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SOME ALTERNATIVE INTERPRETATIONS OF STRONGLY STAR SEMI-ROTHBERGER AND RELATED SPACES

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ABSTRACT. In this article, some under-appreciated characteristics of strongly star-Rothberger subsets are explored. Additionally, with the aid of the SSI property and MSSI property, semi-Rothberger and star semi-Rothberger spaces are represented by families of closed sets. Finally, several selection principle like attributes are produced that can reflect the previously mentioned sequential covering features in an inverted form.

1. Introduction

Semi-open sets in topological spaces were first mentioned by Levine [17] in 1963. Numerous mathematical theories about the topological characteristics of semi-open sets have been developed. A subset A in the space X is said to be semi-open if and only if there is an open set O in X such that $O \subset A \subset \overline{O}$. In other words, the set A is said to be semi-open if and only if $A \subset \overline{\mathrm{Int}(A)}$, where $\overline{O}$ signifies the closure of the set O and $\mathrm{Int}(A)$ denotes the interior of the set A. Semi-closed [10] is the name of the complement of a semi-open set. While arbitrary union of a semi-open set is semi-open, finite intersection of a semi-open set may or may not be semi-open. We think of the semi-Rothberger property to be the most useful among a number of covering properties where covers by semi-open sets are considered as the base.

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Let (X,τ) be a topological space and $A\subset X$. If for every sequence $\{\mathcal{U}_n : n\in\mathbb{N}\}$ of semi-open covers of A there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that

$$U_n \in \mathcal{U}_n \quad \text{for all} \quad n \in N \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} U_n \supset A,$$

then A is called a semi-Rothberger subset [9,15,19]. If $X \subseteq X$ is semi-Rothberger subset, then X is called a semi-Rothberger space.

Another interesting generalization of semi-Rothberger space has been done with the help of star operator which is call strongly star semi-Rothberger space. The star operator is defined as for $A \subset X$, if $\mathcal{U}$ is a family of subsets of X, then

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\} \qquad [11].$$

The star operator was introduced by Douwen in the year 1991 and he used it for the generalization of compactness and Lindelöfness [11]. Later on, it has been used by several researchers. Some recent uses of star operator can be found in [1–3, 5–8, 21, 22].

A space X is called strongly star semi-Rothberger [19] if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open covers of X, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that

$$U_n \in \mathcal{U}_n \quad \text{for each} \;\; n \in \mathbb{N} \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n,\mathcal{U}_n) = X.$$

Our main goal is to investigate the structure of strongly star semi-Rothberger spaces by means of families of closed sets.

2. Preliminaries

Throughout the paper, a space X denotes a topological space X equipped with the topology τ, no specific separation axioms is considered otherwise stated. We adopt the following notions for this paper:

- $\mathrm{SO}(X)$ denotes collection of all semi-open covers of a space X.
- $\mathrm{SC}(X)$ denotes collection of all families of semi-closed set $\mathcal{F}$ for which $\bigcap \mathcal{F} = \emptyset$.

For other notations and terminology, we follow [13].

Definition 2.1 ([16, 18]). Let (X,τ) be a topological space and $A \subset X$. If for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open covers of A there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ is finite for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}(\bigcup \mathcal{V}_n) \supset A$, then A is called semi-Menger subset. If $X \subseteq X$ is semi-Menger subset, then X is called a semi-Menger space.

Definition 2.2 ([16, 18]). Let (X, τ) be a topological space and $A \subset X$. If for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open covers of A there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ is finite for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}\big((\bigcup \mathcal{V}_n), \mathcal{U}_n\big) \supset A$, then A is called strongly star semi-Menger subset. If $X \subseteq X$ is strongly star semi-Menger subset, then X is called a strongly star semi-Menger space.

Another way to depict sequential coverings is to use selection principles. Two of the very basic selection principles related to Rothberger type coverings are $S_1(\mathcal{A}, \mathcal{B})$ and $S_1^*(\mathcal{A}, \mathcal{B})$.

Definition 2.3 ([20]). The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Definition 2.4 ([14]). The symbol $S_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

This should be mentioned that selection principle type of characterization for semi-Rothberger space and strongly star semi-Rothberger space are given by $S_1\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ and $S_1^*\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$, respectively. Some recent investigations in these direction can be found in [1–8, 12, 23].

In our investigation we search for some new selection principles which can characterize semi-Rothberger space and strongly star semi-Rothberger space through the family $\mathrm{SC}(X)$.

3. On semi-Rothberger spaces

Proposition 3.1. *The union of two semi-Rothberger sets is semi-Menger.*

Proof. Let A, B be two semi-Rothberger sets and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of semi-open covers of $A \cup B$.

Therefore, $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a semi-open cover for both A and B. Since A and B are semi-Rothberger sets, there exist sequences $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$, where $U_n, V_n \in \mathcal{U}_n$, for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} U_n \supset A$ and $\bigcup_{n\in\mathbb{N}} V_n \supset B$. If we take $\mathcal{W}_n = \{U_n, V_n\}$ for all $n \in \mathbb{N}$, then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{W}_n \subset \mathcal{U}_n$ is finite for all $n \in \mathbb{N}$ and

$$\bigcup_{n\in\mathbb{N}} (\cup \mathcal{W}_n) = \bigcup_{n\in\mathbb{N}} (U_n \cup V_n) = \left(\bigcup_{n\in\mathbb{N}} U_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} V_n\right) \supset A \cup B.$$

Thus, $A \cup B$ is a semi-Menger subset of X. $\square$

Corollary 3.2. *The finite union of semi-Rothberger sets is semi-Menger.*

Definition 3.3 (Sequential singletonic intersection property(SSIP))**.** A sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is said to have sequentially singletonic intersection property if for every sequence $\{F_n : n \in \mathbb{N}\}$ such that $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Theorem 3.4. *The following statements are equivalent:*

(1) *X is semi-Rothberger space.*

(2) *For every sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of the family of semi-closed sets in X having SSI property, there exists $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{F}_{n_0} \neq \emptyset$.*

Proof.

(1) $\Longrightarrow$ (2): Let X be a semi-Rothberger space; $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of family of semi-closed sets with SSI property and $\bigcap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$.

$$\Longrightarrow X \setminus \left(\bigcap \mathcal{F}_n\right) = X \setminus \emptyset = X.$$
$$\Longrightarrow \bigcup \{X \setminus F : F \in \mathcal{F}_n\} = X.$$

Since F is semi-closed, $X \setminus F$ is semi-open, and therefore, $\mathcal{G}_n = \{G = X \setminus F : F \in \mathcal{F}_n\}$ is a semi-open cover of X for all $n \in \mathbb{N}$ $\therefore$ $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of semi-open covers of X. Therefore, there exists a sequence $\{G_n : n \in \mathbb{N}\}$ such that $G \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} G_n = X$.

$$\Longrightarrow X \setminus \cup_{n\in\mathbb{N}} G_n = X \setminus X = \emptyset.$$
$$\Longrightarrow \cap_{n\in\mathbb{N}} (X \setminus G_n) = \emptyset.$$

Clearly, $(X \setminus G_n) = F_n \in \mathcal{F}_n$ and $\cap_{n\in\mathbb{N}} F_n = \emptyset$, which contradicts the SSI property. Thus, there exists a $n_0 \in \mathbb{N}$ such that $\cap \mathcal{F}_{n_0} \neq \emptyset$.

(2) $\Longrightarrow$ (1): Let condition (2) hold and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of semi-open covers of X. Suppose,

$$\mathcal{F}_n = \{X \setminus U : U \in \mathcal{U}_n\} \quad \text{for all } n \in \mathbb{N}.$$

So, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of the family of semi-closed sets such that $\cap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$. By the contrapositivity of (2), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ does not have SSI property. Therefore, there exists a sequence $\{F_n : n \in \mathbb{N}\}$ with $F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ such that $\cap_{n\in\mathbb{N}} F_n = \emptyset$.

$$\Longrightarrow \bigcap_{n\in\mathbb{N}} X \setminus U_n = \emptyset, \quad \text{where} \quad F_n = X \setminus U_n \text{ for all } n \in \mathbb{N}.$$
$$\Longrightarrow X \setminus \bigcup_{n\in\mathbb{N}} U_n = \emptyset.$$
$$\Longrightarrow \cup_{n\in\mathbb{N}} U_n = X.$$

Therefore, the sequence $\{U_n : n \in \mathbb{N}\}$ witnesses the semi-Rothberger property of X. $\square$

Theorem 3.5. *The following statements are equivalent :*

(1) *X is semi-Rothberger.*

(2) *The selection principle $S_1\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ holds for the space X.*

(3) *The selection principle $S_1\big(\mathrm{SC}(X), \mathrm{SC}(X)\big)$ holds for the space X.*

P r o o f. Here

(1)$\Longleftrightarrow$(2): is obvious.

(2) $\Longrightarrow$ (3): Let the selection principle $S_1\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ hold for the space X and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence such that $\mathcal{F}_n \in \mathrm{SC}(X)$ for all $n \in \mathbb{N}$. Consider $\mathcal{U}_n = \{U = X \setminus F : F \in \mathcal{F}_n\}$ for all $n \in \mathbb{N}$. Then, $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{U}_n \in \mathrm{SO}(X)$ for all $n \in \mathbb{N}$. However, $S_1\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ holds for the space X. Then, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ such that

$$U_n \in \mathcal{U}_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \{U_n : n \in \mathbb{N}\} \in \mathrm{SO}(X)$$

$$\Longrightarrow X \setminus \left(\bigcup_{n \in \mathbb{N}} U_n\right) = \emptyset.$$

$$\Longrightarrow \quad \bigcap_{n \in \mathbb{N}} (X \setminus U_n) = \emptyset.$$

Clearly, $X \setminus U_n = F_n \in \mathcal{F}_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. So, $\{F_n : n \in \mathbb{N}\} \in \mathrm{SC}(X)$. Therefore, X has the selection property

$$S_1\big(\mathrm{SC}(X), \mathrm{SC}(X)\big).$$

(3) $\Longrightarrow$ (2): Let condition (3) hold and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence such that $\mathcal{U}_n \in \mathrm{SO}(X)$ for all $n \in \mathbb{N}$. Suppose $\mathcal{F}_n = \{F = X \setminus U : U \in \mathcal{U}_n\}$ for all $n \in \mathbb{N}$. Therefore, $\bigcap \mathcal{F}_n = \emptyset$, for all $n \in \mathbb{N}$ Therefore, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{F}_n \in \mathrm{SC}(X)$ for all $n \in \mathbb{N}$. However, X has the selection property $S_1\big(\mathrm{SC}(X), \mathrm{SC}(X)\big)$. Therefore, there exists $F_n \in \mathcal{F}_n$, for all $n \in \mathbb{N}$ such that $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$.

$$\Longrightarrow \bigcup_{n \in \mathbb{N}} (X \setminus F_n) = X.$$

$$\Longrightarrow \bigcup_{n \in \mathbb{N}} U_n = X, \text{ where } U_n = X \setminus F_n.$$

So, $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ and $\{U_n : n \in \mathbb{N}\} \in \mathrm{SO}(X)$. Therefore, (X, τ) has the selection property $S_1\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$. $\square$

4. On strongly star semi-Rothberger spaces

Proposition 4.1. *The union of two strongly star semi-Rothberger subset is strongly star semi-Menger subset.*

Proof. Let A and B be two strongly star semi-Rothberger sets and let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of semi-open cover of $A \cup B$ in a space X. So, $\mathcal{U}_n$ is a semi-open cover for both A and B for all $n \in \mathbb{N}$. Since A and B are strongly star semi-Rothberger sets, there exit sequences $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$, where $U_n, V_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n,\mathcal{U}_n) \supset A$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(V_n,\mathcal{U}_n) \supset B$. If we take $\mathcal{W}_n = \{U_n, V_n\}$ for all $n \in \mathbb{N}$, then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{W}_n \subset \mathcal{U}_n$ is finite for all $n \in \mathbb{N}$ and

$$\bigcup_{n\in\mathbb{N}} \mathrm{St}\left(\bigcup \mathcal{W}_n,\mathcal{U}_n\right) = \left(\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n,\mathcal{U}_n)\right) \cup \left(\bigcup_{n\in\mathbb{N}} \mathrm{St}(V_n,\mathcal{U}_n)\right) \supset A \cup B.$$

Therefore, $A \cup B$ is a strongly star semi-Menger subset of X. □

Corollary 4.2. *The finite union of strongly star semi-Rothberger sets is strongly star semi-Menger.*

Definition 4.3 (Modified Sequential Singletonic Intersection Property (MSSI Property))**.** A sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of family of subsets of X is said to have MSSIP if for all sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and $H_n \subset \mathcal{F}_n$ for all $n \in \mathbb{N}$ either $E_n \cup F = X$ for some $F \in \mathcal{H}_n$ and for all $n \in \mathbb{N}$ or $\bigcap_{n\in\mathbb{N}}(\cap\mathcal{H}_n) \neq \emptyset$.

Theorem 4.4. *The following conditions are equivalent:*

(1) *X is strongly star semi-Rothberger space.*

(2) *If a sequence of closed sets $\{\mathcal{F}_n : n \in \mathbb{N}\}$ in X has MSSIP, then there exists a $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{F}_{n_0} \neq \emptyset$.*

Proof. Let (X, τ) be a strongly star semi-Rothberger space; $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a family of semi-closed sets having MSSIP and let $\bigcap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$.

Consider $\mathcal{G}_n = \{G = X \setminus F : F \in \mathcal{F}_n\}$. Then, $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of semi-open covers of X. However, X is strongly star semi-Rothberger space. Therefore, there exists a sequence $\{G_n : n \in \mathbb{N}\}$ with $G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(G_n, \mathcal{G}_n)\} = X$.

$$\implies \bigcup_{n\in\mathbb{N}} \left\{\bigcup(G_n \in \mathcal{G}_n : G_n \cap \mathcal{G}_n \neq \emptyset)\right\} = X.$$

Now, let $E_n = X \setminus G_n$ for all $n \in \mathbb{N}$ and $\mathcal{W}_n = \{G \in \mathcal{G}_n : G \cap G_n \neq \emptyset\}$ for all $n \in \mathbb{N}$. So, $E_n \in \mathcal{F}_n, \mathcal{W}_n \subseteq \mathcal{G}_n$ for all $n \in \mathbb{N}$, and hence, we construct

$$\mathcal{H}_n = \{X \setminus W : W \in \mathcal{W}_n\} \subseteq \mathcal{F}_n.$$

Now,

$$\begin{aligned}
\bigcap_{n\in\mathbb{N}} \left(\bigcap \mathcal{H}_n\right) &= \bigcap_{n\in\mathbb{N}} \left\{\bigcap (X \setminus W : W \in \mathcal{W}_n)\right\} \\
&= \bigcap_{n\in\mathbb{N}} \left\{X \setminus \bigcup (W : W \in \mathcal{W}_n)\right\} \\
&= \bigcap_{n\in\mathbb{N}} \left\{X \setminus \bigcup \{G \in \mathcal{G}_n : G \cap G_n \neq \emptyset\}\right\} \\
&= X \setminus \left\{\bigcup_{n\in\mathbb{N}} \{\cup\{G \in \mathcal{G}_n : G \cap G_n \neq \emptyset\}\}\right\} \\
&= X \setminus \left\{\bigcup_{n\in\mathbb{N}} \{\mathrm{St}(G_n, \mathcal{G}_n)\}\right\} \\
&= X \setminus X \;=\; \emptyset.
\end{aligned}$$

Suppose,for all $n \in \mathbb{N}$, there exists a $F \in \mathcal{H}_n$ such that

$$E_n \cup F = X$$

$$\begin{aligned}
&\implies (X \setminus G_n) \cup (X \setminus W) = X, W \in \mathcal{W}_n. \\
&\implies X \setminus (G_n \cap W) = X, W \in \mathcal{W}_n. \\
&\implies G_n \cap W = \emptyset,
\end{aligned}$$

which contradicts the structure of $\mathcal{W}_n$ for all $n \in \mathbb{N}$. So, there must exit a $n_0 \in \mathbb{N}$ such that $\bigcap F_{n_0} \neq \emptyset$. Conversely, let condition (2) hold and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a family of semi-open covers of X, then

$$\bigcup \mathcal{U}_n = X \quad \text{for all} \quad n \in \mathbb{N}.$$

We construct $\mathcal{F}_n = \{F = X \setminus U : U \in \mathcal{U}_n\}$ for all $n \in \mathbb{N}$, then $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of the family of semi-closed set such that $\bigcap \mathcal{F}_n = \emptyset$ for all $n \in \mathbb{N}$. Now, due to the contrapositivity of condition (2), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ does not have the MSSIP, i.e., there exist sequences $\{E_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $E_n \in \mathcal{F}_n$ and $\mathcal{H}_n \subset \mathcal{F}_n$ and for which $E_n \cup F \neq X$ for any $F \in \mathcal{H}_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}}(\bigcap \mathcal{H}_n) = \emptyset$.

Now, take $\mathcal{W}_n = \{X \setminus F : F \in \mathcal{H}_n\}$ for all $n \in \mathbb{N}$ and $G_n = X \setminus E_n$ for all $n \in \mathbb{N}$. Clearly, $G_n \in \mathcal{U}_n$ and $\mathcal{W}_n \subseteq \mathcal{U}_n$ for all $n \in \mathbb{N}$. So, $E_n \cup F \neq X$ for any $F \in \mathcal{H}_n$ for all $n \in \mathbb{N}$.

$\implies (X \setminus G_n) \cup (X \setminus G) \neq X$ for any $G \in \mathcal{W}_n$, for all $n \in \mathbb{N}$.

$\implies G_n \cap G \neq \emptyset$ for any $G \in \mathcal{W}_n$, for all $n \in \mathbb{N}$.

So, we have

$$\begin{aligned} X \supseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n,\mathcal{U}_n) &\subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n,\mathcal{W}_n) \\ &= \bigcup_{n\in\mathbb{N}} \{G \in \mathcal{W}_n : G \cap G_n \neq \emptyset\} \\ &= \bigcup_{n\in\mathbb{N}} \bigcup \{X \setminus F : F \in \mathcal{H}_n\} \\ &= X \setminus \bigcap_{n\in\mathbb{N}} \bigcap (H)_n = X \setminus \emptyset = X. \end{aligned}$$

$$\implies \bigcup_{n\in\mathbb{N}} \{\mathrm{St}(G_n,\mathcal{U}_n)\} = X.$$

So, (X,τ) is a strongly star semi-Rothberger space. $\square$

Example. There exists a T_0 space in which

$$S_1^*\big(\mathrm{SO}(X),\mathrm{SO}(X)\big) \quad \text{and} \quad S_1^*\big(\mathrm{SC}(X),\mathrm{SC}(X)\big) \text{ are not equivalent.}$$

Let $X = [0,1)$ and $\tau = \{\emptyset, X\} \cup \{[0,\alpha) : \alpha \in [0,1]\}$. So, (X,τ) forms a T_0 space. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be an arbitrary sequence of semi-open covers. Clearly, there exists an $U_n \in \mathcal{U}_n$ such that $0 \in U_n$ for all $n \in \mathbb{N}$, then

$$\mathrm{St}(U_n,\mathcal{U}_n) = \bigcup \mathcal{U}_n = X \quad \text{for all } n \in \mathbb{N}.$$

However, X is semi-open and open. Thus, $\{\mathrm{St}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\} = \{X\} \in \mathrm{SO}(X)$. Therefore, the space X follows the selection principle

$$S_1^*\big(\mathrm{SO}(X),\mathrm{SO}(X)\big).$$

Now, consider a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of a family of closed sets such that

$$\mathcal{F}_n = \{F_{n_m} : m \in \mathbb{N}\} \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad F_{n_m} = \left[1 - \frac{1}{m}, 1\right).$$

Clearly,

$$\bigcap \mathcal{F}_n = \bigcap_{m=1}^{\infty} F_{n_m} = \bigcap_{m=1}^{\infty} \left[1 - \frac{1}{m}, 1\right) = \emptyset.$$

Therefore,

$$\mathcal{F}_n \in \mathrm{SC}(X) \quad \text{for all } n \in \mathbb{N}.$$

Now, for every selection $F_n \in \mathcal{F}_n$,

$$\mathrm{St}\,(F_n,\,\mathcal{F}_n) \;=\; [0,1) \;=\; X.$$

$$\therefore \{\mathrm{St}(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} = \{X\}.$$

However,

$$\bigcap\{\mathrm{St}(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} = X \neq \emptyset,$$

$$\therefore \{\mathrm{St}(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} \notin \mathrm{SC}(X).$$

So, the space X does not follow the selection principle $S_1^*\big(\mathrm{SC}(X), \mathrm{SC}(X)\big)$.

So, $S_1^*\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ and $S_1^*\big(\mathrm{SC}(X), \mathrm{SC}(X)\big)$ are not equivalent.

Now, we desire to have a selection principle that can act on $\mathrm{SC}(X)$ and be equivalent to $S_1^*\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$. So, we introduce the selection principle

$$S_{1,S}(\mathcal{A}, \mathcal{B}).$$

Definition 4.5. The symbol $S_{1,S}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exist sequences $(E_n : n \in \mathbb{N})$ and $(\mathcal{H}_n : n \in \mathbb{N})$ such that $E_n \in \mathcal{U}_n$ and $\mathcal{H}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $E_n \cup F \neq X$ for any $F \in \mathcal{H}_n$ and $\{\bigcap \mathcal{H}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Corollary 4.6. *$S_1^*\big(\mathrm{SO}(X), \mathrm{SO}(X)\big)$ and $S_{1,S}\big(\mathrm{SC}(X), \mathrm{SC}(X)\big)$ are equivalent.*

Proof. The proof is similar to the proof of Theorem 4.4, hence omitted. □

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`Declaration on data availability and conflict of interest.`
In this article, no dataset has been generated or analysed. So data sharing does not apply here. We declare that all of the images/graphics included in this article are the authors' own works. There are no conflicts of interest in any of the topics covered in this article.

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