s_i(T) \leq 1$ in inequality (2.5), then

$$\frac{\left(1\sharp_{p,v}s_i(T)\right)^m - \left(1\sharp_v s_i(T)\right)^m}{\left(1\sharp_{p,\tau}s_i(T)\right)^m - \left(1\sharp_\tau s_i(T)\right)^m} \geq \frac{v}{\tau}, \quad s_i(T) \neq 1, \quad i = 1, 2, \ldots, n.$$

$$\begin{aligned}
\det\left(I\sharp_{p,v}T\right)^{\frac{m}{n}} &= \left(\prod_{i=1}^{n} 1\sharp_{p,v}s_i(T)\right)^{\frac{m}{n}} \\
&\geq \left(\prod_{i=1}^{n}\left[\frac{v}{\tau}\Big(\left(1\sharp_{p,\tau}s_i(T)\right)^m - \left(1\sharp_\tau s_i(T)\right)^m\Big) + \left(1\sharp_v s_i(T)\right)^m\right]\right)^{\frac{1}{n}} \\
&\geq \prod_{i=1}^{n}\left[\frac{v}{\tau}\Big(\left(1\sharp_{p,\tau}s_i(T)\right)^m - \left(1\sharp_\tau s_i(T)\right)^m\Big)\right]^{\frac{1}{n}} + \prod_{i=1}^{n}\left(1\sharp_v s_i(T)\right)^{\frac{m}{n}} \\
&\quad \text{(by Lemma 3.6)} \\
&\geq \frac{v}{\tau}\prod_{i=1}^{n}\left(1\sharp_{p,\tau}s_i(T) - 1\sharp_\tau s_i(T)\right)^{\frac{m}{n}} + \prod_{i=1}^{n}\left(1\sharp_v s_i(T)\right)^{\frac{m}{n}} \\
&= \frac{v}{\tau}\det\left(I\sharp_{p,\tau}T - I\sharp_\tau T\right)^{\frac{m}{n}} + \det\left(I\sharp_v T\right)^{\frac{m}{n}}.
\end{aligned}$$

Since $a \geq b > 0$ and $m \in \mathbb{N}^+$, $a^m - b^m \geq (a-b)^m$, we can get the last inequality. Then, multiply the both sides of the above inequalities by $\left(\det A^{\frac{1}{2}}\right)^{\frac{m}{n}}$, we can obtain the desired results. Inequality (3.16) can be proved analogously as above. □

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