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THE NEMYTSKIĬ OPERATOR AND VECTOR MEASURE SOLUTIONS FOR NON-LINEAR INITIAL VALUE PROBLEMS

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ABSTRACT. We define and study a Banach-space-valued Nemytskiĭ operator and we find vector measure solutions for associated non-linear initial value problems.

1. Introduction

We develop and study in detail a vector-valued Nemytskiĭ operator that is an operator with values in a Banach space. Then, we solve an associated non--linear initial value problem in the context of vector measures, that is, measures that take values in a Banach space. Our work extends [2] to vector measures and it includes an expanded treatment of results already considered in an unpublished manuscript [1].

The Nemytskiĭ operator, let us recall, is a variable-coefficient composition operator of the form $f(t) \to G\big(t, f(t)\big)$. Defined by V. V. Nemytskiĭ [28], it has been studied and used, among others, by M. M. Vainberg ([29], [30]), Nemytskiĭ's student at Moscow State University, and M. A. Krasnosel'kiĭ [20]. More recently, the Nemytskiĭ operator has been extensively studied in ([16, Chapters 6 and 7]), and has been used in the context of various problems involving non-linear, functional, integral and differential equations, (see, for instance, [5–9, 12, 18, 21, 23–27]).

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In some applications, the need arises to study the Fréchet and Gâteaux differentiability of particular Nemytskiĭ operators (see, for instance, [13, p. 267]; [14, pp. 318, 342]; [19, pp. 96–97]; [29]). Other results on differentiability can be formulated using Sobolev spaces ([14, p. 342]). A detailed treatment of the differentiability of a general Nemytskiĭ operator is found in ([4, Chapter 1], [16, Chapter 6]), while continuity properties in various functional spaces are studied in ([16, Chapter 7]).

By necessity, we use a number of definitions and results pertaining to the theory of vector measures. This background material can be found, for instance, in the expository article [3], which provides a fairly detailed study of vector measures, including numerous examples and plenty of commentary, some of a historical nature, as well as a comprehensive list of references. We will cite this article often.

2. The Nemytskiĭ operator

We fix a complete and σ-finite measure space (S, Σ, μ). With X and Y we indicate real Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

We begin with a definition (see [20, p. 20] and [2, p. 73,] for the real-valued case).

Definition 2.1. A function $G : S \times X \to Y$ is called a vector-valued Carathéodory function when it satisfies the following conditions, called Carathéodory conditions:

(1) For each $u \in X$, the function $t \to G(t, u)$ is strongly measurable (for the definition, see [3, p. 27, Definition 12]).

(2) The function $u \to G(t, u)$ is continuous for μ-a.a. $t \in S$. That is to say, there is a μ-null set $N \in \Sigma$ such that for each $t \in S \backslash N$, the function $u \to g(t, u)$ is continuous.

Proposition 2.2. *Given a Carathéodory function G, the following statements hold:*

(1) *The application $f \to G(\cdot, f)$ maps $B^0(S, X)$ into $B^0(S, Y)$ (for the definition, see [3, p. 45, Definition 22]).*

(2) *If μ is finite, the application $f \to G(\cdot, f)$ is continuous from $B^0(S, X)$ into $B^0(S, Y)$.*

P r o o f. To prove 1), we begin with an X-valued simple function $\varphi = \sum_j x_j \chi_{M_j}$. Then, if we fix an open set $O \subseteq Y$,

$$\{t \in S : G(t, \varphi(t)) \in O\} = \left\{ t \in S \setminus \bigcup_j M_j : G(t,0) \in O \right\}$$
$$\bigcup \left(\bigcup_j \{t \in M_j : G(t, x_j) \in O\} \right),$$

thus, it belongs to Σ, according to Definition 2.1. Furthermore, the image of the function $G(\cdot, \varphi(\cdot))$ is the set $\{0\} \bigcup ((\bigcup_j)\{x_j\}_j)$. Thus, $G(\cdot, \varphi(\cdot))$ is separably valued (for the definition, see [3, p. 28, Definition 14]). So, according to [3, p. 30, Remark 20], $G(\cdot, \varphi(\cdot)) \in B^0(S, Y)$.

Now, if $f \in B^0(S, X)$ and $\{\varphi_j\}_{j \geq 1}$ is a sequence of X-valued simple functions, converging pointwise to f on $S \backslash N'$, for some $N' \in \Sigma$ μ-null set, the sequence $\{G(t, \varphi_j(t))\}_{j \geq 1}$ converges to $G(t, f(t))$ in Y, for each $t \in S \backslash (N \bigcup N')$, where N is the μ-null set in 2) of Definition 2.1. Thus, the function $G(\cdot, f(\cdot)) \in B^0(S, Y)$. So, we have proved 1).

As for the proof of 2), let us assume that the application $f \to G(\cdot, f)$ is not continuous from $B^0(S, X)$ into $B^0(S, Y)$. That is to say, for some sequence $\{f_j\}_{j \geq 1}$ converging to a function f in $B^0(S, X)$, the sequence $\{G(\cdot, f_j(\cdot))\}_{j \geq 1}$ does not converge to $G(\cdot, f(\cdot))$ in $B^0(S, Y)$. In other words, there is $\varepsilon > 0$, so for each $k \geq 1$, we can select $j_k \geq 1$, with $j_{k+1} \geq j_k$, for which

$$d_{B^0(Y)} \Big(G(\cdot, f_{j_k}(\cdot)) - G(\cdot, f_k(\cdot)) \Big) > \varepsilon. \tag{1}$$

Since the subsequence $\{f_{j_k}\}_{k \geq 1}$ converges to f in $B^0(S, X)$, that is, it converges to f in μ-measure, according to [3, p. 45, Proposition 19], there is a subsubsequence $\{f_{j_{k_l}}\}_{l \geq 1}$ that converges to f, μ-a.e. on S. Then, as in 1), $\{G(\cdot, f_{j_{k_l}}(\cdot))\}_{l \geq 1}$ converges to $G(\cdot, f(\cdot))$, μ-a.e. on S. Since the measure μ is finite, $\{G(\cdot, f_{j_{k_l}}(\cdot))\}_{l \geq 1}$ also converges to $G(\cdot, f(\cdot))$, in μ-measure. However, this conclusion contradicts (1).

This proves 2) and completes the proof of the proposition. □

Definition 2.3. A function $G : S \times X \to Y$ is called bi-measurable if it satisfies 1) in Proposition 2.2. According to Proposition 2.2, a Carathéodory function is bi-measurable. For more general conditions on G that imply bi-measurability, we refer to ([16, p. 336]), where the notion of Shragin function is presented.

Remark 1. When the function G only depends on u, the operator N_G is called autonomous. In this case, N_G reduces to a simple composition $G \circ f$.

It was C. Carathéodory who studied the measurability of this composition in the real valued case [11]. He observed that the composition of two measurable functions might not be measurable and proved that $G \circ f$ is measurable if f is measurable and G is continuous, thus suggesting the correct assumptions, as stated in Definition 2.1, on a general function G. This is the reason for the name Carathéodory function and for the two conditions formulated in Definition 2.1, which are referred to as Carathéodory conditions.

The map $f \to G(\cdot, f)$ established in Proposition 2.2 is called, in the real-valued case, a Nemytskiĭ operator, usually denoted N_G. For this reason, the function G introduced in Definition 2.1 is sometimes called, in the real-valued case, an N-function. We will sometimes use this name in the vector-valued case as well.

Besides the differentiability properties already mentioned in the introduction, the real-valued Nemytskiĭ operator has other interesting continuity and boundedness properties, for which bi-measurability is a necessary condition. As an example, we mention the following result (see [30, p. 154, Theorem 19.1]).

Theorem 2.4. *Let $(\mathbb{R}^n, \mathcal{L}, \lambda)$ be the Lebesgue measure space. If we fix $1 \le p_1$, $p_2 < \infty$, the following statements are equivalent:*

(1) *The Nemytskiĭ operator N_G is continuous and bounded from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$.*

(2) *There is a non-negative function $a \in L^{p_2}(\mathbb{R}^n)$ and a real number $b \ge 0$, so that*

$$|G(t,u)| \le a(t) + b|u|^{p_1/p_2}.$$

Remark 2. According to a footnote inserted by the translator in [30, p. 155], the proof of Theorem 2.4, as presented in [30], is due to M. A. Krasnosel'skiĭ (see [20, p. 20]) and differs considerably from the original proof by Vainberg]. The translator, M. Feinstein, also points out that the result applies as well to the case $0 < p_1, p_2 < 1$.

For an extension of Theorem 2.4 to the vector-valued case, we refer to ([22, Theorem 3.1]). Let us observe that, in the vector-valued case, the spaces X and Y are assumed to be separable and the measure space (S, Σ, μ) is atomless (for the definition, see [3, p. 19, Definition 6]).

Theorem 2.4 suggests the following non-autonomous example of a Carathéodory function:

Example. Let us consider the finite sum:

$$G(t,u) = \sum_j a_j(t) \|d_j u - b_j(t)\|_X + T(u), \tag{2}$$

under the following assumptions:

(1) For each j, $a_j : S \to Y$ and $b_j : S \to X$, are strongly measurable.

(2) For each j, $d_j \in \mathbb{R}$.

(3) $T \in L(X,Y)$ (see in [3, p. 40 the statement of Theorem 5]).

Then, it should be clear that G satisfies Definition 2.1. By analogy with the real-valued case (for the real case, see [2, p. 77, Definition 13]), we call (2) a piecewise linear vector-valued N-function.

The proof of the following result is quite straightforward, so we will omit it.

Proposition 2.5. *Given a piecewise linear vector-valued N-function G, the associated Nemytskiĭ operator N_G is well defined, continuous, and bounded, from $B^1(S,X)$ into $B^1(S,Y)$, if $a_j \in B^\infty(S,Y)$ and $b_j \in B^1(S,X)$, for every j (for the definitions, see [3, Section 4]).*

In the next few sections, we define natural extensions of the Nemytskiĭ operator N_G to vector measures. As expected, there will be a balance between how general the operator can be and how general the vector measure can be.

3. First extension of the Nemytskiĭ operator to vector measures

We fix a complete and σ-finite measure space (S,Σ,μ).

We begin by assuming that the Banach space X has the Radon-Nikodým property with respect to the measure space (S,Σ,μ) (see [3, p. 54, Definition 24]). Then,

Proposition 3.1. *Let G be an N-function satisfying the growth condition*

$$\|G(t,u)\|_Y \le a(t) + b\|u\|_X, \tag{3}$$

μ-a.e. on S with $a \in L^1(S)$ and $b \ge 0$. Then, the associated Nemytskiĭ operator N_G is bounded and continuous from $B^1(S,X)$ into $B^1(S,Y)$.

Furthermore, there exists a unique operator $\widetilde{N}_G : \mathcal{M}_{f,a}(X) \to \mathcal{M}_{f,a}(Y)$ such that

$$\Lambda_Y \circ N_G(f) = \widetilde{N}_G \circ \Lambda_X(f), \tag{4}$$

for all $f \in B^1(S,X)$. Let us recall that the space $\mathcal{M}_{f,a}$ and the operator Λ, were defined in the statement [3, p. 54, Proposition 24].

The equality (4) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} B^1(S,X) & \xrightarrow{N_G} & B^1(S,Y) \\ \Lambda_X \downarrow & & \downarrow \Lambda_Y \\ \mathcal{M}_{f,a}(X) & \xrightarrow{\widetilde{N}_G} & \mathcal{M}_{f,a}(Y) \end{array} . \tag{5}$$

Proof. We propose

$$\widetilde{N}_G(fd\mu) = G\big(\cdot, f(\cdot)\big)d\mu. \tag{6}$$

The Radon-Nikodým property determines the function f in (6) only μ-a.e. in S, so, we need to verify that $\widetilde{N}_G$ is well defined. Let $h \in B^1(S,X)$ be so that $h = f$ on $S \backslash U$, where $U \in \Sigma$ is a μ-null set. Then, given $A \in \Sigma$,

$$\begin{aligned}\widetilde{N}_G(hd\mu)(A) &= \int_A G\big(\cdot, h(\cdot)\big)d\mu = \int_{(S\backslash U)\cap A} G\big(\cdot, f(\cdot)\big)d\mu \\ &= \int_A G\big(\cdot, f(\cdot)\big)d\mu \\ &= \widetilde{N}_G(fd\mu)(A),\end{aligned}$$

where we have used the completeness of the measure space (S,Σ,μ).

Next, we show that $\widetilde{N}_G$ makes the diagram in (5) commutative. In fact,

$$\Lambda_Y \circ N_G(f) = G\big(\cdot, f(\cdot)\big)d\mu = \widetilde{N}_G\big(fd\mu\big) = \widetilde{N}_G \circ \Lambda_X(f),$$

for all $f \in B^1(S,X)$.

As for the uniqueness, let $H : \mathcal{M}_{f,a}(X) \to \mathcal{M}_{f,a}(Y)$ be another operator that also makes the diagram in (5) commutative. That is to say,

$$\Lambda_Y \circ N_G(f) = H \circ \Lambda_X(f),$$

for all $f \in B^1(S,X)$. Then,

$$H(fd\mu) = H \circ \Lambda_X(f) = \Lambda_Y \circ N_G(f) = \widetilde{N}_G \circ \Lambda_X(f) = \widetilde{N}_G(fd\mu).$$

This completes the proof of the proposition. □

4. Properties of the map $G \to \widetilde{N}_G$

Once again, we fix a complete and σ-finite measure space (S,Σ,μ).

Although it is not necessary, to avoid notational complications, we assume now that $X = Y$. Still, we suppose that the Banach space X has the Radon-Nikodým property with respect to the measure space (S,Σ,μ).

We begin with a definition.

Definition 4.1. Let us call $\mathcal{N}$ the family of those vector-valued N-functions G that satisfy the growth condition (3). Moreover, let

$$\widetilde{\mathcal{N}} = \{\widetilde{N}_G : G \in \mathcal{N}\}.$$

In the theorem that follows, we show that the map $G \to \widetilde{N}_G$ has quite a few properties resembling those of a functional calculus. By the way, it might have been M. Fréchet who first used the words "functional calculus" in his doctoral thesis presented at the Faculté des Sciences de Paris on April 2, 1906 [17].

Theorem 4.2. *The following statements hold:*

(1) *The spaces $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are real linear spaces and the map $G \to \widetilde{N}_G$ from $\mathcal{N}$ into $\widetilde{\mathcal{N}}$ is linear.*

(2) *The space $B^1(S)$ is contained in $\mathcal{N}$. In fact, each function $g \in B^1(S)$ defines an N-function g that satisfies (3). Furthermore, given $g \in B^1(S)$,*

$$\widetilde{N}_g(f d\mu) = g d\mu.$$

(3) *The space $\mathcal{N}$ is closed under the composition operation*

$$(G_2 \circ G_1)(t,u) = G_2\big(t, G_1(t,u)\big).$$

Moreover,

$$\widetilde{N}_{G_2 \circ G_1} = \widetilde{N}_{G_2} \circ \widetilde{N}_{G_1}.$$

Proof. It should be clear that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are real linear spaces.

If $\alpha, \beta \in \mathbb{R}$ and $G_1, G_2 \in \mathcal{N}$,

$$\begin{aligned}\alpha \widetilde{N}_{G_1}(f d\mu) + \beta \widetilde{N}_{G_2}(f d\mu) &= \alpha\Big(G_1\big(\cdot, f(\cdot)\big)d\mu + \beta G_2\big(\cdot, f(\cdot)\big)d\mu\Big)\\ &= \Big(\alpha G_1\big(\cdot, f(\cdot)\big) + \beta G_2\big(\cdot, f(\cdot)\big)\Big)d\mu\\ &= \widetilde{N}_{\alpha G_1 + \beta G_2}(f d\mu).\end{aligned}$$

This proves 1).

As for 2), it should be clear that functions in $B^1(S)$ define N-functions. Moreover, given

$$g \in B^1(X), \quad N_g(f) = g \quad \text{and} \quad \widetilde{N}_g(f d\mu) = g d\mu$$

for all $f \in B^1(S)$, according to Proposition 3.1.

To prove 3), let us recall that there are μ-null sets U and V in Σ, so that the function $u \to G_1(t,u)$ is continuous for each $t \in S \backslash U$, and the function $v \to G_2(t,v)$ is continuous for each $t \in S \backslash V$.

So, the function $u \to G_2\big(t, G_1(t,u)\big)$ is continuous for each $t \in S \backslash (U \bigcup V)$.

If we fix $u \in X$, the function $t \to G_1(t,u)$ is strongly measurable. So, according to 1) in Proposition 2.2, the function $t \to G_2\big(t, G_1(t,u)\big)$ is strongly measurable.

Thus, $G_2 \circ G_1$ is an N-function. As for the growth condition,

$$\begin{aligned}\left\|G_2\big(t, G_2(t,u)\big)\right\| &\leq a_2(t) + b_2 \left\|G_2(t,u)\right\| \\ &\leq a_2(t) + b_2\big(a_1(t) + b_1\|u\|\big) \\ &= a_2(t) + b_2 a_1(t) + b_2 b_1 \|u\|.\end{aligned}$$

So, $G_2 \circ G_1 \in \mathcal{N}$.

Finally,

$$\begin{aligned}\big(\widetilde{N}_{G_2} \circ \widetilde{N}_{G_1}\big)(f d\mu) = \widetilde{N}_{G_2}\big(\widetilde{N}_{G_1}(f) d\mu\big) = \widetilde{N}_{G_2}\Big(G_1\big(\cdot, f(\cdot)\big) d\mu\Big) = \\ G_2\Big(\cdot, G_1\big(\cdot, f(\cdot)\big)\Big) d\mu = \widetilde{N}_{G_2 \circ G_1}(f d\mu).\end{aligned}$$

This completes the proof of the proposition. □

Remark 3. If $G(\cdot) : S \to \mathbb{R}$ is the real-valued and autonomous N-function defined as $G(u) = \|u\|$, then

$$N_G(f)(\cdot) = \|f(\cdot)\|.$$

Thus, according to [3, p. 47, Theorem 6, 3]

$$\widetilde{N}_G(f d\mu) = \|f\| d\mu = |f d\mu|. \tag{7}$$

5. Second extension of the Nemytskiĭ operator to vector measures

As before, we fix a complete and σ-finite measure space (S, Σ, μ). However, this time we do not assume that X has the Radon-Nikodým property with respect to (S, Σ, μ). Moreover, we take $X = Y$.

Our goal is to extend the Nemytskiĭ operator associated with a particular kind of piecewise linear N-function (see Example 2) to the space $\mathcal{M}_f$ (see the statement in [3, p. 21, Proposition 7]).

More precisely

Definition 5.1. We consider N-functions of the form

$$G(t,u) = \sum_j a_j(t) \left\|d_j u - b_j(t)\right\| + T(u),$$

assuming that

$$d_j \in \mathbb{R}, \quad a_j \in B^\infty(S), \quad b_j \in B^1(S) \quad \text{for all } j, \text{ and } T \in L(X),$$

that is to say, T is a linear and continuous operator from X into itself.

Lemma 5.2. *The following statements are true:*

(1) *Given $m \in \mathcal{M}_f$, the set function $T \circ m$ defined as*

$$(T \circ m)(A) = T\big(m(A)\big)$$

for all $A \in \Sigma$, is a vector measure.

(2) *Moreover,*

$$|T \circ m|\,(A) \leq \|T\|_{L(X)}\,|m|(A) \tag{8}$$

for all $A \in \Sigma$.

(3) *The map $m \to T \circ m$ is a linear and continuous function from $\mathcal{M}_f$ into itself.*

P r o o f. Let us prove 1):

For starters, $(T \circ m)\,(\varnothing) = T\,(0) = 0$.

Next, if $\{A_j\}_{j \geq 1}$ is a countable family of disjoint sets in Σ,

$$\begin{aligned}(T \circ m)\left(\bigcup_{j \geq 1} A_j\right) &= T\left(\lim_{k \to \infty} \sum_{j=1}^{k} m(A_j)\right) \\ &= \lim_{j \to \infty} \sum_{j=1}^{k} T\big(m(A_j)\big) \\ &= \sum_{j \geq 1} (T \circ m)(A_j).\end{aligned}$$

So, $T \circ m$ is a vector measure.

As for 2), given $A \in \Sigma$ and $\varepsilon > 0$, there exists a finite partition $\{A_j\}_j \subseteq \Sigma$ of the set A for which

$$\begin{aligned}|T \circ m|\,(A) - \varepsilon &< \sum_j \|(T \circ m)\,(A_j)\| \\ &\leq \|T\|_{L(X)} \sum_j \|m\,(A_j)\| \\ &\leq \|T\|_{L(X)}\,|m|(A).\end{aligned}$$

So, (8) holds. In particular, $T \circ m \in \mathcal{M}_f$.

Finally, it should be clear that 3) is an immediate consequence of 1) and 2). This completes the proof of the lemma. □

The following definition is justified by Theorem 4.2 and the properties of the variation discussed in [3, Section 2].

Definition 5.3. For each j, we denote $b_j\mu$ the vector measure defined as

$$(b_j\mu)(A) = \int_A b_j d\mu$$

for each $A \in \Sigma$, while given $m \in \mathcal{M}_f$, $a_j |d_j m - b_j\mu|$ is the vector measure defined as

$$\int_A a_j(t) d\,|d_j m - b_j\mu| \tag{9}$$

for each $A \in \Sigma$.

Let us observe that $b_j\mu$, $d_j m$, and $|d_j m - b_j\mu|$, all have finite variation (see [3, Section 2]).

Then, given $m \in \mathcal{M}_f$, we denote $\widetilde{N}_G(m)$ the vector measure defined as

$$\big(\widetilde{N}_G(m)\big)(A) = \sum_j (a_j|d_j m - b_j\mu|)(A) + (T \circ m)(A) \tag{10}$$

for all $A \in \Sigma$.

Remark 4. Since $\mu(A) = \int_A d\mu$ for all $A \in \Sigma$, we can write $d\mu$ as in 2) of Theorem 4.2 or μ as in (9).

Proposition 5.4. *The map $\widetilde{N}_G$ defined by* (10) *is bounded from $\mathcal{M}_f$ into itself.*

Proof. First, let us recall that $\mathcal{M}_f$ is a linear normed space with the total variation as norm (see [3, p. 21, Proposition 7]).

It should be clear from Lemma 5.2 and Definition 5.3 that $\widetilde{N}_G$ maps $\mathcal{M}_f$ into itself. For simplicity, we will work with one term in (10). Moreover, since the map $m \to T \circ m$ is bounded from $\mathcal{M}_f$ into itself, it suffices to assume that

$$\widetilde{N}_G(m) = a|dm - b\mu|,$$

with

$$a \in L^\infty(S), \quad b \in B^1(S) \quad \text{and} \quad d \in \mathbb{R}.$$

If $\mathcal{B}$ is a bounded subset of $\mathcal{M}_f$ and $m \in \mathcal{B}$,

$$|\widetilde{N}_G(m)|(S) \le \|a\|_{B^\infty(S)}\big(|d|_{\mathbb{R}}|m|(S) + \|b\|_{B^1(S)}\big),$$

which shows that $\{\|\widetilde{N}_G(m)\|_{\mathcal{M}_f}\}_{m\in\mathcal{B}}$ is bounded in $\mathcal{M}_f$. To avoid confusion, we have denoted $|d|_{\mathbb{R}}$ the absolute value of the real number d.

This completes the proof of the proposition. □

We now want to establish the relationship between the Lebesgue decomposition of $\widetilde{N}_G(m)$ and the Lebesgue decomposition of m, assuming that the measure space (S, Σ, μ) is complete and finite. To attain this goal fully, we also assume that X has the Radon-Nikodým property with respect to (S, Σ, μ).

According to [3, p. 26, Theorem 1 and p. 54, Definition 24], given $m \in \mathcal{M}_f$,

$$m = f d\mu + m_s,$$

where $f \in B^1(S)$ and $m_s \perp \mu$ (for the definition, see [3, p. 21, Definition 7]). Then,
Theorem 5.5.

$$\widetilde{N}_G(m) = G\big(\cdot, f(\cdot)\big)d\mu + \left(\sum_j a_j\right)|m_s| + T \circ m_s,$$

where

$$\left(\sum_j a_j\right)|m_s| + T \circ m_s \perp G\big(\cdot, f(\cdot)\big)d\mu.$$

P r o o f. We begin by writing

$$\widetilde{N}_G\,(m) = \sum_j a_j\,|d\,(f d\mu + m_s) - b_j d\mu| + T\big(f(\cdot)\big)d\mu + T \circ m_s.$$

It should be clear that $m_s \perp \mu$ implies that the measures m_s and $(f - b_j)\,d\mu$ are also mutually singular. Moreover, according to [3, p. 22, Lemma 6],

$$\begin{aligned}\widetilde{N}_G(m) &= \sum_j a_j|(f - b_j)\mu| + \left(\sum_j a_j\right)|m_s| + T\big(f(\cdot)\big)\mu + T \circ m_s\\ &= \widetilde{N}_G\,(f d\mu) + \left(\sum_j a_j\right)|m_s| + T \circ m_s.\end{aligned}$$

By definition, $|m_s|$ and μ are mutually singular. So, in [3, p. 22, Lemma 5], tells us that there is a partition $S = A \bigcup B$, $A, B \in \Sigma$, such that

$$\begin{aligned}|m_s|(A') &= 0 \quad \text{for all} \quad A' \subseteq A, \quad A' \in \Sigma,\\ \mu(B') &= 0 \quad \text{for all} \quad B' \subseteq B, \quad B' \in \Sigma.\end{aligned}$$

Then,

$$\left[\left(\sum_j a_j\right)|m_s|\right](A') = \int_{A'} \left(\sum_j a_j\right) d|m_s| = 0.$$

So, the measures $(\sum_{i=1}^n a_i)|m_s|$ and μ are also mutually singular. It remains to prove that the measures $T \circ m_s$ and μ are mutually singular, as well.

Since $|m_s| \perp \mu$, there is a partition $S = A \bigcup B$, $A, B \in \Sigma$, such that $|m_s|(A) = 0$ and $\mu(B) = 0$. Then, (8) implies that $|T \circ m_s|(A) = 0$ and, therefore,

$$\left(\sum_{i=1}^n a_i\right)|m_s| + T \circ m_s$$

and μ are mutually singular.

According to Proposition 3.1, $G\big(\cdot, f(\cdot)\big)\in B^1(S)$. Thus $\widetilde{N}_G(fd\mu)=G\big(\cdot, f(\cdot)\big)d\mu$ is absolutely continuous with respect to μ.

Finally, $(\sum_{i=1}^{n} a_i)|m_s| + T \circ m_s$ and $\widetilde{N}_G(fd\mu)$ are mutually singular.

This completes the proof of the theorem. □

Remark 5. The operator $\widetilde{N}_G$ given by (10) makes the following diagram commutative:

$$\begin{array}{ccc} B^1(S) & \xrightarrow{N_G} & B^1(S) \\ \Lambda\downarrow & & \downarrow\Lambda \\ \mathcal{M}_f & \xrightarrow{\widetilde{N}_G} & \mathcal{M}_f \end{array} . \tag{11}$$

Let us observe that, this time, Λ is an isometric isomorphism between $B^1(S)$ and the proper closed subspace $\mathcal{M}_{f,a}$ of $\mathcal{M}_f$, introduced in [3, p. 54, Proposition 24]. Let us recall that $\mathcal{M}_{f,a}$ consists of those vector measures of the form $fd\mu$, for $f \in B^1(S)$. Therefore, unlike the case in Proposition 3.1, the operator $\widetilde{N}_G$ in (11) is not uniquely determined.

Actually, there are interesting vector measures that have finite variation and are mutually singular with respect to a measure μ. We present the following example:

For $n \geq 2$, let us consider the Lebesgue measure spaces

$$(\mathbb{R}^n, \mathcal{L}_n, \lambda_n) \quad \text{and} \quad \big(\mathbb{R}^{n-1}, \mathcal{L}_{n-1}, \lambda_{n-1}\big)$$

and let us fix a real Banach space X.

Given $f \in B^1\left(\mathbb{R}^{n-1}\right)$, we define a set function $m : \mathcal{L}_n \to X$ as

$$m(A) = \int_{\mathbb{R}^{n-1}} f\chi_A(\cdot, 0)d\lambda_{n-1},$$

where χ_A denotes the characteristic function of A.

It should be clear that m is a vector measure of finite variation that is continuous (see [3, p. 19, Definition 5 and Remark 12]). Moreover, m and λ_n are mutually singular. In fact, if

$$\mathbb{X}_{n-1} = \left\{(x', 0) : x' \in \mathbb{R}^{n-1}\right\},$$

we have

$$\lambda_n\left(\mathbb{X}_{n-1}\right) = 0 \quad \text{and} \quad m(A) = 0,$$

for every

$$A \in \mathbb{R}^n, \quad A \subseteq \mathbb{R}^n \backslash \mathbb{X}_{n-1}.$$

Thus, we conclude our presentation of the Nemytskiĭ operator in the context of vector measures.

In the next and last section, we will put to use most of the material presented so far.

6. A non-linear initial value problem

We fix a complete and σ-finite measure space (S, Σ, μ) and a real Banach space X that has the Radon-Nikodým property with respect to (S, Σ, μ). In what follows, $B^1(S)$ will be the Banach space of equivalence classes as defined in [3, p. 37, Remark 25]. We will now use $\|f\|_{B^1(S)}$ to denote the norm in the Banach space $B^1(S)$ of the equivalence class f.

Definition 6.1. For $T > 0$ fixed, $C\,[0, T; \mathcal{M}_{f,a}]$ is the linear space of continuous functions $m : [0, T] \to \mathcal{M}_{f,a}$.

The space $C\,[0, T; \mathcal{M}_{f,a}]$ becomes a Banach space with the norm

$$\|m\| = \sup_{0 \le t \le T} \|m(t)\|_{\mathcal{M}_{f,a}}.$$

Likewise, the space $C^1\,[0, T; \mathcal{M}_{f,a}]$ of continuously differentiable functions $m : [0, T] \to \mathcal{M}_{f,a}$ is a Banach space with the norm $\|m\| + \|m'\|$.

Let us observe that in the previous sections, t has indicated a variable taking value in S. In this section, that variable will be denoted x, while t will be a variable taking value in the interval $[0, T]$.

Remark 6. According to [3, p. 54, Proposition 24 and p. 37, Remark 25], $C[0, T; \mathcal{M}_{f,a}]$ is isometrically isomorphic to $C[0, T; B^1(S)]$ endowed with the norm

$$\|f\| = \sup_{0 \le t \le T} \|f(t)\|_{B^1(S)}.$$

Given $m_0 \in \mathcal{M}_{f,a}$, we consider the initial value problem

$$\begin{cases} \frac{dm}{dt} + \mathcal{A}(m)(t) = 0 & \text{for } 0 < t < T, \\ m(0) = m_0. \end{cases} \tag{12}$$

We want to formulate conditions on the operator $\mathcal{A}$ so that (12) has a unique solution $m \in C^1\,[0, T; \mathcal{M}_{f,a}]$.

Lemma 6.2. *Let $G : [0, T] \times S \times X \to X$ be a function $G(t, x, u)$ satisfying the following conditions:*

(1) *The function $G(\cdot, x, \cdot)$ is continuous from $[0, T] \times X$ into X, for μ-a.a. $x \in S$.*

(2) *For some non-negative $a \in L^1(S)$ and $b \ge 0$,*

$$\|G(t, x, u)\|_X \le a(x) + b\,\|u\|_X$$

for all $t \in [0, T], u \in X$, and for μ-a.a. $x \in S$.

(3) *The function $x \to G(t, x, u)$ is strongly measurable for each $t \in [0, T], u \in X$.*

(4) *There exists* $C > 0$ *such that*

$$\|G(t,x,u_1) - G(t,x,u_2)\|_X \le C\|u_1 - u_2\|_X,$$

for all $t \in [0,T]$ *and* $u_1, u_2 \in X$, *and for* μ*-a.a.* $x \in S$.

Then, the following properties hold:

a) *The function* $G_t = G(t,\cdot,\cdot)$ *is an* N*-function for each* $t \in [0,T]$.

b) *The Nemystkiĭ operator* N_{G_t} *maps* $B^1(S)$ *into itself for each* $t \in [0,T]$.

c) *The operator* $f(t,\cdot) \to N_{G_t}\big(f(t,\cdot)\big)(\cdot)$ *maps* $C[0,T;B^1(S)]$ *continuously into itself.*

P r o o f. Condition a) is a direct application of 1) and 3), while b) follows from 2).

To prove c), we begin by observing that given $f \in C[0,T;B^1(S)]$, the function $N_{G_t}\big(f(t,\cdot)\big)(x)$ belongs to $B^1(S)$ for each $t \in [0,T]$, as a consequence of b).

Moreover, we claim that $N_G\big(f(t,\cdot)\big)$ belongs to $C[0,T;B^1(S)]$. Indeed, if the sequence $\{t_j\}_{j\ge1}$ converges to t in $[0,T]$, the sequence $\{f(t_j,\cdot)\}_{j\ge1}$ converges to $f(t,\cdot)$ in $B^1(S)$. Hence, according to 4),

$$N_{G_t}\big(f(t_j,\cdot)\big) \underset{j\to\infty}{\longrightarrow} N_{G_t}\big(f(t_j,\cdot)\big)$$

in $B^1(S)$.

Finally, if the sequence $\{f_n\}_{n\ge1}$ converges to f in $C[0,T;B^1(S)]$, we use 4) to write

$$\begin{aligned}\big\|N_{G_t}\big(f_n(t,\cdot)\big) - N_{G_t}\big(f(t,\cdot)\big)\big\|_{B^1(S)} &\le C\,\|f_n(t,\cdot) - f(t,\cdot)\|_{B^1(S)}\\ &\le C\,\|f_n - f\|.\end{aligned}$$

Thus,

$$\sup_{0\le t\le T}\big\|N_{G_t}\big(f_n(t,\cdot)\big) - N_{G_t}\big(f(t,\cdot)\big)\big\|_{B^1(S)} \underset{n\to\infty}{\longrightarrow} 0.$$

This completes the proof of the lemma. □

We are now ready to define the operator $\mathcal{A}$ in (12).

Definition 6.3. Given a function G satisfying the hypotheses of Lemma 6.2, we define, for $m \in C[0,T;\mathcal{M}_{f,a}]$,

$$\mathcal{A}(m)(t) = \widetilde{N}_{G_t}\big(m(t)\big), \tag{13}$$

where $\widetilde{N}_{G_t}$, for each $t \in [0,T]$, is given by (6). That is to say,

$$\widetilde{N}_{G_t}\big(m(t)\big) = G\big(t,\cdot,f_{m(t)}(\cdot)\big)d\mu.$$

Lemma 6.4. *The operator* $\mathcal{A}$ *is well defined and continuous from* $C[0,T;\mathcal{M}_{f,a}]$ *into itself.*

P r o o f. According to b) in Lemma 6.2, $G\big(t,\cdot,f_{m(t)}(\cdot)\big) \in B^1(S)$ for each $t\in[0,T]$. Therefore, for each $t\in[0,T]$, the function $\widetilde{N}_{G_t}\big(m(t)\big)$ belongs to $\mathcal{M}_{f,a}$. Condition c) in Lemma 6.2 implies that

$$G\big(t,\cdot,f_{m(t)}(\cdot)\big) \in C[0,T;B^1(S)].$$

So,

$$\widetilde{N}_{G_t}\big(m(t)\big) \in C[0,T;\mathcal{M}_{f,a}].$$

That is to say, the operator $\mathcal{A}$ is well defined from $C[0,T;\mathcal{M}_{f,a}]$ into itself. As for the continuity, if the sequence $\{m_j\}_{j\geq 1}$ converges to m in $C\,[0,T;\mathcal{M}_{f,a}]$, according to Remark 6,

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\mathcal{A}(m_j)(t)-\mathcal{A}(m)(t)\|_{\mathcal{M}_{f,a}}\\ &= \sup_{0\leq t\leq T}\big\|G\big(t,\cdot,f_{m_j(t)}(\cdot)\big)-G\big(t,\cdot,f_{m(t)}(\cdot)\big)\big\|_{B^1(S)}\end{aligned}$$

and $\{f_{m_j(t)}\}_{j\geq 1}$ converges to $f_{m(t)}$ in $C[0,T;B^1(S)]$. Finally,

$$\sup_{0\leq t\leq T}\big\|G\big(t,\cdot,f_{m_j(t)}(\cdot)\big)-G\left(t,\cdot,f_{m(t)}(\cdot)\right)\big\|_{B^1(S)}\underset{j\to\infty}{\longrightarrow}0$$

as a consequence of c) in Lemma 6.2.

This completes the proof of the lemma. □

Remark 7. According to 4) in Lemma 6.2 and (13), given $m_1,m_2\in C[0,T;\mathcal{M}_{f,a}]$,

$$\begin{aligned}\big\|\big(\mathcal{A}(m_1)(t)-\mathcal{A}(m_2)(t)\big)\big\|_{\mathcal{M}_{f,a}} &= \Big\|\Big(G\big(t,\cdot,f_{m_1(t)}(\cdot)\big)-G\big(t,\cdot,f_{m_2(t)}(\cdot)\big)\Big)d\mu\Big\|_{\mathcal{M}_{f,a}}\\ &= \Big\|\Big(G\big(t,\cdot,f_{m_1(t)}(\cdot)\big)-G\big(t,\cdot,f_{m_2(t)}(\cdot)\big)\Big)\Big\|_{B^1(S)}\\ &\leq C\left\|f_{m_1(t)}-f_{m_2(t)}\right\|_{B^1(S)}\\ &= C\left\|m_1(t)-m_2(t)\right\|_{\mathcal{M}_{f,a}}\end{aligned}$$

for all $t\in[0,T]$, where C is the positive constant appearing in 4) of Lemma 6.2.

In each of the following two examples, we define a function G that satisfies the hypothesis of Lemma 6.2.

Example. To begin with, we fix a function $H: X\to X$ satisfying the following two conditions:

1. There exists $C_1>0$ such that $\|H(r)\|_X\leq C_1\|r\|_X$ for all $r\in X$.
2. H is a Lipschitz function; that is to say, there exists $C_2>0$ such that

$$\|H(r_1)-H(r_2)\|_X\leq C_2\|r_1-r_2\|_X \quad \text{for all } r_1,r_2\in X.$$

Then, given $b \in B^1(S)$, we define $G : [0,T] \times S \times X \to X$ as

$$G_t(x,u) = G(t,x,u) = H\big(b(x) + tu\big). \tag{14}$$

We claim that G satisfies conditions 1) − 4) in Lemma 6.2.

In fact, if $\{t_j\}_{j\geq 1}$ converges to t in $[0,T]$ and $\{u_j\}_{j\geq 1}$ converges to u in X, the product $\{t_j u_j\}_{j\geq 1}$ will converge to tu in X. Hence,

$$\big\|H\big(b(x) + t_j u_j\big) - H\big(b(x) + tu\big)\big\|_X \leq C_2 \, \|t_j u_j - tu\|_X \underset{j\to\infty}{\longrightarrow} 0$$

for μ-a.a. $x \in S$.

So, condition 1) is satisfied.

$$\begin{aligned}\|G(t,x,u)\|_X &= \|H(b(x) + tu)\|_X \\ &\leq C_1 \, \|b(x) + tu\|_X \\ &\leq C_1\big(\|b(x)\|_X + T\|u\|_X\big)\end{aligned}$$

for all $t \in [0,T], u \in X$, and for μ-a.a. $x \in S$. Therefore, 2) holds.

If we fix $t \in [0,T]$ and $u \in X$, the function $x \to H(b(x) + tu)$ is strongly measurable, because it is the composition, in the required order, of the strongly measurable function $x \to b(x)$ and the continuous function $r \to H(r + tu)$. So, condition 3) holds.

We can write

$$\begin{aligned}\|G(t,x,u_1) - G(t,x,u_2)\|_X &= \big\|H\big(b(x) + tu_1\big) - H\big(b(x) + tu_2\big)\big\|_X \\ &\leq C_2 t\|u_1 - u_2\|_X \\ &\leq C_2 T\|u_1 - u_2\|_X,\end{aligned}$$

which is condition 4).

Therefore, Lemma 6.4 implies that

$$\mathcal{A}(m)(t) = \widetilde{N}_{G_t}\big(m(t)\big) = H\big(b(\cdot) + tf(t,\cdot)\big)\,d\mu$$

is a well-defined and continuous operator from $C\,[0,T;\mathcal{M}_{f,a}]$ into itself.

Example. This time, we consider a function $g : S \times X \to X$ defined as a finite sum,

$$g(x,u) = \sum_j a_j(x)\, \|d_j u - b_j(x)\|_X + A(u),$$

where

$$d_j \in \mathbb{R}, \quad a_j \in B^\infty(S), \quad b_j \in B^1(S) \quad \text{and} \quad A \in L(X),$$

meaning that A is a linear and continuous operator from X into itself.

Next, we define the function $G : [0,T] \times S \times X \to X$ as

$$G_t(x,u) = G(t,x,u) = g(x,tu).$$

We claim that G satisfies conditions 1) – 4) in Lemma 6.2. Indeed, if $\{t_k\}_{k\geq 1}$ converges to t in $[0,T]$ and $\{u_k\}_{k\geq 1}$ converges to u in X, the product $\{t_k u_k\}_{k\geq 1}$ will converge to tu in X. Hence,

$$\begin{aligned}&\|g(x,t_k u_k) - g(x,tu)\|_X \\ &\leq \sum_j \|a_j\|_{B^\infty(S)} \left| \|d_j t_k u_k - b_j(x)\|_X - \|d_j tu - b_j(x)\|_X \right| \\ &\quad + \|A\,(t_k u_k - tu)\|_X\end{aligned}$$

for μ-a.a. $x \in S$.

If we recall that

$$|\|v\|_X - \|w\|_X| \leq \|v - w\|_X\,,$$

we can write

$$\begin{aligned}&\|g\,(x,t_k u_k) - g(x,tu)\|_X \\ &\leq \left(\sum_j \|a_j\|_{B^\infty(S)} |d_j| + \|A\|_{L(X)} \right) \|t_k u_k - tu\|_X \underset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus, 1) holds.

$$\begin{aligned}&\|G(t,x,u)\|_X \\ &\leq \sum_j \|a_j\|_{B^\infty(S)} \left(|d_j|\, t \|u\|_X + \|b_j\,(x)\|_X \right) \\ &\quad + \ t\,\|A\|_{L(X)}\,\|u\|_X \\ &\leq T \left(\sum_j \|a_j\|_{B^\infty(S)} |d_j| + \|A\|_{L(X)} \right) \|u\|_X \\ &\quad + \left(\sum_j \|a_j\|_{B^\infty(S)} \right) \sup_j \|b_j(x)\|_X\,,\end{aligned}$$

for μ-a.a. $x \in S$, which shows that 2) holds.

Condition 3) is true as well, since for each $t \in [0,T]$ and $u \in X$, the function

$$x \to a_j(x)\,\|d_j tu - b_j(x)\|_X + tA\,(u)$$

is strongly measurable for each j.

To prove 4), we fix u_1, u_2 in X. Then,

$$\begin{aligned}\|G(t,x,u_1)-G(t,x,u_2)\|_X &\leq \sum_j \|a_j(x)\|d_j t u_1 - b_j(x)\|_X + tA(u_1) \\ &\quad - a_j(x)\left\|d_j t u_2 - b_j\left(x\right)\right\|_X - tA\left(u_2\right)\big\|_X \\ &\leq \sum_j \|a_j(x)\|_X \big| \|d_j t u_1 - b_j(x)\|_X - \|d_j t u_2 - b_j(x)\|_X \big| \\ &\quad + T\left\|A\right\|_{L(X)}\left\|u_1-u_2\right\|_X \\ &\leq T\left(\sum_j \|a_j\|_{B^\infty(S)}\left|d_j\right| + \|A\|_{L(X)}\right)\|u_1-u_2\|_X,\end{aligned}$$

for μ-a.a. $x \in S$, which is 4).

Then, we can define the operator $\mathcal{A}$ as in Example 6.

Theorem 6.5. *The initial value problem* (12) *has one and only one solution in* $C^1[0,T;\mathcal{M}_{f,a}]$, *if we assume that* $m_0 \in \mathcal{M}_{f,a}$ *and the operator* $\mathcal{A}$ *is given by* (13), *where the function* G *satisfies 1) – 4) in Lemma 6.2.*

We will give two proofs of this theorem. The first proof relies on the following extension of the Banach fixed point theorem:

Proposition 6.6. *(For the proof see, for instance, [15, p. 286, Corollary 4]), Let* (M,d) *be a complete metric space and let* f *be a map from* M *into* M. *If there is* $k \geq 1$, *so that the composite map* $f^{(k)}$ *is a contraction, then the map* f *has a unique fixed point.*

The first proof of Theorem 6.5. We begin by observing that the solutions in $C^1[0,T;\mathcal{M}_{f,a}]$ of the initial value problem (12) are exactly the functions $m(t)$ that are solutions of the integral equation

$$m(t) = m_0 + \int_0^t \mathcal{A}(m)(s)ds. \tag{15}$$

We will show that (15) has one and only one solution in $C^1[0,T;\mathcal{M}_{f,a}]$ by proving that the operator $\mathcal{T}$ defined on $C[0,T;\mathcal{M}_{f,a}]$ as

$$\mathcal{T}(m) = m_0 + \int_0^t \mathcal{A}(m)(s)ds \tag{16}$$

has a unique fixed point. According to Proposition 6.6, it suffices to show that $\mathcal{T}^{(k)}$ is a contraction from $C\left[0,T;\mathcal{M}_{f,a}\right]$ to itself, for some $k \geq 1$.

We claim that, for some $C > 0$,

$$\left\|\mathcal{T}^{(k)}(m_1)(t) - \mathcal{T}^{(k)}(m_2)(t)\right\| \leq \frac{C^k t^k}{k!}\|m_1 - m_2\|, \tag{17}$$

for

$$t \in [0, T], \quad m_1, m_2 \in C[0, T; \mathcal{M}_{f,a}] \quad \text{and} \quad k \geq 1.$$

In fact, when $k = 1$, if $m_i(t) = f_i\,(t, \cdot)\,d\mu$ for $i = 1, 2$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\|\mathcal{T}(m_1)(t) - \mathcal{T}(m_2)(t)\|_{\mathcal{M}_{f,a}} \\
&\leq \int_0^t \|\mathcal{A}(m_1)(s) - \mathcal{A}(m_2)(s)\|_{\mathcal{M}_{f,a}}\, ds \\
&= \int_0^t \left\|G\big(\cdot, f_1(s, \cdot)\big) - G\big(s, \cdot, f_2(s, \cdot)\big)\right\|_{B^1(S)} ds \\
&\leq Ct \sup_{0 \leq s \leq T} \|f_1(s, \cdot) - f_2(s, \cdot)\|_{B^1(S)} \\
&= Ct \sup_{0 \leq s \leq T} \|m_1(s) - m_2(s)\|_{\mathcal{M}_{f,a}} = Ct\|m_1 - m_2\|,
\end{aligned}$$

where C is the positive constant in 3) of Lemma 6.2. Therefore,

$$\|\mathcal{T}(m_1) - \mathcal{T}(m_2)\| \leq CT\|m_1 - m_2\|.$$

We prove now that (17) holds for $k = n+1$, assuming that it holds for $k = n$. According to Remark 7,

$$\begin{aligned}
&\left\|\mathcal{T}^{(n+1)}(m_1)(t) - \mathcal{T}^{(n+1)}(m_2)(t)\right\|_{\mathcal{M}_{f,a}} \\
&= \left\|\int_0^t (\mathcal{A}\big(\mathcal{T}^{(n)}(m_1)\big)(s) - \mathcal{A}\big(\mathcal{T}^{(n)}(m_2)\big)(s))ds\right\|_{\mathcal{M}_{f,a}} \\
&\leq C \int_0^t \|\mathcal{T}^{(n)}(m_1)(s) - \mathcal{T}^{(n)}(m_2)(s)\|_{\mathcal{M}_{f,a}} ds \\
&\leq C \int_0^t \frac{C^n s^n}{n!}\|m_1 - m_2\| ds \\
&= \frac{C^{n+1} t^{n+1}}{(n+1)!}\|m_1 - m_2\|.
\end{aligned}$$

Hence, if we pick $k \geq 1$, so that $\frac{C^k T^k}{k!} < 1$, the composite operator $\mathcal{T}^{(k)}$ will be a contraction in $C\left[0, T; \mathcal{M}_{f,a}\right]$. According to Proposition 6.6, the operator $\mathcal{T}$ has a unique fixed point m in $C\left[0, T; \mathcal{M}_{f,a}\right]$, which implies that m is the unique solution in $C^1\left[0, T; \mathcal{M}_{f,a}\right]$ of (12).

This completes the first proof of Theorem 6.5. $\square$

The second proof relies on a "trick" mentioned in [10, p. 267], in the real-valued case.

Let B be a real Banach space and, for $T > 0$ fixed, let $g : [0, T] \times B \to B$ be a continuous function $g\left(t, u\right)$ that is Lipschitz in u, uniformly on $t \in [0, T]$. That is to say,

$$\left\| g\left(t, u_1\right) - g\left(t, u_2\right) \right\|_B \leq L \left\| u_1(t) - u_2(t) \right\|_B ,$$

for some $L > 0$ and all $t \in [0, T]$.

We consider the following two norms on $C[0, T; B]$:

$$\|u\| = \sup_{t \in [0,T]} \left\| u\left(t\right) \right\|_B$$

and

$$\|u\|_* = \sup_{t \in [0,T]} \left(e^{-Lt} \|u(t)\|_B \right) .$$

These norms are equivalent. In fact, given $u \in C[0, T; B]$ and $t \in [0, T]$,

$$e^{-LT} \|u(t)\|_B \leq e^{-Lt} \|u(t)\|_B \leq \|u(t)\|_B .$$

Therefore,

$$e^{-LT} \|u\| \leq \|u\|_* \leq \|u\| .$$

Now, if we fix $u_0 \in B$, we consider the operator $\mathcal{G}$ defined on $C[0, T; B]$

$$\mathcal{G}(u)(t) = u_0 + \int_0^t g\big(s, u(s)\big) ds,$$

for $t \in [0, T]$. We claim that

$$\left\| \mathcal{G}(u_1) - \mathcal{G}\left(u_2\right) \right\|_* \leq \left(1 - e^{-LT}\right) \left\| u_1 - u_2 \right\|_* , \tag{18}$$

for all $u_1, u_2 \in C[0, T; B]$.

Indeed, if we fix $t \in [0, T]$,

$$\begin{aligned}\left\|\big(\mathcal{G}(u_1)\big)(t) - \big(\mathcal{G}\,(u_2)\big)(t)\right\|_B &\le \int_0^t \left\|g\big(s, u_1(s)\big) - g\big(s, u_2(s)\big)\right\|_B ds \\ &\le L\int_0^t \|u_1(s) - u_2(s)\|_B\, ds \\ &= L\|u_1 - u_2\|_* \int_0^t e^{Ls} ds \\ &= \left(e^{Lt} - 1\right)\|u_1 - u_2\|_*.\end{aligned}$$

Hence,

$$e^{-Lt}\left\|\big(\mathcal{G}(u_1)\big)(t) - \big(\mathcal{G}(u_2)\big)(t)\right\|_B \le \left(1 - e^{-Lt}\right)\|u_1 - u_2\|_*\,.$$

Since

$$0 < 1 - e^{-Lt} \le 1 - e^{-LT} < 1$$

for every $t \in [0, T]$, we conclude that (18) holds and therefore, the operator $\mathcal{G}$ is a contraction.

If we invoke the Banach fixed point theorem (see, for instance, [15, p. 284, Theorem 1]), we conclude that there is a unique $u \in C\,[0, T; B]$ so that $\mathcal{G}(u) = u$. Equivalently, there is a unique $u \in C^1[0, T; B]$ that solves the initial value problem

$$\begin{cases} \frac{du}{dt} + g(t, u) &= 0 \qquad \text{for } 0 < t < T, \\ u(0) &= u_0. \end{cases}$$

The second proof of Theorem 6.5 follows quite easily from this discussion. Indeed,

The second proof of Theorem 6.5. Remark 7 implies that the operator $\mathcal{A}$ is Lipschitz with constant C. Therefore, we only need to use the "trick" with $B = \mathcal{M}_{f,a}$, $\mathcal{G} = \mathcal{T}$ in (16) and $L = C$, to conclude that (12) has one and only one solution in $C^1[0, T; \mathcal{M}_{f,a}]$.

This completes the second proof of Theorem 6.5. □

Remark 8. When the space X is $\mathbb{R}^n$, the equation $\frac{dm}{dt} + \mathcal{A}(m)(t) = 0$ becomes a system of non-linear real equations.

We will end with a very simple example of a different nature, where measures are not assumed to be absolutely continuous with respect to μ.

Remark 9. If the space X is $\mathbb{R}$, we claim that the initial value problem (12) has a unique solution when, for $m \in C(0,T;\mathcal{M}_{\mathbb{R}})$,

$$\mathcal{A}(m)(t) = \big(ctm(t) - b(\cdot)\mu\big), \tag{19}$$

where $t \in [0,T], c \in \mathbb{R}, b \in B^1(S)$.

Let us recall that, according to [3, p. 21, Proposition 7 and p. 55, Proposition 25], $\mathcal{M}_{\mathbb{R}}$ is the Banach space consisting of the signed measures $m : \Sigma \to \mathbb{R}$ with the norm $C_{\mathcal{M}_{\mathbb{R}}} = |m|(S)$, where $|m|$ denotes the variation of m.

It should be clear that given $m \in \mathcal{M}_{\mathbb{R}}$, $\mathcal{A}(m)(t) \in \mathcal{M}_{\mathbb{R}}$ for each $t \in [0,T]$. Moreover,

(1) $\mathcal{A}$ is continuous as a function from $[0,T] \times \mathcal{M}_{\mathbb{R}}$ into $\mathcal{M}_{\mathbb{R}}$.

(2) There exists $C > 0$ such that

$$\left\|\mathcal{A}\left(m_1\right)(t) - \mathcal{A}\left(m_2\right)(t)\right\|_{\mathcal{M}_{\mathbb{R}}} \leq C\left\|m_1 - m_2\right\|_{\mathcal{M}_{\mathbb{R}}},$$

for all $t \in [0,T]$ and $m_1, m_2 \in \mathcal{M}_{\mathbb{R}}$.

As a consequence, $\mathcal{A}$ is continuous from $C[0,T;\mathcal{M}_{\mathbb{R}}]$ into itself.

The proof of these statements involve calculations similar to those performed in previous examples, so we will omit it.

From these statements we can conclude, as in both proofs of Theorem 6.5, that (12) has a unique solution in $C^1\left(0,T;\mathcal{M}_{\mathbb{R}}\right)$ when $m_0 \in \mathcal{M}_{\mathbb{R}}$ and $\mathcal{A}$ is given by (19).

REFERENCES

[1] ÁLVAREZ, J.—EYDENBERG, M.—MARIANI, M. C.: *The Nemytsky operator on vector valued measures* (preprint).

[2] ÁLVAREZ, J.—GUZMÁN-PARTIDA, M.: *Nonlinear initial value problems with measure solutions*, Electronic Journal of Mathematical Analysis and Applications, **5** (2017), no. 2, 69–88, `http://math-frac.org/Journals/EJMAA/Vol5(2)_July_2017/Vol5(2)_Papers/09_EJMAA_Vol5(2)_July_2017_pp_68-88.pdf`.

[3] ÁLVAREZ, J.—GUZMÁN-PARTIDA, M.: *A study of vector measures* Lecturas Matemáticas, **44** (2023), no. 1, 5–61, `http://scm.org.co/archivos/revista/Articulos/1361.pdf`.

[4] AMBROSETTI, A.—PRODI, G.: *A Primer of Nonlinear Analysis*. Corrected reprint of the 1993 original. Cambridge Studies in Advanced Mathematics, Vol. 34, Cambridge University Press, Cambridge, 1995.

[5] APPELL, J.—ZABREĬKO, P. P.: *Nonlinear Superposition Operators*. Cambridge University Press 1990, `https://doi.org/10.1017/CBO9780511897450`.

[6] APPELL, J.—ZABREĬKO, P. P.: *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics, Vol. 95, Cambridge University Press, Cambridge, 1990. `www.bsu.by/sm.aspx?guid=65573`.

[7] BERKOVIT, J.—FABRY, C.: *An extension of the topological degree in Hilbert space,* Abstract and Applied Analysis, **4** (2005), no. 2, 581–597.

[8] BERGOUNIOUX, M.—TRÖLTZSCH, F.: *Optimal control of semilinear parabolic equations with state-constraints of bottleneck type*, ESAIM: Control Optim. Calc. Var. **4** (1999), 595–608.

[9] BERKOVITS, J.—MAWHIN, J.: *Diophantine approximation, Bessel functions and radially symmetric periodic solutions of semilinear wave equations in a ball*, Trans. Amer. Math. Soc. **353** (2001), no. 12, 5041–5055.

[10] BROWN, R.: *Book Review*. Bull. Amer. Math. Soc. **41** (2004), no. 2, 267–271.

[11] CARATHÉODORY, C.: *Vorlesungen über reelle Funktionen*. BG Teubner, Leipzig and Berlin, 1918.

[12] CASTILLO, R. E.—MERENTES, N.—TROUSSELOT, E.: *The Nemytskii operator on bounded ϕ-variation in the mean spaces*, Proyecciones J. of Math. **32** (2013), no. 2, 119–142, `www.revistaproyecciones.cl/article/view/1121/1161`

[13] CHABROWSKI, J.: *Variational Methods for Potential Operator Equations. With applications to nonlinear elliptic equations*. De Gruyter Studies in Mathematics, Vol. 24. Walter de Gruyter & Co., Berlin, 1997.

[14] CIARLET, P. G.: *Mathematical elasticity. Volume I: Three-dimensional elasticity*. Stud. Math. Appl. Vol. 20. Amsterdam etc.: North- Holland, 1988.

[15] DEPREE, J.—SWARTZ, C.: *Introduction to Real Analysis*. John Wiley & Sons, New York, 1988.

[16] DUDLEY, R. M.—NORVAIŠA, R.: *Concrete Functional Calculus*. Springer-Verlag, Berlin, 2013.

[17] FRÉCHET, M.: *Sur quelques points du calcul fonctionnel*, Rend. Circ. Matem. Palermo **22** (1906), no. 1, 1–72.

[18] GAINES, R. E.—MAWHIN, J.: *Coincidence Degree and Nonlinear Differential Equations*. Lecture Notes in Math. Vol. 568, Springer-Verlag, Berlin, 1977.

[19] GATICA, G. N.—HSIAO, G. C.: *Boundary-Field Equation Methods for a Class of Nonlinear Problems*. Pitman Research Notes in Mathematics Series, Vol. 331. Longman, Harlow, 1995.

[20] KRASNOSEL'SKIĬ, M. A.: *Topological Methods in the Theory of Nonlinear Integral Equations*. (Translated by A. H. Armstrong; Translation edited by J. Burlak.) A Pergamon Press Book, The Macmillan Company, New York, 1964.

[21] KRASNOSEL'SKIĬ, M. A.—J. MAWHIN, J.: *The index at infinity of some twice degenerate compact vector fields*, Discrete Contin. Dynamical Systems **1** (1995), no. 2, 207–216.

[22] LUCCHETTI, R. E.—PADRONE, F.: *On Nemytskii's operator and its application to the lower continuity of integral functionals*, Indiana Univ. Math. J. **29** (1980), no. 5, 703–713.

[23] MATKOWSKI, J.: *On Nemytskii operator*, Math. Japon, **33** (1988), no. 1, 81–86.

[24] MATKOWSKI, J.: *Functional equations and Nemytskii operator*, Funkcial Ekvac. **25** (1982), no. 2, 127–132,

[25] MAWHIN, J.: *Topological Degree Methods in Nonlinear Boundary Value Problems*, In: Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977. CBMS Regional Conference Series in Mathematics, Vol. 40. Amer. Math. Soc., Providence, RI, 1979.

[26] MAWHIN, J.: *Topological degree and boundary value problems for nonlinear differential equations.* In: (M. Furi and P. Zecca, eds.), Topological Methods for Ordinary Differential Equations, Lect. Notes in Math. Vol. 1537, Springer-Verlag, Berlin, 1993, pp. 74–142.

[27] MOREIRA, D. R.—TEIXEIRA, E. V. O.: *On the behavior of weak convergence under nonlinearity and applications,* Proc. Amer. Math.Soc. **133** (2005), no. 6, 1647–1656, `www.ams.org/proc/2005-133-06/S0002-9939-04-07876-1/S0002-9939-04-07876-1.pdf`.

[28] NEMYTSKIĬ, V. V.: *Théoremès d'éxistence et unicité des solutions de quelques équations intégrales non-linéaires*, Mat. Sb. **41** (1934), 438–452.

[29] VAĬNBERG, M. M.: *On the differential and gradient of a functional*, Uspehi Mat. Nauk **7** (1952), no. 3, 139–143.

[30] VAĬNBERG, M. M.: *Variational Methods for the Study of Nonlinear Operators*, (Translated by A. Feinstein), Holden-Day Series in Mathematical Physics, San Francisco--London-Amsterdam: Holden-Day, Inc. x, 323 pp. (1964).

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