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ON O'MALLEY LOWER POROUSCONTINUOUS FUNCTIONS

Jakub Bilski—Anna Kamińska—Stanisław Kowalczyk—Małgorzata Turowska

Institute of Exact and Technical Sciences, Pomeranian University in Słupsk, Słupsk, POLAND

ABSTRACT. In 2014, J. Borsík and J. Holos defined porouscontinuous functions. Using the notion of density in the sense of O'Malley, new definitions of porouscontinuity were introduced in 2021, namely $\mathcal{MO}_r$ and $\mathcal{SO}_r$-continuity. These kinds of porouscontinuity used upper porosity. We consider lower porouscontinuity in the sense of O'Malley, where lower porosity is used instead of standard (upper) porosity.

1. Preliminaries

In 2014, J. Borsík and J. Holos [1] defined classes of porouscontinuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Porouscontinuity is an example of generalized continuity defined by a family of sets that are "large" in some sense in the neighbourhood of the point. Namely the function f is continuous at x if every preimage of the open interval to which $f(x)$ belongs contains a set from this family. J. Borsík and J. Holos used a family of sets whose complements have a specific upper porosity at a point in their definition. Porouscontinuity was studied in many papers (see, for example, [5, 6]). It turned out that these classes of functions can be supplemented with porouscontinuous functions in the sense of O'Malley, which was shown in [3]. The aim of the presented paper is to transfer these results to lower porouscontinuous functions. Throughout the paper, we consider only functions from $\mathbb{R}$ to $\mathbb{R}$. In the first section, we recall the properties of the upper porosity, the upper porouscontinuous functions and the O'Malley upper porouscontinuous functions. These results are summarized in Theorem 1.11.

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The second section is devoted to the lower porosity. An important result is Theorem 2.3 showing that if x belongs to the closure of A, then the lower porosity of A at x is not greater than a half. Theorem 2.10 and Corollary 2.12 show a method of constructing interval sets with a predetermined lower porosity.

In the last section, replacing the upper porosity with the lower porosity, we define classes of lower porouscontinuous functions $\underline{\mathcal{S}}_r$, $\underline{\mathcal{P}}_r$, $\underline{\mathcal{M}}_r$ and $\underline{\mathcal{N}}_r$. Then, applying O'Malley's ideas, we introduce O'Malley lower porouscontinuous functions $\underline{\mathcal{SO}}_r$ and $\underline{\mathcal{MO}}_r$. In the last part of the paper, the relationships between considered types of lower porouscontinuity are investigated. These relationships are summarized in Theorem 3.16 showing a sequence of equalities and proper inclusions.

$$\begin{aligned}\mathcal{C} = \underline{\mathcal{M}}_1 = \underline{\mathcal{P}}_{\frac{1}{2}} = \underline{\mathcal{S}}_{\frac{1}{2}} = \underline{\mathcal{SO}}_{\frac{1}{2}} = \underline{\mathcal{MO}}_{\frac{1}{2}} \subset \underline{\mathcal{M}}_{\frac{1}{2}} \subset& \\ \underline{\mathcal{P}}_t \subset \underline{\mathcal{S}}_t \subset \underline{\mathcal{SO}}_t \subset \underline{\mathcal{MO}}_t \subset \underline{\mathcal{M}}_t \subset& \\ \underline{\mathcal{P}}_r \subset \underline{\mathcal{S}}_r \subset \underline{\mathcal{SO}}_r \subset \underline{\mathcal{MO}}_r \subset \underline{\mathcal{M}}_r \subset& \\ \underline{\mathcal{P}}_0 \subset \underline{\mathcal{S}}_0 \subset \underline{\mathcal{SO}}_0 = \mathcal{Q}^{bil}&\end{aligned}$$

for $0 < r < t < \frac{1}{2}$.

Now, we present the basic notations and recall the necessary results from previous research. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural and the set of all real numbers, respectively. By $\operatorname{cl}(A)$ and A^d we denote a closure and a set of accumulation points of a set $A \subset \mathbb{R}$, respectively. By $f_{\restriction A}$ we denote the restriction of f to $A \subset \mathbb{R}$. For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$, let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A. Then according to [1, 9, 10], the right (upper) porosity of the set A at $x \in \mathbb{R}$ is defined as

$$p^+(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left (upper) porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the (upper) porosity of A at x is defined as

$$p(A, x) = \max\left\{p^-(A, x), p^+(A, x)\right\}.$$

In 2014, J. Borsik and J. Holos defined the families of porouscontinuous functions.

Definition 1.1 ([1]). Let $r \in [0, 1)$. A point $x \in \mathbb{R}$ will be called a point of π_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) > r$.

Definition 1.2 ([1]). Let $r \in (0, 1]$. A point $x \in \mathbb{R}$ will be called a point of μ_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) \geq r$.

Definition 1.3 ([1]). Let $r \in [0,1)$ and $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ will be called

- $\mathcal{P}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f_{\restriction A}$ is continuous at x;
- $\mathcal{S}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f(A) \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$.

Let $r \in (0,1]$ and $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ will be called

- $\mathcal{M}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f_{\restriction A}$ is continuous at x;
- $\mathcal{N}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f(A) \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$.

All these functions are called porouscontinuous functions.

The symbols $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ denote the set of all points at which $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{P}_r$-continuous, $\mathcal{S}_r$-continuous, $\mathcal{M}_r$-continuous and $\mathcal{N}_r$--continuous, respectively. In [1], the equality $\mathcal{M}_r(f) = \mathcal{N}_r(f)$ for every f and every $r \in (0,1]$ was proved. Observe that if f is right-hand continuous or left--hand continuous at some x, then f is porouscontinuous at x.

In [8], R. J. O'Malley modified the notion of preponderant continuity. Preponderant continuity is an another type of generalized continuity similar to porouscontinuity, in which a lower density of a Lebesgue measurable set at a point is used instead of porosity. R. J. O'Malley showed that one can replace the density of a set with another condition involving the Lebesgue measure, [4,8]. Combining the notion of porouscontinuity defined by J. Borsík and J. Holos and using the concept of R. J. O'Malley, we obtained other types of porouscontinuity. O'Malley's idea is that instead of examining porosity or density, we demand that the ratio of the measure of a given set intersected with $(x, x+h)$ or $\Lambda\big(A, (x, x+h)\big)$ to the value h is specified.

Definition 1.4 ([3]). Let $r \in [0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. A point x will be called a point of πO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0,\eta)$ and an open interval $(a,b) \subset A \cap \big((x-\delta, x+\delta) \setminus \{x\}\big)$ such that $\frac{b-a}{\delta} > r$.

Definition 1.5 ([3]). Let $r \in (0,1]$, $x \in \mathbb{R}$, $A \subset \mathbb{R}$. A point x will be called a point of μO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0,\eta)$ and $(a,b) \subset A \cap \big((x-\delta, x+\delta) \setminus \{x\}\big)$ such that $\frac{b-a}{\delta} \geq r$.

Directly from the above definitions and Definitions 1.1 and 1.2, we obtain the following remarks.

Remark 1.6 ([3]). Let $r \in (0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of πO_r-density of A, then x is a point of μO_r-density of A.

Remark 1.7 ([3])**.** Let $r \in [0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of π_r-density of A, then x is a point of πO_r-density of A.

Remark 1.8 ([3])**.** Let $r \in (0,1]$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of μO_r-density of A, then x is a point of μ_r-density of A.

Definition 1.9 ([3])**.** Let $r \in [0,1)$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{SO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of πO_r-density of a set $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$.

Definition 1.10 ([3])**.** Let $r \in (0,1]$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{MO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of μO_r-density of a set $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$.

The symbols $\mathcal{SO}_r(f)$ and $\mathcal{MO}_r(f)$ denote the set of all points at which f is $\mathcal{SO}_r$-continuous, $\mathcal{MO}_r$-continuous, respectively, for corresponding r.

We say that $f \colon \mathbb{R} \to \mathbb{R}$ is quasicontinuous at $x \in \mathbb{R}$ if for every $\varepsilon > 0$ and $\delta > 0$ there exists an open interval $(a,b) \subset (x-\delta, x+\delta)$ such that $|f(x) - f(y)| < \varepsilon$ for every $y \in (a,b)$. Some properties of quasicontinuity can be found, for example, in [2, 7].

$\mathcal{Q}(f)$ and $\mathcal{C}(f)$ denote the set of points at which f is quasicontinuous and continuous, respectively.

We introduce the following notations:

- $\mathcal{C} = \{f \colon \mathcal{C}(f) = \mathbb{R}\}$, $\mathcal{Q} = \{f \colon \mathcal{Q}(f) = \mathbb{R}\}$ and $\mathcal{C}^{\pm}$ is the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that at every $x \in \mathbb{R}$, f is right-hand continuous or left-hand continuous (obviously $\mathcal{C} \subsetneq \mathcal{C}^{\pm}$),
- for $r \in (0,1]$, let $\mathcal{M}_r = \{f \colon \mathcal{M}_r(f) = \mathbb{R}\}$, $\mathcal{MO}_r = \{f \colon \mathcal{MO}_r(f) = \mathbb{R}\}$,
- for $r \in [0,1)$, let $\mathcal{P}_r = \{f \colon \mathcal{P}_r(f) = \mathbb{R}\}$, $\mathcal{S}_r = \{f \colon \mathcal{S}_r(f) = \mathbb{R}\}$ and $\mathcal{SO}_r = \{f \colon \mathcal{SO}_r(f) = \mathbb{R}\}$.

Theorem 1.11 ([3])**.** *Let $0 < r < t < 1$. Then,*

$$\begin{aligned}&\mathcal{C}^{\pm} = \mathcal{MO}_1 \subset \mathcal{M}_1 \subset \\ &\quad \mathcal{P}_t \subset \mathcal{S}_t \subset \mathcal{SO}_t \subset \mathcal{MO}_t \subset \mathcal{M}_t \subset \mathcal{P}_r \subset \mathcal{S}_r \subset \mathcal{SO}_r \subset \mathcal{MO}_r \subset \mathcal{M}_r \subset \\ &\qquad \mathcal{P}_0 \subset \mathcal{S}_0 \subset \mathcal{SO}_0 = \mathcal{Q}\end{aligned}$$

and all inclusions are proper.

2. Lower porosity in $\mathbb{R}$

By replacing the upper limit with the lower limit in the definition of porosity, we obtain the right lower porosity of the set $A \subset \mathbb{R}$ at $x \in \mathbb{R}$ as

$$\underline{p}^{+}(A,x) = \liminf_{h\to 0^{+}} \frac{\Lambda(A,(x,x+h))}{h},$$

the left lower porosity as

$$\underline{p}^{-}(A,x) = \liminf_{h\to 0^{+}} \frac{\Lambda(A,(x-h,x))}{h},$$

and the lower porosity as

$$\underline{p}\,(A,x) = \min\left\{\underline{p}^{-}(A,x), \underline{p}^{+}(A,x)\right\}.$$

Remark 2.1. Obviously, if a set $A \subset \mathbb{R}$ is symmetric about x, then $\underline{p}(A,x) = \underline{p}^{+}(A,x) = \underline{p}^{-}(A,x)$.

Clearly, $\underline{p}^{+}(A,x) \le p^{+}(A,x)$ and $\underline{p}^{-}(A,x) \le p^{-}(A,x)$ for every $A \subset \mathbb{R}$ and $x \in \mathbb{R}$.

Example 2.2. We construct a set $A \subset \mathbb{R}$ such that $\underline{p}^{+}(A,0)=0$ and $p^{+}(A,0)=1$.

Let $x_n = \frac{1}{n!}$, $A = \{0\} \cup \bigcup_{n=1}^{\infty}(x_{2n+1}, x_{2n})$. Clearly,

$$\underline{p}^{+}(A,0) \le \liminf_{n\to\infty} \frac{\Lambda(A,(0,x_{2n}))}{x_{2n}} \le \lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n}} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

and

$$p^{+}(A,0) \ge \limsup_{n\to\infty} \frac{\Lambda(A,(0,x_{2n-1}))}{x_{2n-1}} \ge \lim_{n\to\infty} \frac{x_{2n-1}-x_{2n}}{x_{2n-1}}$$

$$= \lim_{n\to\infty} \frac{\frac{2n-1}{(2n)!}}{\frac{1}{(2n-1)!}} = \lim_{n\to\infty} \frac{2n-1}{2n} = 1.$$

Theorem 2.3. *Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. If there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ with the terms belonging to A converging to x, then for every $\varepsilon > 0$ there exists $h \in (0,\varepsilon)$ such that $\frac{\Lambda(A,(x,x+h))}{h} < \frac{1}{2}$.*

Proof. Take any $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $x_n - x < \frac{\varepsilon}{2}$ and let $\alpha = \Lambda\big(A,(x,x_n)\big)$. Clearly, $\alpha < x_n - x$ and $\Lambda\big(A,(x,x_n+\alpha)\big) = \alpha$. Put $h = x_n + \alpha - x$. Then, $2\alpha < h < 2(x_n - x) < \varepsilon$ and $\frac{\Lambda(A,(x,x+h))}{h} = \frac{\alpha}{h} < \frac{1}{2}$. □

Corollary 2.4. *Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. If there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ with terms belonging to A converging to x, then $\underline{p}^{+}(A,x) \le \frac{1}{2}$.*

Similarly, we can show that if there exists an increasing sequence $(x_n)_{n\in\mathbb{N}}$ with terms belonging to A converging to x, then $\underline{p}^-(A,x) \le \frac{1}{2}$.

Corollary 2.5. *For every $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ we have*

$$\underline{p}(A,x) \le \frac{1}{2} \ \text{ if } x \in \operatorname{cl}(A) \quad \text{ and } \quad \underline{p}(A,x) = 1 \ \text{ if } x \notin \operatorname{cl}(A).$$

Now, we prove a number of technical theorems that allow us to construct sets with the desired porosity. These theorems are the main tool in proofs of relationships between classes of lower porouscontinuous functions, which are the main topic of the paper.

Lemma 2.6. *For every $[a,b)$ there exist two sequences $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$ such that $a < \cdots < b_{n+1} < a_n < b_n < \cdots < b$, $\lim_{n\to\infty} a_n = a$ and*

$$\underline{p}^+\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (a_n, b_n), a\right) = \frac{1}{2}.$$

P r o o f. Without loss of generality we assume that $b-a > 1$. Let $a_n = a + \frac{1}{(n+1)!}$ and $b_n = a + \frac{1}{n!} - \frac{1}{((n+1)!)^2}$ for $n \ge 1$ and let $A = \mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n)$. Choose $n \ge 2$. Clearly, $\Lambda\big(A,(a,y)\big) = b_n - a_n = \frac{n}{(n+1)!} - \frac{1}{((n+1)!)^2}$ for $y \in [b_n, a_{n-1} + b_n - a_n]$ and $\Lambda\big(A,(a,y)\big) = y - a_{n-1}$ for $y \in [a_{n-1} + b_n - a_n, b_{n-1}]$. Hence,

$$\frac{\Lambda(A,(a,y))}{y-a} \ge \frac{b_n - a_n}{a_{n-1} + b_n - a_n - a}$$

for $y \in [b_n, a_{n-1} + b_n - a_n]$ and

$$\frac{\Lambda(A,(a,y))}{y-a} = \frac{y - a_{n-1}}{y-a} \ge \frac{b_n - a_n}{a_{n-1} + b_n - a_n - a}$$

for $y \in [a_{n-1} + b_n - a_n, b_{n-1}]$. Therefore, for every $y \in [b_n, b_{n-1}]$ we have

$$\begin{aligned}\frac{\Lambda(A,(a,y))}{y-a} &\ge \frac{b_n - a_n}{b_n - a_n + a_{n-1} - a} \\ &= \frac{\frac{1}{n!} - \frac{1}{(n+1)!} - \frac{1}{((n+1)!)^2}}{\frac{1}{n!} - \frac{1}{(n+1)!} - \frac{1}{((n+1)!)^2} + \frac{1}{n!}} = \frac{1 - \frac{1}{n+1} - \frac{1}{(n+1)!(n+1)}}{2 - \frac{1}{n+1} - \frac{1}{(n+1)!(n+1)}}.\end{aligned}$$

Finally,

$$\underline{p}^+(A,a) = \liminf_{n\to\infty} \frac{1 - \frac{1}{n+1} - \frac{1}{(n+1)!(n+1)}}{2 - \frac{1}{n+1} - \frac{1}{(n+1)!(n+1)}} = \frac{1}{2}.$$

□

We can show the following lemma in a similar way.

Lemma 2.7. *For every $(a,b]$ there exist two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ such that $a < \cdots < a_n < b_n < a_{n+1} < \cdots < b$, $\lim_{n\to\infty} a_n = b$ and*

$$\underline{p}^-\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n), b\right) = \frac{1}{2}.$$

Lemma 2.8. *Let $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$ be such that $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$ and let $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$. Then,*

$$\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = \frac{a_{n+1}-b_{n+2}}{b_{n+1}-x+a_{n+1}-b_{n+2}}.$$

Proof. If $r \in [a_{n+1}, b_{n+1} + a_{n+1} - b_{n+2}]$, then $\Lambda\big(A,(x,r)\big) = a_{n+1} - b_{n+2}$ and

$$\frac{\Lambda(A,(x,x+(r-x)))}{r-x} \geq \frac{\Lambda(A,(x,b_{n+1}+a_{n+1}-b_{n+2}))}{b_{n+1}-x+a_{n+1}-b_{n+2}}.$$

If $r \in [b_{n+1} + a_{n+1} - b_{n+2}, a_n)$, then

$$\begin{aligned}\Lambda\big(A,(x,r)\big) &= r - b_{n+1}\\ &= \Lambda\big(A,(x,b_{n+1}+a_{n+1}-b_{n+2})\big) + \Big(r - \big(b_{n+1}+a_{n+1}-b_{n+2}\big)\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\Lambda(A,(x,r))}{r-x} &= \frac{\Lambda(A,(x,b_{n+1}+a_{n+1}-b_{n+2})) + (r-(b_{n+1}+a_{n+1}-b_{n+2}))}{b_{n+1}-x+a_{n+1}-b_{n+2}+(r-(b_{n+1}+a_{n+1}-b_{n+2}))}\\ &\geq \frac{\Lambda(A,(x,b_{n+1}+a_{n+1}-b_{n+2}))}{b_{n+1}-x+a_{n+1}-b_{n+2}}.\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{h\in[a_{n+1}-x,a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} &= \frac{\Lambda(A,(x,b_{n+1}+a_{n+1}-b_{n+2}))}{b_{n+1}-x+a_{n+1}-b_{n+2}}\\ &= \frac{a_{n+1}-b_{n+2}}{b_{n+1}-x+a_{n+1}-b_{n+2}},\end{aligned}$$

which completed the proof. □

Corollary 2.9. *Let $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$, where $x < \cdots < b_{n+1} < a_n < b_n < \cdots \cdots < a_1 < b_1$, $\lim_{n\to\infty} a_n = x$ be such that $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$. Then,*

$$\underline{p}^+(A,x) = \liminf_{n\to\infty} \frac{a_{n+1}-b_{n+2}}{b_{n+1}-x+a_{n+1}-b_{n+2}}.$$

Theorem 2.10. *For every $x \in \mathbb{R}$ and for every sequence $(c_n)_{n\geq 1}$ from $(0, \frac{1}{2})$ there exist two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $\lim_{n\to\infty} a_n = x$, $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$, $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$ and $\inf_{h\in[a_{n+1}-x,a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = c_n$, where $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$.*

P r o o f. Take any $x \in \mathbb{R}$ and $(c_n)_{n\geq 1}$ from $(0, \frac{1}{2})$. We construct the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ inductively. Let $b_1 > x$ be arbitrary. We can find $\gamma_1 \in (0, b_1 - x)$ such that $\frac{\gamma_1}{\gamma_1+(b_1-x)} = c_1$. Define b_2, a_1 such that $x < b_2 < a_1 < b_1$, $a_1 - b_2 = \gamma_1$ and $b_2 - x < \gamma_1$. Then, $\frac{a_1-b_2}{(b_1-x)+(a_1-b_2)} = c_1$. Assume that $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n, b_{n+1}$ satisfying $x < b_{n+1} < a_n < b_n < \cdots \cdots < a_1 < b_1$, $\frac{a_i-b_{i+1}}{b_i-x+a_i-b_{i+1}} = c_i$ and $b_{i+1} - x < a_i - b_{i+1}$ for $i = 1, 2, \dots, n$ are chosen. We can find $\gamma_{n+1} \in (0, b_{n+1} - x)$ such that $\frac{\gamma_{n+1}}{\gamma_{n+1}+(b_{n+1}-x)} = c_{n+1}$. Define b_{n+2}, a_{n+1} such that $x < b_{n+2} < a_{n+1} < b_{n+1}$, $a_{n+1} - b_{n+2} = \gamma_{n+1}$ and $b_{n+2} - x < \gamma_{n+1}$. Then, $\frac{a_{n+1}-b_{n+2}}{b_{n+1}-x+a_{n+1}-b_{n+2}} = c_{n+1}$. Thus, the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfying $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$ and $a_{n+2} < a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$ are defined. Then, the equality $\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = c_n$ for $n \geq 1$ follows from Lemma 2.8. $\square$

Theorem 2.11. *For every $x \in \mathbb{R}$ and for every sequence $(c_n)_{n\geq 1}$ from $(0, \frac{1}{2})$ there exist two sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \alpha_n = x$, $x < \cdots < \beta_{n+1} < \alpha_n < \beta_n < \cdots < \alpha_1 < \beta_1 < \alpha_0$, $\beta_{n+1} - \alpha_{n+1} < \beta_n - \alpha_n$ for every $n \geq 1$ and $\inf_{h\in[\beta_{n+1}-x, \beta_n-x]} \frac{\Lambda(\mathbb{R}\setminus B,(x,x+h))}{h} = c_n$, where $B = \bigcup_{n=1}^{\infty}(\alpha_n, \beta_n)$.*

P r o o f. Take any $x \in \mathbb{R}$ and $(c_n)_{n\geq 1}$ from $(0, \frac{1}{2})$. By Theorem 2.10, there exist two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $\lim_{n\to\infty} a_n = x$, $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$, $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$ and $\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = c_n$, where $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Define two new sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ by $\alpha_n = b_{n+1}$ for $n \geq 0$ and $\beta_n = a_n$ for $n \geq 1$. Then, $\beta_{n+1} - \alpha_{n+1} = a_{n+1} - b_{n+2} < a_n - b_{n+1} = \beta_n - \alpha_n$ and

$$\begin{aligned}
c_n &= \inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} \\
&= \inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(\bigcup_{n=1}^{\infty}(a_n,b_n),(x,x+h))}{h} \\
&= \inf_{h\in[\beta_{n+1}-x, \beta_n-x]} \frac{\Lambda(\bigcup_{n=1}^{\infty}(\beta_n,\alpha_{n-1}),(x,x+h))}{h} \\
&= \inf_{h\in[\beta_{n+1}-x, \beta_n-x]} \frac{\Lambda(\bigcup_{n=1}^{\infty}[\beta_n,\alpha_{n-1}],(x,x+h))}{h} \\
&= \inf_{h\in[\beta_{n+1}-x, \beta_n-x]} \frac{\Lambda(\mathbb{R}\setminus\bigcup_{n=1}^{\infty}(\alpha_n,\beta_n),(x,x+h))}{h}.
\end{aligned}$$

$\square$

Corollary 2.12. *For every $c, d \in [0, \frac{1}{2}]$ and $x \in \mathbb{R}$ there exists $A \subset \mathbb{R}$ such that $\underline{p}^{+}(A,x) = c$ and $\underline{p}^{-}(A,x) = d$.*

Proof. For every $n \geq 1$ let us take $c_n = \frac{1}{2n+1}$ if $c = 0$, $c_n = c$ if $c \in (0, \frac{1}{2})$ and $c_n = \frac{1}{2} - \frac{1}{2n+1}$ if $c = \frac{1}{2}$. By Theorem 2.10, there exist two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $\lim_{n\to\infty} a_n = x$, $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$, $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ for every $n \geq 1$ and $\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = c_n$, where $A = \bigcup_{n=1}^{\infty}(a_n, b_n)$. By Lemma 2.8 and Corollary 2.9, $\underline{p}^+(A, x) = \lim_{n\to\infty} c_n = c$.

Clearly, for every $x \in \mathbb{R}$ and for every sequence $(d_n)_{n\geq 1}$ from $(0, \frac{1}{2})$ there exist two sequences $(a'_n)_{n\geq 1}$ and $(b'_n)_{n\geq 1}$ such that $\lim_{n\to\infty} a'_n = x$, $a'_1 < b'_1 < \cdots \cdots < a'_n < b'_n < a'_{n+1} < \cdots < x$, $a'_{n+2} - b'_{n+1} < a'_{n+1} - b'_n$ for every $n \geq 1$ and $\inf_{h\in[x-b'_n, x-b'_{n+1}]} \frac{\Lambda(A',(x-h,x))}{h} = d_n$, where $A' = \bigcup_{n=1}^{\infty}(a'_n, b'_n)$. This is just a theorem analogous to Theorem 2.10 for the left-hand neighbourhood of x. Again, taking $d_n = \frac{1}{2n+1}$ if $d = 0$, $d_n = d$ if $d \in (0, \frac{1}{2})$ and $d_n = \frac{1}{2} - \frac{1}{2n+1}$ if $d = \frac{1}{2}$ and applying the mentioned fact, we obtain $\underline{p}^-(A', x) = \lim_{n\to\infty} d_n = d$. □

Theorem 2.13. *Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Assume that*

$$x \in \big(A \cap (x, \infty)\big)^d \quad \text{and} \quad \Lambda\big(A, (x, x+h)\big) > 0 \quad \text{for every } h > 0.$$

Then, there exist a sequence of open intervals $((a_n, b_n))_{n\geq 1}$ from the set $(x, \infty) \setminus A$ and $\delta > 0$ such that $\lim_{n\to\infty} a_n = x$, $x < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$ and $\Lambda\big(A, (x, x+h)\big) = \Lambda\big(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n), (x, x+h)\big)$ for every $h \in (0, \delta)$.

Proof. Let

$$\mathcal{A} = \big\{(a,b) \subset (x,\infty) \setminus A : (a-\varepsilon, b) \cap (A \cup \{x\}) \neq \emptyset \neq (a, b+\varepsilon) \cap A \text{ for every } \varepsilon > 0\big\}.$$

Since $\Lambda\big(A, (x, x+h)\big) > 0$ for every $h > 0$, there exists a sequence $\big((c_k, d_k)\big)_{k\geq 1} \subset \mathcal{A}$ such that $x < \cdots < d_{k+1} < c_k < d_k < \cdots < c_1 < d_1$ and $\lim_{k\to\infty} c_k = x$. For every $k \geq 1$ let

$$B_k = \{(a,b) \subset [c_{k+1}, c_k] : (a,b) \in \mathcal{A} \quad \text{and} \quad b - a \geq d_{k+1} - c_{k+1}\}.$$

Certainly, every B_k is finite and consists of pairwise disjoint intervals. Let $\mathcal{B} = \bigcup_{k=1}^{\infty} B_k$. Obviously, $\mathcal{B}$ can be represented in the form $\mathcal{B} = \bigcup_{n=1}^{\infty}(a_n, b_n)$, where $x < \cdots < b_{n+1} < a_n < b_n < \cdots < b_1$ and $(a_n, b_n) \in \mathcal{A}$ for $n \geq 1$. Let $\delta = b_1 - x$ and take any $h \in (0, \delta)$ and $(a,b) \subset (x, x+h) \setminus A$. Then, $(a,b) \subset (c_{k_0+1}, c_{k_0})$ for some $k_0 \geq 1$.

- If $(a,b) \subset (c_{k_0+1}, d_{k_0+1})$, then

$$b - a \leq b - c_{k_0+1} \leq \Lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n), (x, x+h)\right).$$

- If $(a,b) \cap (c_{k_0+1}, d_{k_0+1}) = \emptyset$, then $(c_{k_0+1}, d_{k_0+1}) \subset (x, x+h)$ and either $b-a < d_{k_0+1} - c_{k_0+1} \le \Lambda\big(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n,b_n),(x,x+h)\big)$ or $(a,b) \subset (a_n,b_n)$ for some n such that $(a_n,b_n) \in B_{k_0}$, and again
$$b-a \le \Lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n,b_n),(x,x+h)\right).$$

Since $(a,b) \subset (x,x+h) \setminus A$ is arbitrary, we obtain an inequality
$$\Lambda\big(A,(x,x+h)\big) \le \Lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n,b_n),(x,x+h)\right).$$

The opposite inequality
$$\Lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n,b_n),(x,x+h)\right) \le \Lambda\big(A,(x,x+h)\big)$$
is obvious, because $A \subset \mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n,b_n)$. The proof is completed. □

Corollary 2.14. *For every $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ there exists an open set $G \subset (x,\infty) \cap (\mathbb{R} \setminus A)$ such that $\underline{p}^+(A,x) = \underline{p}^+(\mathbb{R} \setminus G, x)$, and for $c > x$ the set (c,∞) intersects only a finite number of components of G.*

3. Lower O'Malley porouscontinuous function

Applying lower porosity and O'Malley concept of preponderant continuity, we can define other types of porouscontinuity. We define these types of porouscontinuity only for $r \in [0, \frac{1}{2}]$ and $r = 1$, because of Corollary 2.5.

Definition 3.1. Let $r \in [0, \frac{1}{2}]$. A point $x \in \mathbb{R}$ will be called a point of $\underline{\pi}_r$-density of a set $A \subset \mathbb{R}$ if $\underline{p}(\mathbb{R} \setminus A, x) > r$.

Definition 3.2. Let $r \in (0, \frac{1}{2}] \cup \{1\}$. A point $x \in \mathbb{R}$ will be called a point of $\underline{\mu}_r$-density of a set $A \subset \mathbb{R}$ if $\underline{p}(\mathbb{R} \setminus A, x) \ge r$.

Definition 3.3. Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. A point x will be called a point of $\underline{\pi O}_r$-density of a set A if there exists $\eta > 0$ such that for every $\delta \in (0,\eta)$ we can find open intervals $(a_1,b_1) \subset A \cap (x-\delta, x)$ and $(a_2,b_2) \subset A \cap (x, x+\delta)$ for which $\frac{b_1-a_1}{\delta} > r$ and $\frac{b_2-a_2}{\delta} > r$.

Definition 3.4. Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$, $A \subset \mathbb{R}$. A point x will be called a point of $\underline{\mu O}_r$-density of a set A if there exists $\eta > 0$ such that for every $\delta \in (0,\eta)$ we can find open intervals $(a_1,b_1) \subset A \cap (x-\delta, x)$ and $(a_2,b_2) \subset A \cap (x, x+\delta)$ for which $\frac{b_1-a_1}{\delta} \ge r$ and $\frac{b_2-a_2}{\delta} \ge r$.

Directly from the above definitions, we obtain the following remark.

Remark 3.5. Let $x \in \mathbb{R}$ and $A \subset \mathbb{R}$.

- If x is a point of $\underline{\pi}_r$-density of A, then x is a point of $\underline{\pi O}_r$-density of A for $r \in [0, \frac{1}{2}]$.
- If x is a point of $\underline{\pi O}_r$-density of A, then x is a point of $\underline{\mu O}_r$-density of A for $r \in (0, \frac{1}{2}]$.
- If x is a point of $\underline{\mu O}_r$-density of A, then x is a point of $\underline{\mu}_r$-density of A for $r \in (0, \frac{1}{2}] \cup \{1\}$.

Definition 3.6. Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is

- $\underline{\mathcal{SO}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\pi O}_r$-density of a set $f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big)$;
- $\underline{\mathcal{S}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\pi}_r$-density of a set $f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big)$;
- $\underline{\mathcal{P}}_r$-continuous at x if there exists $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $\underline{\pi}_r$-density of A and $f_{\restriction A}$ is continuous at x.

Definition 3.7. Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. We will say that f is

- $\underline{\mathcal{MO}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\mu O}_r$-density of a set $f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big)$;
- $\underline{\mathcal{M}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\mu}_r$-density of a set $f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big)$;
- $\underline{\mathcal{N}}_r$-continuous at x if there exists $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $\underline{\mu}_r$-density of A and $f_{\restriction A}$ is continuous at x.

The symbols

$$\underline{\mathcal{SO}}_r(f), \quad \underline{\mathcal{MO}}_r(f), \quad \underline{\mathcal{S}}_r(f), \quad \underline{\mathcal{M}}_r(f), \quad \underline{\mathcal{P}}_r(f) \quad \text{and} \quad \underline{\mathcal{N}}_r(f)$$

denote the set of all points at which f is

$$\underline{\mathcal{SO}}_r\text{-continuous}, \quad \underline{\mathcal{MO}}_r\text{-continuous}, \quad \underline{\mathcal{S}}_r\text{-continuous},$$
$$\underline{\mathcal{M}}_r\text{-continuous}, \quad \underline{\mathcal{P}}_r\text{-continuous}, \quad \underline{\mathcal{N}}_r\text{-continuous},$$

respectively, for the corresponding r.

Similarly, the symbols

$$\underline{\mathcal{SO}}_r, \quad \underline{\mathcal{MO}}_r, \quad \underline{\mathcal{S}}_r, \quad \underline{\mathcal{M}}_r, \quad \underline{\mathcal{P}}_r \quad \text{and} \quad \underline{\mathcal{N}}_r$$

denote the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$, which are

$$\underline{\mathcal{SO}}_r\text{-continuous}, \quad \underline{\mathcal{MO}}_r\text{-continuous}, \quad \underline{\mathcal{S}}_r\text{-continuous},$$
$$\underline{\mathcal{M}}_r\text{-continuous}, \quad \underline{\mathcal{P}}_r\text{-continuous}, \quad \underline{\mathcal{N}}_r\text{-continuous},$$

respectively, for the corresponding r.

Proposition 3.8. *For every $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the following properties hold:*

(1) *f is $\underline{\mathcal{N}}_r$-continuous at x if and only if f is $\underline{\mathcal{M}}_r$-continuous at x for every $r \in (0, \frac{1}{2}] \cup \{1\}$;*

(2) *if f is $\underline{\mathcal{S}}_r$-continuous at x, then f is $\underline{\mathcal{SO}}_r$-continuous at x for every $r \in [0, \frac{1}{2})$;*

(3) *if f is $\underline{\mathcal{SO}}_r$-continuous at x, then f is $\underline{\mathcal{MO}}_r$-continuous at x for every $r \in (0, \frac{1}{2})$;*

(4) *if f is $\underline{\mathcal{MO}}_r$-continuous at x, then f is $\underline{\mathcal{M}}_r$-continuous at x for every $r \in (0, \frac{1}{2}]$;*

(5) *if f is $\underline{\mathcal{P}}_r$-continuous at x, then f is $\underline{\mathcal{S}}_r$-continuous at x for every $r \in [0, \frac{1}{2})$;*

(6) *if f is $\underline{\mathcal{M}}_t$-continuous at x, then f is $\underline{\mathcal{P}}_r$-continuous at x for every $0 \le r < t \le \frac{1}{2}$.*

Proof.

(1) Choose $r \in (0, \frac{1}{2}] \cup \{1\}$. First, assume that f is $\underline{\mathcal{N}}_r$-continuous at x. Then, there exists $E \subset \mathbb{R}$ such that $x \in E$, $f_{\restriction E}$ is continuous at x and x is a point of $\underline{\mu}_r$-density of E. Take $\varepsilon > 0$. By continuity of $f_{\restriction E}$ at x, there exists $\delta_\varepsilon > 0$ such that $|f(t) - f(x)| < \varepsilon$ for each $t \in E \cap (x - \delta_\varepsilon, x + \delta_\varepsilon)$. Since x is a point of $\underline{\mu}_r$--density of E, $\underline{p}(\mathbb{R} \setminus \{y\colon |f(x) - f(y)| < \varepsilon\}, x) \ge \underline{p}(\mathbb{R} \setminus \big(E \cap (x - \delta, x + \delta)\big), x) \ge r$. By arbitrary of ε, f is $\underline{\mathcal{M}}_r$-continuous at x.

Now, assume that f is $\underline{\mathcal{M}}_r$-continuous at x. Let $E_n = \{y\colon |f(x) - f(y)| < \frac{1}{n}\}$ for $n \ge 1$. By assumption, $\underline{p}(\mathbb{R} \setminus E_n, x) \ge r$ for every n. Therefore, we can find a decreasing sequence of positive reals $(\delta_n)_{n \ge 1}$ such that

$$\frac{\Lambda(\mathbb{R} \setminus E_n, (x, x + \eta))}{\eta} > r - \frac{r}{n+1}$$

and

$$\frac{\Lambda(\mathbb{R} \setminus E_n, (x - \eta, x))}{\eta} > r - \frac{r}{n+1}$$

for every $\eta \in (0, \delta_n)$ and $n \ge 1$. Let

$$E = \{x\} \cup \bigcup_{n=1}^{\infty} \Big(E_n \cap \Big(\big(x - \delta_n, x - \tfrac{\delta_{n+1}}{n+1}\big) \cup \big(x + \tfrac{\delta_{n+1}}{n+1}, x + \delta_n\big) \Big) \Big).$$

Obviously, $x \in E$ and $f_{\restriction E}$ is continuous at x. Take any $h \in (\delta_{n+1}, \delta_n]$. Then,

$$\frac{\Lambda(\mathbb{R} \setminus E, (x, x+h))}{h} \ge \frac{\Lambda\Big(\mathbb{R} \setminus \Big(E_n \cap (x + \frac{\delta_{n+1}}{n+1}, x + h]\Big), (x, x+h)\Big)}{h} \ge$$

$$\frac{\Lambda(\mathbb{R} \setminus E_n, (x, x+h)) - \frac{\delta_{n+1}}{n+1}}{h} \ge r - \tfrac{r}{n+1} - \tfrac{\delta_{n+1}}{h(n+1)} \ge r - \tfrac{r}{n+1} - \tfrac{1}{n+1}$$

and

$$\frac{\Lambda(\mathbb{R}\setminus E,(x-h,x))}{h} \geq \frac{\Lambda\left(\mathbb{R}\setminus\left(E_n\cap\left[x-h,x-\frac{\delta_{n+1}}{n+1}\right)\right),(x-h,x)\right)}{h} \geq$$

$$\frac{\Lambda(\mathbb{R}\setminus E_n,(x-h,x))-\frac{\delta_{n+1}}{n+1}}{h} \geq r-\tfrac{r}{n+1}-\tfrac{\delta_{n+1}}{h(n+1)} \geq r-\tfrac{r}{n+1}-\tfrac{1}{n+1}.$$

Hence,

$$\underline{p}^{+}(\mathbb{R}\setminus E,x) \geq \liminf_{h\to 0^+}\frac{\Lambda(\mathbb{R}\setminus E,(x,x+h))}{h} \geq \liminf_{n\to\infty}\left(r-\tfrac{r}{n+1}-\tfrac{1}{n+1}\right)=r$$

and

$$\underline{p}^{-}(\mathbb{R}\setminus E,x) \geq \liminf_{h\to 0^+}\frac{\Lambda(\mathbb{R}\setminus E,(x-h,x))}{h} \geq \liminf_{n\to\infty}\left(r-\tfrac{r}{n+1}-\tfrac{1}{n+1}\right)=r.$$

That is why $\underline{p}(\mathbb{R}\setminus E,x)\geq r$, which completes the proof of (1).

Conditions (2), (3) and (4) follow directly from the definitions of different types of porouscontinuity and Remark 3.5.

(5) Choose $r\in[0,\frac{1}{2})$. Assume that f is $\underline{\mathcal{P}}_r$-continuous at x. There exists $E\subset\mathbb{R}$ such that $x\in E$, $f_{\restriction E}$ is continuous at x and x is a point of $\underline{\pi}_r$-density of E. Take $\varepsilon>0$. By continuity of $f_{\restriction E}$ at x, there exists $\delta_\varepsilon>0$ such that $|f(t)-f(x)|<\varepsilon$ for each $t\in E\cap(x-\delta_\varepsilon,x+\delta_\varepsilon)$. Since x is a point of $\underline{\pi}_r$-density of E, $\underline{p}\big(\mathbb{R}\setminus\{y\colon |f(x)-f(y)|<\varepsilon\},x\big)\geq\underline{p}\big(\mathbb{R}\setminus(E\cap(x-\delta_\varepsilon,x+\delta_\varepsilon)),x\big)>r$. By arbitrarily of ε, f is $\underline{\mathcal{S}}_r$-continuous at x.

(6) Choose $0\leq r<t\leq\frac{1}{2}$. Assume that f is $\underline{\mathcal{M}}_t$-continuous at x. By (1), there exists $E\subset\mathbb{R}$ such that $x\in E$, $f_{\restriction E}$ is continuous at x and x is a point of $\underline{\mu}_t$-density of E. Obviously, x is a point of $\underline{\pi}_r$-density of E. Hence, f is $\underline{\mathcal{P}}_r$-continuous at x.

The proof is completed. □

Theorem 3.9. *Let $r\in[0,\frac{1}{2})$, $x\in\mathbb{R}$ and $f\colon\mathbb{R}\to\mathbb{R}$. Then, f is $\underline{\mathcal{P}}_r$-continuous at x if and only if* $\lim_{\varepsilon\to 0^+}\underline{p}\big(\mathbb{R}\setminus\{y\in\mathbb{R}\colon |f(x)-f(y)|<\varepsilon\},x\big)>r$.

Proof. First, assume that $x\in\underline{\mathcal{P}}_r(f)$ and let E be a set witnessing this fact. For every $\varepsilon>0$, $E\cap(x-h,x+h)\subset\{y\in\mathbb{R}\colon |f(x)-f(y)|<\varepsilon\}$ for some $h>0$. Therefore,

$$\underline{p}\big(\mathbb{R}\setminus\{y\in\mathbb{R}\colon |f(x)-f(y)|<\varepsilon\},x\big)\geq\underline{p}(\mathbb{R}\setminus E,x).$$

Thus,

$$\lim_{\varepsilon\to 0^+}\underline{p}\big(\mathbb{R}\setminus\{y\in\mathbb{R}\colon |f(x)-f(y)|<\varepsilon\},x\big)\geq\underline{p}(\mathbb{R}\setminus E,x)>r.$$

Assume that $\lim_{\varepsilon\to 0^+}\underline{p}\big(\mathbb{R}\setminus\{y\in\mathbb{R}\colon |f(x)-f(y)|<\varepsilon\},x\big)>r$. We will construct $E=E_-\cup E_+\cup\{x\}\subset\mathbb{R}$ such $E_+\subset(x,\infty)$, $E_-\subset(-\infty,x)$, x is a point of $\underline{\pi}_r$-density of E and $f_{\restriction E}$ is continuous at x.

Let $E_n = \{y \in \mathbb{R} \colon |f(y) - f(x)| < \frac{1}{n}\}$ for every $n \geq 1$. By assumption,

$$\lim_{n\to\infty} \underline{p}^+(\mathbb{R} \setminus E_n, x) = s_1 > r \quad \text{and} \quad \lim_{n\to\infty} \underline{p}^-(\mathbb{R} \setminus E_n, x) = s_2 > r.$$

If f is right-hand continuous at x, then we can take $E_+ = (x, \infty)$. Otherwise, $x \in \big((x,\infty) \setminus E_n\big)^d$ for almost all n. Without loss of generality we may assume that $x \in \big((x,\infty)\setminus E_n\big)^d$ for every $n \geq 1$. By assumptions, we can find a decreasing sequence $(x_n)_{n\geq 1}$ tending to x such that for every $n \geq 1$ and for every $\delta \in (x, x_n]$ we can find an open interval $(a,b) \subset E_n \cap (x,\delta)$ for which

$$\frac{b-a}{\delta - x} > \frac{s_1 + r}{2}.$$

Therefore, for every $n \geq 1$ and for every $\delta \in [x_{n+1}, x_n]$ we can find $(a,b) \subset E_n \cap (x,\delta)$ for which $b - a > \frac{s_1+r}{2}(x_{n+1} - x)$. For every $n \geq 1$ choose $y_n \in \big(x, x + r(x_{n+1} - x)\big) \setminus E_n$. Then, for every $\delta \in (x_{n+1}, x_n]$ we can find an open interval $(a,b) \subset E_n \cap (y_n, \delta)$ for which $\frac{b-a}{\delta-x} > \frac{s_1+r}{2}$.

Let $E_+ = \bigcup_{n=1}^{\infty}\big(E_n \cap (y_n, x_n)\big)$. Obviously, $f_{\restriction E_+ \cup \{x\}}$ is continuous at x. Moreover, for every $\delta \in [x_{n+1}, x_n]$ there exists $(a,b) \subset E_n \cap (y_n, \delta) \subset E_+$ such that $\frac{b-a}{\delta-x} > \frac{s_1+r}{2}$. Hence, $\underline{p}^+(\mathbb{R} \setminus E_+, x) \geq \frac{s_1+r}{2}$.

In the same way, we can find $E_- \subset (-\infty, x)$ such that $f_{\restriction E_- \cup \{x\}}$ is continuous at x and $\underline{p}^-(\mathbb{R} \setminus E_-, x) \geq \frac{s_2+r}{2} > r$.

It means that x is a point of $\underline{\pi}_r$-density of $E = \{x\} \cup E_+ \cup E_-$ and $f_{\restriction E}$ is continuous at x. □

Theorem 3.10. *Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$. A function $f \colon \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{SO}}_r$-continuous at x if and only if there exists a set $E \subset \mathbb{R}$ such that $x \in E$, x is a point of $\underline{\pi O}_r$-density of E and $f_{\restriction E}$ is continuous at x.*

Proof. Assume that f is $\underline{\mathcal{SO}}_r$-continuous at x. Let

$$E_n = \left\{t \in \mathbb{R} \colon |f(t) - f(x)| < \frac{1}{n}\right\} \quad \text{for } n \geq 1.$$

We will construct

$$E = E_- \cup E_+ \cup \{x\} \subset \mathbb{R} \quad \text{such that} \quad E_+ \subset (x,\infty), \quad E_- \subset (-\infty, x),$$

x is a point of $\underline{\pi O}_r$-density of E and $f_{\restriction E}$ is continuous at x.

If f is right-hand continuous at x, then we can take $E_+ = (x,\infty)$. Otherwise, $x \in \big((x,\infty) \setminus E_n\big)^d$ for almost every n. Without loss of generality we may assume that $x \in \big((x,\infty) \setminus E_n\big)^d$ for every $n \geq 1$. By assumptions, we can find a decreasing sequence $(x_n)_{n\geq 1}$ tending to x such that for every $n \geq 1$ and for every $\delta \in (x, x_n]$ we can find an open interval $(a,b) \subset E_n \cap (x,\delta)$ for which $\frac{b-a}{\delta-x} > r$. Therefore, for every $n \geq 1$ and for every $\delta \in [x_{n+1}, x_n]$ we can find $(a,b) \subset E_n \cap (x,\delta)$ for which $b - a > r(x_{n+1} - x)$. For every $n \geq 1$ choose

$y_n \in \big(x, x + r(x_{n+1} - x)\big) \setminus E_n$. Then, for every $\delta \in (x_{n+1}, x_n]$ we can find an open interval $(a, b) \subset E_n \cap (y_n, \delta)$ for which $\frac{b-a}{\delta - x} > r$.

Let $E_+ = \bigcup_{n=1}^{\infty}\big(E_n \cap (y_n, x_n)\big)$. Obviously, $f_{\restriction E_+ \cup \{x\}}$ is continuous at x. Moreover, for every $\delta \in [x_{n+1}, x_n]$ there exists $(a, b) \subset E_n \cap (y_n, \delta) \subset E_+$ such that $\frac{b-a}{\delta-x} > r$.

In the same way, we can find $E_- \subset (-\infty, x)$ such that $f_{\restriction E_- \cup \{x\}}$ is continuous at x and there exists $\eta > 0$ such that for every $\delta \in (0, \eta)$ there exists $(a, b) \subset E_- \cap (x - \delta, x)$ for which $\frac{b-a}{\delta-x} > r$. It means that x is a point of $\underline{\pi O}_r$-density of $E = \{x\} \cup E_+ \cup E_-$ and $f_{\restriction E}$ is continuous at x.

Now, assume that there exists $E \subset \mathbb{R}$ such that $x \in E$, $f_{\restriction E}$ is continuous at x and x is a point of $\underline{\pi O}_r$-density of E. Take $\varepsilon > 0$. By continuity of $f_{\restriction E}$ at x, there exists $\delta_\varepsilon > 0$ such that $|f(t) - f(x)| < \varepsilon$ for each $t \in E \cap (x - \delta_\varepsilon, x + \delta_\varepsilon)$. Since x is a point of $\underline{\pi O}_r$-density of E, there exists $\delta \in (0, \delta_\varepsilon)$ such that for every $\eta \in (0, \delta)$ we can find intervals $(a_1, b_1) \subset E \cap (x - \eta, x)$, $(a_2, b_2) \subset E \cap (x, x + \eta)$ such that $\frac{b_1 - a_1}{\eta} > r$ and $\frac{b_2 - a_2}{\eta} > r$. Therefore, f is $\underline{\mathcal{SO}}_r$-continuous at x. □

In a similar way, we can proof the following theorem.

Theorem 3.11. *Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{MO}}_r$-continuous at x if and only if there exists a set $E \subset \mathbb{R}$ such that $x \in E$, x is a point of $\underline{\mu O}_r$-density of E and $f_{\restriction E}$ is continuous at x.*

By Corollary 2.5, we obtain the following proposition.

Proposition 3.12. *For every $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the following conditions are equivalent:*

(1) *f is $\underline{\mathcal{M}}_1$-continuous at x;*
(2) *f is $\underline{\mathcal{MO}}_{\frac{1}{2}}$-continuous at x;*
(3) *f is $\underline{\mathcal{SO}}_{\frac{1}{2}}$-continuous at x;*
(4) *f is $\underline{\mathcal{S}}_{\frac{1}{2}}$-continuous at x;*
(5) *f is $\underline{\mathcal{P}}_{\frac{1}{2}}$-continuous at x;*
(6) *f is continuous at x.*

Proposition 3.13. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{SO}}_0$-continuous at $x \in \mathbb{R}$ if and only if f is bilaterally quasicontinuous at x. In particular, $\mathcal{Q} \supset \mathcal{Q}^{bil} = \underline{\mathcal{SO}}_0$, where $\mathcal{Q}^{bil}$ denotes the family of functions which are bilaterally quasicontinuous at every point ($f\colon \mathbb{R} \to \mathbb{R}$ is bilaterally quasicontinuous at x if for every $\varepsilon > 0$ and $\delta > 0$ there exist $(a, b) \subset (x - \delta, x)$ and $(c, d) \subset (x, x + \delta)$ such that $f\big((a, b) \cup (c, d)\big) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$).*

Directly from Proposition 3.8 and Propositin 3.12, we obtain the following theorem.

Theorem 3.14. *Let* $r \in (0, \frac{1}{2}]$. *Then,* $\underline{\mathcal{P}}_r \subset \underline{\mathcal{S}}_r \subset \underline{\mathcal{SO}}_r \subset \underline{\mathcal{MO}}_r \subset \underline{\mathcal{M}}_r$.

By Proposition 3.12 and Proposition 3.13, we get the following equalities and inclusions.

Theorem 3.15. $\mathcal{C} = \underline{\mathcal{M}}_1 = \underline{\mathcal{MO}}_{\frac{1}{2}} = \underline{\mathcal{S}}_{\frac{1}{2}} = \underline{\mathcal{SO}}_{\frac{1}{2}} = \underline{\mathcal{P}}_{\frac{1}{2}}$ *and* $\underline{\mathcal{S}}_0 \subset \underline{\mathcal{SO}}_0 = \mathcal{Q}^{bil}$.

Finally, we can obtain the full chain of equalities and inclusions between different kinds of lower porouscontinuity.

Theorem 3.16. *Let* $0 < r < t < \frac{1}{2}$. *Then,*

$$\begin{aligned}\mathcal{C} = \underline{\mathcal{M}}_1 = \underline{\mathcal{P}}_{\frac{1}{2}} = \underline{\mathcal{S}}_{\frac{1}{2}} = \underline{\mathcal{SO}}_{\frac{1}{2}} = \underline{\mathcal{MO}}_{\frac{1}{2}} \subset \underline{\mathcal{M}}_{\frac{1}{2}} \subset \\ \underline{\mathcal{P}}_t \subset \underline{\mathcal{S}}_t \subset \underline{\mathcal{SO}}_t \subset \underline{\mathcal{MO}}_t \subset \underline{\mathcal{M}}_t \subset \underline{\mathcal{P}}_r \subset \underline{\mathcal{S}}_r \subset \underline{\mathcal{SO}}_r \subset \underline{\mathcal{MO}}_r \subset \underline{\mathcal{M}}_r \subset \\ \underline{\mathcal{P}}_0 \subset \underline{\mathcal{S}}_0 \subset \underline{\mathcal{SO}}_0 = \mathcal{Q}^{bil}\end{aligned}$$

and all inclusions are proper.

P r o o f. All inclusions and equalities follow directly from Theorem 3.14, Theorem 3.15 and Proposition 3.8 (6). We just have to prove that all inclusions are proper. We do this by constructing examples of appropriate functions. Ideas of constructing some of these examples are analogous to those from [3].

(1) First, we show $\underline{\mathcal{M}}_{\frac{1}{2}} \setminus \mathcal{C} \neq \emptyset$. Applying Lemma 2.6, we can find two sequences of open intervals

$$((a_n, b_n))_{n\geq 1}, \qquad ((c_n, d_n))_{n\geq 1}$$

such that

$$0 < \dots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots, \quad \lim_{n\to\infty} a_n = 0$$

and

$$\underline{p}^+\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n), 0\right) = \frac{1}{2}.$$

Define $f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} 0, & x \in \{0\} \cup (-\infty, -d_1] \cup [d_1, \infty) \cup \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), \\ 1, & x \in \bigcup_{n=1}^{\infty}([d_{n+1}, c_n] \cup [-c_n, -d_{n+1}]), \\ \text{affine} & \text{in every interval } [-d_n, -b_n], [-a_n, -c_n], [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

Obviously, f_1 is not continuous at 0 and f_1 is continuous at every point $x \neq 0$. Moreover,

$$\underline{p}\big(\mathbb{R} \setminus \{x : f_1(x) = 0\}, 0\big) = \underline{p}\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), 0\right) = \frac{1}{2}.$$

Therefore, $f_1 \in \underline{\mathcal{M}}_{\frac{1}{2}} \setminus \mathcal{C}$.

(2) Now, we show $\underline{\mathcal{P}}_r \setminus \underline{\mathcal{M}}_t \neq \emptyset$ for $0 \le r < t < \frac{1}{2}$.

Fix $0 \le r < t < \frac{1}{2}$. Let $s = \frac{r+t}{2}$. By Theorem 2.11, we can find two sequences

$$(a_n)_{n\ge1} \quad \text{and} \quad (b_n)_{n\ge1}$$

such that

$$0 < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1, \qquad \lim_{n\to\infty} a_n = 0$$

and

$$\inf_{h\in[b_{n+1},b_n]} \frac{\Lambda(A,(0,h))}{h} = s$$

for every n, where

$$A = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Obviously, we can find sequences

$$(c_n)_{n\ge1} \quad \text{and} \quad (d_n)_{n\ge1}$$

such that

$$0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots < d_1$$

and

$$\inf_{h\in[b_{n+1},b_n]} \frac{\Lambda(B,(0,h))}{h} < s + \frac{1}{n} \qquad \text{for every } n,$$

where

$$B = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (c_n, d_n).$$

Define $f_2\colon \mathbb{R} \to \mathbb{R}$ by

$$f_2(x) = \begin{cases} 0, & x \in \{0\} \cup (-\infty, -d_1] \cup [d_1, \infty) \cup \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), \\ 1, & x \in \bigcup_{n=1}^{\infty}([d_{n+1}, c_n] \cup [-c_n, -d_{n+1}]), \\ \text{affine} & \text{in every interval } [-d_n, -b_n], [-a_n, -c_n], [c_n, a_n], [b_n, d_n], n \ge 1. \end{cases}$$

Obviously, f_2 is continuous at every point, except at 0. Moreover,

$$\underline{p}\big(\mathbb{R} \setminus \{x\colon f_2(x) = 0\}, 0\big) = \underline{p}\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), 0\right) = \lim_{n\to\infty} s = s > r.$$

Hence, $f_2 \in \underline{\mathcal{P}}_r$. On the other hand,

$$\underline{p}\big(\mathbb{R} \setminus \{x\colon f_2(x) < 1\}, 0\big) = \underline{p}\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}([c_n, d_n] \cup [-d_n, -c_n]), 0\right) \le \lim_{n\to\infty}\left(s + \frac{1}{n}\right) = s < t.$$

It follows, $f_2 \notin \underline{\mathcal{M}}_t$. Finally, $f_2 \in \underline{\mathcal{P}}_r \setminus \underline{\mathcal{M}}_t$.

(3) Next, we prove $\underline{\mathcal{S}}_t \setminus \underline{\mathcal{P}}_t \neq \emptyset$ for every $r \in (0, \frac{1}{2})$.

Choose any $r \in (0, \frac{1}{2})$. Let

$$a_n = \frac{1}{n!} \quad \text{and} \quad b_n = \frac{1+rn}{(n+1)!(1-r)} \quad \text{for } n \geq 1.$$

Then,

$$\alpha_n = b_n - a_{n+1} = \frac{1+rn-1+r}{(n+1)!(1-r)} = \frac{r}{n!(1-r)} \quad \text{for every } n.$$

Hence,

$$0 < \cdots < a_{n+1} < b_n < a_n < \cdots \quad \text{and} \quad \frac{\alpha_n}{a_n + \alpha_n} = r \quad \text{for every } n.$$

Next, let $c_n = \frac{b_n + a_n}{2}$ and $\beta_n = \frac{a_n - b_n}{2}$ for $n \geq 1$. Observe that

$$\frac{\beta_n}{\alpha_n} = \frac{\frac{\frac{1}{n!} - \frac{1+rn}{(n+1)!(1-r)}}{2}}{\frac{r}{n!(1-r)}} = \frac{n(1-2r)-r}{2(n+1)r} \quad \text{for } n \geq 1.$$

Hence,

$$\frac{\beta_n}{\alpha_n} > \frac{n(1-2r) - n\frac{1-2r}{2}}{4nr} = \frac{1-2r}{8r} \quad \text{for sufficiently large } n$$

and

$$\frac{\beta_n}{\alpha_n} < \frac{(n+1)(1-2r)}{2(n+1)r} = \frac{1-2r}{2r} = 4 \cdot \frac{1-2r}{8r} \quad \text{for all } n.$$

Let $\lambda = \frac{1-2r}{8r} > 0$. Then, $\beta_n > \lambda\alpha_n$ for almost all n and $\beta_n < 4\lambda\alpha_n$ for all n. Define $f_3 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_3(x) = \begin{cases} 0, & x \in \{0\} \cup (-\infty, -a_1] \cup [a_1, \infty) \cup \bigcup_{n=1}^{\infty}([a_{n+1}, b_n] \cup [-b_n, -a_{n+1}]), \\ 1, & x \in \bigcup_{n=1}^{\infty}\{c_n, -c_n\}, \\ \text{affine} & \text{in every interval } [-a_n, -c_n], [-c_n, -b_n], [b_n, c_n], [c_n, a_n], n \geq 1. \end{cases}$$

Obviously, f_3 is continuous at every point, except at 0. Let

$$E_\varepsilon = \{x \in \mathbb{R} \colon |f_3(x)| < \varepsilon\} \quad \text{for every } \varepsilon > 0.$$

Then,

$$E_\varepsilon = \{0\} \cup \bigcup_{n=1}^{\infty} \big((a_{n+1} - \varepsilon\beta_{n+1}, b_n + \varepsilon\beta_n) \cup (-b_n - \varepsilon\beta_n, -a_{n+1} + \varepsilon\beta_{n+1})\big).$$

Thus,

$$\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) = \underline{p}^+(\mathbb{R} \setminus E_\varepsilon, 0) = \underline{p}^-(\mathbb{R} \setminus E_\varepsilon, 0)$$

and

$$\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) = \liminf_{n\to\infty} \frac{b_n - a_{n+1} + \varepsilon(\beta_n + \beta_{n+1})}{a_n - \varepsilon\beta_n + b_n - a_{n+1} + \varepsilon(\beta_n + \beta_{n+1})}$$

$$= \liminf_{n\to\infty} \frac{\alpha_n + \varepsilon(\beta_n + \beta_{n+1})}{a_n + \alpha_n + \varepsilon\beta_{n+1}} \quad \text{for every } \varepsilon > 0.$$

Obviously,

$$\alpha_{n+1} \leq \frac{a_n + \alpha_n}{8} \quad \text{for all} \quad n \geq 3.$$

Therefore,

$$\begin{aligned}\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) &\geq \liminf_{n\to\infty} \frac{\alpha_n + \varepsilon\beta_n}{a_n + \alpha_n + \varepsilon\beta_{n+1}} \geq \liminf_{n\to\infty} \frac{\alpha_n + \varepsilon\lambda\alpha_n}{a_n + \alpha_n + 4\varepsilon\lambda\alpha_{n+1}} \\ &\geq \liminf_{n\to\infty} \frac{\alpha_n(1+\varepsilon\lambda)}{a_n + \alpha_n + \frac{\varepsilon\lambda}{2}(a_n + \alpha_n)} = \liminf_{n\to\infty} \frac{1+\varepsilon\lambda}{1+\frac{\varepsilon\lambda}{2}} \cdot \frac{\alpha_n}{a_n + \alpha_n} \\ &= \frac{1+\varepsilon\lambda}{1+\frac{\varepsilon\lambda}{2}} r\end{aligned}$$

and

$$\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) \leq \liminf_{n\to\infty} \frac{\alpha_n + 2\varepsilon\beta_n}{\alpha_n + a_n} \leq \liminf_{n\to\infty} \frac{(1+8\lambda\varepsilon)\alpha_n}{\alpha_n + a_n} = r(1+8\lambda\varepsilon).$$

Hence, $\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) > r$ for every $\varepsilon > 0$ and $\limsup_{\varepsilon\to 0^+} \underline{p}^+(\mathbb{R} \setminus E_\varepsilon, 0) \leq r$. By Theorem 3.9, $f_3 \in \underline{\mathcal{S}}_t \setminus \underline{\mathcal{P}}_t$.

(4) We show $\underline{\mathcal{SO}}_r \setminus \underline{\mathcal{S}}_r \neq \emptyset$ for every $r \in [0, \frac{1}{2})$.

Fix $r \in [0, \frac{1}{2})$. Applying Theorem 2.11, we can find two sequences

$$(a_n)_{n\geq 1} \quad \text{and} \quad (b_n)_{n\geq 1}$$

such that $0 < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$, $\lim_{n\to\infty} a_n = 0$ and $\inf_{h\in[b_{n+1},b_n]} \frac{\Lambda(A,(0,h))}{h} = r + \frac{\frac{1}{2}-r}{2(n+1)}$ for every n, where $A = \mathbb{R} \setminus \bigcup_{n=1}^\infty (a_n, b_n)$. Obviously, we can find sequences

$$(c_n)_{n\geq 1} \quad \text{and} \quad (d_n)_{n\geq 1}$$

such that $0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots < d_1$ and

$$\inf_{h\in[b_{n+1},b_n]} \frac{\Lambda(B,(0,h))}{h} < r + \frac{\frac{1}{2}-r}{n+1}$$

for every n, where $B = \mathbb{R} \setminus \bigcup_{n=1}^\infty (c_n, d_n)$. Define $f_4 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_4(x) = \begin{cases} 0, & x \in \{0\} \cup (-\infty, -d_1] \cup [d_1, \infty) \cup \bigcup_{n=1}^\infty ([a_n, b_n] \cup [-b_n, -a_n]), \\ 1, & x \in \bigcup_{n=1}^\infty ([d_{n+1}, c_n] \cup [-c_n, -d_{n+1}]), \\ \text{affine in every interval } [-d_n, -b_n], [-a_n, -c_n], [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

Obviously, f_4 is continuous at every point, except at 0. Moreover,

$$\underline{p}\big(\mathbb{R} \setminus \{x \colon f_4(x) < 1\}, 0\big) = \underline{p}\left(\mathbb{R} \setminus \bigcup_{n=1}^\infty ([c_n, d_n] \cup [-d_n, -c_n]), 0\right) = r.$$

Hence, $f_4 \notin \underline{\mathcal{S}}_r$. On the other hand, 0 is a point of $\underline{\pi O}_r$ density of $\{x \colon f_4(x) = 0\}$. Therefore, $f_4 \in \underline{\mathcal{SO}}_r$. Finally, $f_4 \in \underline{\mathcal{SO}}_r \setminus \underline{\mathcal{S}}_r$.

(5) We prove $\underline{\mathcal{S}}_0 \setminus \underline{\mathcal{P}}_0 \neq \emptyset$.

Let $a_n = \frac{1}{n!}$, $b_n = \frac{a_n + a_{n+1}}{2}$ and $\alpha_n = \frac{a_n - a_{n+1}}{2} = b_n - a_{n+1} = a_n - b_n$ for $n \geq 1$. Then,

$$\alpha_n = \frac{1}{2}\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = \frac{n}{2(n+1)!} \quad \text{for every } n \geq 1.$$

Define $f_5 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_5(x) = \begin{cases} 0, & x \in \{0\} \cup (a_1, \infty) \cup \{a_n \colon n \geq 1\}, \\ 1, & x \in \{b_n \colon n \geq 1\}, \\ \text{affine in every interval } [a_{n+1}, b_n], [b_n, a_n], n \geq 1 & \end{cases}$$

and $f_5(x) = f_5(-x)$ for $x \in (-\infty, 0)$.

Obviously, f_5 is continuous at every point, except at 0. Let

$$E_\varepsilon = \{x \in \mathbb{R} \colon |f_5(x)| < \varepsilon\} \quad \text{for } \varepsilon > 0.$$

Then,

$$E_\varepsilon = \{0\} \cup (-\infty, -a_1 + \alpha_1) \cup (a_1 - \alpha_1, \infty) \cup \bigcup_{n=2}^{\infty} \big((a_n - \varepsilon\alpha_n, a_n + \varepsilon\alpha_{n-1}) \cup (-a_n - \varepsilon\alpha_{n-1}, -a_n + \varepsilon\alpha_n)\big).$$

Thus, $\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) = \underline{p}^+(\mathbb{R} \setminus E_\varepsilon, 0) = \underline{p}^-(\mathbb{R} \setminus E_\varepsilon, 0)$ and

$$\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) = \liminf_{n\to\infty} \frac{\varepsilon(\alpha_n + \alpha_{n+1})}{a_n - \varepsilon\alpha_n + \varepsilon(\alpha_n + \alpha_{n+1})}$$
$$= \liminf_{n\to\infty} \frac{\varepsilon\left(\frac{n}{2(n+1)!} + \frac{n+1}{2(n+2)!}\right)}{\frac{1}{n!} + \varepsilon\frac{n+1}{2(n+2)!}} = \liminf_{n\to\infty} \frac{\varepsilon\left(\frac{n}{2(n+1)} + \frac{1}{2(n+2)}\right)}{1 + \frac{\varepsilon}{2(n+2)}} = \frac{\varepsilon}{2}$$

for every $\varepsilon > 0$. Therefore, $\underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) > 0$ for every $\varepsilon > 0$ and finally, $\lim_{\varepsilon\to 0^+} \underline{p}(\mathbb{R} \setminus E_\varepsilon, 0) = 0$. By Theorem 3.9, $f_5 \in \underline{\mathcal{S}}_0 \setminus \underline{\mathcal{P}}_0$.

(6) Next, we prove $\underline{\mathcal{MO}}_r \setminus \underline{\mathcal{M}}_r \neq \emptyset$ for every $r \in (0, \frac{1}{2})$.

Choose $r \in (0, \frac{1}{2})$. By Theorem 2.11, we can find two sequences

$$(a_n)_{n\geq 1} \quad \text{and} \quad (b_n)_{n\geq 1}$$

such that

$$0 < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1, \quad \lim_{n\to\infty} a_n = 0$$

and

$$\inf_{h\in[b_{n+1}, b_n]} \frac{\Lambda(A, (0,h))}{h} = r - \frac{r}{n+1} \quad \text{for every } n, \quad \text{where } A = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Obviously, we can find sequences $(c_n)_{n\geq 1}$ and $(d_n)_{n\geq 1}$ such that

$$0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots < d_1 \quad \text{and} \quad \inf_{h\in[b_{n+1}, b_n]} \frac{\Lambda(B, (0,h))}{h} < r$$

for every n, where $B = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (c_n, d_n)$.

Define $f_6 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_6(x) = \begin{cases} 0, & x \in \{0\} \cup (-\infty, -d_1] \cup [d_1, \infty) \cup \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), \\ 1, & x \in \bigcup_{n=1}^{\infty}([d_{n+1}, c_n] \cup [-c_n, -d_{n+1}]), \\ \text{affine in every interval } [-d_n, -b_n], [-a_n, -c_n], [c_n, a_n], [b_n, d_n], n \geq 1. \end{cases}$$

Obviously, f_6 is continuous at every point, except at 0. Moreover,

$$\underline{p}(\mathbb{R} \setminus \{x \in \mathbb{R} \colon f_6(x) = 0\}, 0) = \underline{p}\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), 0\right) = r.$$

Therefore, $f_6 \notin \underline{\mathcal{M}}_r$. On the other hand, 0 is a point of $\underline{\mu O}_r$-density of $\{x \in \mathbb{R} \colon f_6(x) < 1\}$. It follows, $f_6 \in \underline{\mathcal{MO}}_r$. Finally, $f_6 \in \underline{\mathcal{MO}}_r \setminus \underline{\mathcal{M}}_r$.

(7) Finally, we show $\underline{\mathcal{MO}}_r \setminus \underline{\mathcal{SO}}_r \neq \emptyset$ for every $r \in (0, \frac{1}{2})$.

Let $r \in (0, \frac{1}{2})$. By Theorem 2.11, we can find two sequences

$$(a_n)_{n \geq 1} \quad \text{and} \quad (b_n)_{n \geq 1}$$

such that $0 < \cdots < b_{n+1} < a_n < b_n < \cdots < a_1 < b_1$, $\lim_{n \to \infty} a_n = 0$ and $\inf_{h \in [b_{n+1}, b_n]} \frac{\Lambda(A, (0,h))}{h} = r$ for every n, where

$$A = \mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n).$$

Fix any sequences $(c_n)_{n \geq 1}$ and $(d_n)_{n \geq 1}$ such that

$$0 < \cdots < d_{n+1} < c_n < a_n < b_n < d_n < \cdots < d_1.$$

Applying Lemma 2.6, for every $n \geq 1$ we can find $g_n \colon [c_n, a_n] \to \mathbb{R}$ such that $g_n(c_n) = 1$, $g_n(a_n) = 0$, g_n is discontinuous only at a_n, $a_n \in \underline{\mathcal{M}}_{\frac{1}{2}}(g_n)$ and $g_n(x_k) = 1$ for every point of a sequence $(x_k)_{k \geq 1}$ tending to a_n from the left. Similarly, by Lemma 2.7, for every $n \geq 1$ we can find $h_n \colon [b_n, d_n] \to \mathbb{R}$ such that $h_n(d_n) = 1$, $h_n(b_n) = 0$, h_n is discontinuous only at b_n, $b_n \in \underline{\mathcal{M}}_{\frac{1}{2}}(h_n)$ and $h_n(y_k) = 1$ for every point of a sequence $(y_k)_{k \geq 1}$ tending to b_n from the right. Define $f_7 \colon \mathbb{R} \to \mathbb{R}$ by

$$f_7(x) = \begin{cases} 0, & x \in \{0\} \cup \bigcup_{n=1}^{\infty}([a_n, b_n] \cup [-b_n, -a_n]), \\ 1, & x \in (-\infty, -d_1) \cup (d_1, \infty) \cup \bigcup_{n=1}^{\infty}([d_{n+1}, c_n] \cup [-c_n, -d_{n+1}]), \\ g_n(x), & x \in [c_n, a_n], \ n \geq 1, \\ g_n(-x), & x \in [-a_n, -c_n], \ n \geq 1, \\ h_n(x), & x \in [b_n, d_n], \ n \geq 1, \\ h_n(-x), & x \in [-d_n, -b_n], \ n \geq 1. \end{cases}$$

Obviously, f_7 is $\underline{\mathcal{M}}_{\frac{1}{2}}$-continuous at every point, except at 0. Moreover,

$$\Lambda\big(\mathbb{R} \setminus \{x \in \mathbb{R} \colon |f_7(x)| < \varepsilon\}, (0,h)\big) = \Lambda\big(\{x \in \mathbb{R} \colon |f_7(x)| > 0\}, (-h,0)\big)$$

for every $h > 0$, $\varepsilon > 0$ and $\inf_{h\in[b_{n+1},b_n]} \frac{\Lambda(\mathbb{R}\setminus\{x\in\mathbb{R}\colon |f_7(x)|<\varepsilon\},(0,h))}{h} = r$ for every n. Hence, 0 is a point of $\underline{\mu O}_r$-density of the set $\{x \in \mathbb{R} \colon |f_7(x)| < \varepsilon\}$ for every $\varepsilon > 0$ and is not a point of $\underline{\pi O}_r$-density of this set for every $\varepsilon > 0$. Thus,

$$f_7 \in \underline{\mathcal{MO}}_r \setminus \underline{\mathcal{SO}}_r \,.$$

The proof is completed. □

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Institute of Exact and Technical Sciences
Pomeranian University in Słupsk
ul. Arciszewskiego 22d
PL 76–200 Słupsk
POLAND

E-mail: jakub.bilski@upsl.edu.pl
anna.kaminska@upsl.edu.pl
stanislaw.kowalczyk@upsl.edu.pl
malgorzata.turowska@upsl.edu.pl