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NEARLY μ-LINDELÖFNESS VIA HEREDITARY CLASS

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ABSTRACT. In this paper, we define and study the notion of hereditary class on nearly μ-Lindelöf space. Moreover, we study the effects of some types of continuity of hereditary class on nearly μ-Lindelöf space by properties of the function. Also, more variations between these spaces and some known spaces are investigated.

1. Introduction

There are many generalizations of the ordinary notion of topological spaces. Among them, the most important and the best-known are in Császár space [1] which are studied in this paper, infra-topological spaces [2], per-topologies [3], minimal spaces [4], weak structures [5] and, finally, generalized weak structures [6] (which are just arbitrary collections of sets). Some other generalizations have been done on covering properties in different ways as [8–13].

By the definition, generalized topology μ on a non-empty set X is a collection of subsets of X where $\emptyset \in \mu$ and $\bigcup_\alpha A_\alpha \in \mu$ for all $A_\alpha \in \mu$.

In this paper, we consider $X \in \mu$. A subset B is μ-open if $B \in \mu$, and B is μ-closed if $X \backslash B \in \mu$. In particular, the concept of nearly μ-Lindelöf spaces has been introduced as an analogous work of nearly countably μ-compact which

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means that $U_\lambda \in \mu$ for all $\lambda \in \Lambda$ and Λ is a countable index set, then there is a finite sub-collection

$$\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\} \quad \text{such that} \quad X = \bigcup_{\lambda \in \Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda),$$

which is presented in [10]. Some prescriptions of the new definition, under different kinds of continuity, have been examined in a section of the current paper. Moreover, the concept of nearly μ-Lindelöfness has been implemented to define soft nearly μ-Lindelöf spaces. Similar applications can be studied in fuzzy set theory as an future work. The interior of B in μ is

$$\mathrm{Int}_\mu(B) = \bigcup_{O_\alpha \subseteq B} O_\alpha \quad \text{for all} \quad O_\alpha \in \mu,$$

and the closure is

$$\mathrm{Cl}_\mu(B) = \bigcap_{B \subseteq S_\alpha} S_\alpha \quad \text{for all} \quad X \backslash S_\alpha \in \mu.$$

Also, a subset B is called μ-regular open, whenever $\mathrm{Int}_\mu \mathrm{Cl}_\mu(B) = B$, and it is called μ-regular closed, whenever $\mathrm{Cl}_\mu \mathrm{Int}_\mu(B) = B$.

In this paper, the notation X_μ stands for the pair (X, μ). Recall that $\mathcal{H}$ is a hereditary class if $\mathcal{H} \subseteq P(X)$ and $\emptyset \in \mathcal{H}$ and whenever $A \in \mathcal{H}$ and $B \subseteq A$, then $B \in \mathcal{H}$ [14].

Definition 1.1 ([15])**.** Let X be a set. The space X_μ is said to be $\mathcal{N}\mu$-Lindelöf whenever $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, where $U_\lambda \in \mu$ for all $\lambda \in \Lambda$, then there is a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $X = \bigcup_{\lambda \in \Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(U_\lambda)$.

Definition 1.2 ([16])**.** Let $(X_\mu, \mathcal{H})$ be a space with respect to $\mathcal{H}$. The pair $(X_\mu, \mathcal{H})$ is said to be $\mu\mathcal{H}$-Lindelöf whenever $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, where $U_\lambda \in \mu$ for all $\lambda \in \Lambda$, then there is a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $X \backslash \bigcup_{\lambda \in \Lambda_0} U_\lambda \in \mathcal{H}$.

Definition 1.3 ([17])**.** Let X be a set. The space X_μ is said to be μ-regular whenever, for each μ-open subset U of X and for each $x \in U$, there exist a μ-open subset V of X and a μ-closed subset F of X such that $x \in V \subset F \subset U$.

Definition 1.4. [17] If $C \subseteq X_\mu$ and $x \in X$, then x is called θ_μ-cluster point of C if $\mathrm{Cl}_\mu(V) \cap C \neq \emptyset$ for all $V \in \mu$ and $x \in V$. The set $(\mathrm{Cl}_\mu)_\theta(C) = \{x \in X : x$ is θ_μ-cluster point of $C\}$ if $(\mathrm{Cl}_\mu)_\theta(C) = C$, then C is called μ_θ-closed. The set C is μ_θ-open if $X \backslash C$ is μ_θ-closed.

Definition 1.5. [18] Let $(X_\mu, \mathcal{H})$ be a space with respect to $\mathcal{H}$. The pair $(X_\mu, \mathcal{H})$ is said to be weakly $\mu\mathcal{H}$-Lindelöf (denoted by $\mathcal{W}\mu\mathcal{H}$-Lindelöf) whenever $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, where $U_\lambda \in \mu$ for all $\lambda \in \Lambda$, then there is a countable sub--collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $X \backslash \bigcup_{\lambda \in \Lambda_0} \mathrm{Cl}_\mu(U_\lambda) \in \mathcal{H}$.

Á. Császár introduced the notions of continues function in generalized topological spaces in 2002 [1]. Let μ and β be generalized on X_μ and Y_β, respectively. Then, a function f : $X_\mu \to Y_\beta$ from a μ-space X_μ into a β-space Y_β is called (μ, β)-continuous if and only if $U \in \beta$ implies that $f^{-1}(U) \in \mu$. Let $f : X_\mu \to Y_\beta$ be said to be open if image of every μ-open set is β-open set.

Lemma 1.6 ([17])**.** *Let X_μ be μ-space and Y_β be β-space, and $f : X_\mu \to Y_\beta$ be a function. Then, the following are equivalent:*

(1) *f is (μ, β)-continuous;*

(2) *For every $x \in X$ and for every β-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subset V$;*

(3) *$f\big(\mathrm{Cl}_\mu(A)\big) \subset \mathrm{Cl}_\beta\big(f(A)\big)$ for every subset A of X;*

(4) *$\mathrm{Cl}_\mu f^{-1}(B) \subset f^{-1}\big(\mathrm{Cl}_\beta(B)\big)$ for every subset B of Y.*

Lemma 1.7 ([19])**.** *Let $f : X_\mu \to Y_\beta$ be a function. If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H}) = \{f(E) : E \in \mathcal{H}\}$ is a hereditary class on Y.*

Lemma 1.8. *Let $f : X_\mu \to Y_\beta$ be a function. If for each $t \in X$ and $f(t) \in V \in \beta$, there exists $U \in \mu$ containing t such that:*

(1) *$f\big(\mathrm{Cl}_\mu(U)\big) \subseteq V$, then f is said to be strongly $\emptyset(\mu, \beta)$-continuous [20].*

(2) *$f\big(\mathrm{Int}_\mu \mathrm{Cl}_\mu(U)\big) \subseteq V$, then f is said to be super (μ, β)- continuous [20].*

(3) *$f\big(\mathrm{Int}_\mu \mathrm{Cl}_\mu(U)\big) \subseteq \mathrm{Int}_\beta \mathrm{Cl}_\beta(V)$, then f is said to be (δ, δ')- continuous [21].*

(4) *$f(U) \subseteq \mathrm{Int}_\beta \mathrm{Cl}_\beta(V)$, then f is said to be almost (μ, β)- continuous [22].*

2. Nearly $\mu\mathcal{H}$-Lindelöfness

Definition 2.1. Let X_μ be a μ-space. A subset A of X is said to be a $\mathcal{N}\mu\mathcal{H}$--Lindelöf set whenever $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, where $U_\lambda \in \mu$ for all $\lambda \in \Lambda$ and Λ is index set, then there is a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $X \setminus \bigcup_{\lambda \in \Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(U_\lambda) \in \mathcal{H}$.

Theorem 2.2. *If X is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf space, then X is a $\mathcal{W}\mu\mathcal{H}$-Lindelöf space.*

Proof. Suppose μ-space $(X_\mu, \mathcal{H})$ with respect to $\mathcal{H}$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf. Then for every μ-open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$ such that $X \setminus \bigcup_{\lambda \in \Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(U_\lambda) \in \mathcal{H}$. However, $X \setminus \bigcup_{\lambda \in \Lambda_0} \mathrm{Cl}_\mu(U_\lambda) \subseteq X \setminus \bigcup_{\lambda \in \Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(U_\lambda) \in \mathcal{H}$. So, $(X_\mu, \mathcal{H})$ is a $\mathcal{W}\mu\mathcal{H}$-Lindelöf. □

In Example 2.1, we show that the converse of Theorem 2.2 is not always true.

Example 2.1. Let Ψ be the smallest uncountable ordinal number and $A = [0,\Psi)$, see that for each $\alpha \in A$ the set $A = [0,\alpha)$ is countable. Let $X = \{a_{ij}, b_{ij}, c_i, a, b\}$ where $i \in A$ and $j \in \mathbb{N}$, and the generalized topology μ is given by taking $\{a_{ij}\}, \{b_{ij}\}$ are isolated, and the local base of the points $\{c_i\}, \{a\}$ and $\{b\}$ are $B^n_{c_i} = \{c_i, a_{ij}, b_{ij}\}_{i>n}$, $B^\alpha_a = \{a, a_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$ and $B^\alpha_b = \{b, b_{ij}\}_{i\geq\alpha, j\in\mathbb{N}}$, respectively, and $\mathcal{H}_c$ is the set of all countable subsets. Thus, $(X,\mu,\mathcal{H}_c)$ is $\mathcal{W}\mu\mathcal{H}$-Lindelöf, but it is not $\mathcal{N}\mu\mathcal{H}_c$-Lindelöf. For more details, see Example 3.5 of Cammaroto paper [23].

Corollary 2.3. *If X is a $\mu\mathcal{H}$-Lindelöf space, then X is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf space.*

Proof. Suppose $(X_\mu, \mathcal{H})$ is a $\mu\mathcal{H}$-Lindelöf. Then for every μ-open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $X \backslash \bigcup_{\lambda\in\Lambda_0} U_\lambda \in \mathcal{H}$. However, $X\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname{Int}_\mu \operatorname{Cl}_\mu(U_\lambda) \subseteq X\backslash \bigcup_{\lambda\in\Lambda_0} U_\lambda \in \mathcal{H}$. Hence, $(X_\mu,\mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf. □

In Example 2.2, we show that the converse of Corollary 2.3 is not always true.

Example 2.2. Let $X = \mathbb{R}$ and choose $n \in \mathbb{R}$, $\mathcal{B} = \{\{n,t\} : t \in X, a \neq t\}$, and a hereditary class $\mathcal{H} = \{\emptyset, \mathbb{R}\}$. If the $\mathcal{GT}$ $\mu(\mathcal{B})$ is generated on X by the μ-base $\mathcal{B}$. Thus, only $\{X\}$ is μ-regular open cover of itself, so a $\mathcal{GTS}$ $\big(X,\mu(\mathcal{B})\big)$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf since $X\backslash \bigcup(\{X\}) \in \mathcal{H}$. However, it is not $\mu\mathcal{H}$-Lindelöf, since if $\big\{\{1,t\} : t \in X, 1 \neq t\big\}$ is μ-open cover, then there exists a countable sub-collection such that $\big\{\{1,t_n\} : t \in X, n \in \mathbb{N}\big\}$, but $X\backslash \bigcup\big\{\{1,t_n\} : t \in X, n \in \mathbb{N}\big\} \notin \mathcal{H}$.

Theorem 2.4. *Let X_μ be a μ-regular space. Then, the following are equivalent:*

(1) $(X_\mu,\mathcal{H})$ *is $\mathcal{N}\mu\mathcal{H}$-Lindelöf;*

(2) $(X_\mu,\mathcal{H})$ *is $\mu\mathcal{H}$-Lindelöf;*

(3) $(X_\mu,\mathcal{H})$ *is $\mathcal{W}\mu\mathcal{H}$-Lindelöf.*

Proof.

(1) → (2) : Suppose X is μ-regular, $\mathcal{N}\mu\mathcal{H}$-Lindelöf and $\{U_\lambda : \lambda\in\Lambda\}$ are μ-open cover of X. Then for each $x \in X$, there exists $\lambda_x \in \Lambda$ such that $x \in U_{\lambda_x}$. Thus, there exists μ-open set M_x such that $x \in M_x \subset \operatorname{Int}_\mu\big(\operatorname{Cl}_\mu(M_x)\big) \subseteq \operatorname{Cl}_\mu(M_x) \subset U_{\lambda_x}$. Then, the sub-collection $\{M_{x_n} : x \in X\}$ is μ-open cover of X. Since X is $\mathcal{N}\mu\mathcal{H}$- -Lindelöf, so $X\backslash \bigcup_{k\in\mathbb{N}} \operatorname{Int}_\mu \operatorname{Cl}_\mu(M_{x_k}) \in \mathcal{H}$. However,

$$X \Big\backslash \bigcup_{k\in\mathbb{N}} U_{\lambda_{x_k}} \subseteq X \Big\backslash \bigcup_{k\in\mathbb{N}} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(M_{x_k}) \in \mathcal{H}.$$

Thus, $X\backslash \bigcup_{k\in\mathbb{N}} U_{\lambda_{x_k}} \in \mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf.

(2) → (3): $X\backslash\bigcup_{k\in\mathbb{N}} \operatorname{Cl}_\mu(M_{x_k}) \subseteq X\backslash\bigcup_{k\in\mathbb{N}} M_{x_k} \in \mathcal{H}$. Thus, $X\backslash \bigcup_{k\in\mathbb{N}} \operatorname{Cl}_\mu(M_{x_k}) \in \mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mathcal{W}\mu\mathcal{H}$-Lindelöf.

(1) $\to$ (3): Notice that $X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Cl}_\mu(M_{x_k}) \subseteq X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Int}_\mu\,\mathrm{Cl}_\mu(M_{x_k}) \in \mathcal{H}$. Thus, $X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Cl}_\mu(M_{x_k})\in\mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mathcal{W}\mu\mathcal{H}$-Lindelöf.

(2)$\to$(1) : $X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Int}_\mu\,\mathrm{Cl}_\mu(M_{x_k}) \subseteq X\backslash\bigcup_{k\in\mathbb{N}} M_{x_k} \in \mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf.

(3)$\to$(2) : $X\backslash\bigcup_{k\in\mathbb{N}} U_{\lambda_{x_k}} \subseteq X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Cl}_\mu(M_{x_k}) \in \mathcal{H}$. Thus, $X\backslash\bigcup_{k\in\mathbb{N}} U_{\lambda_{x_k}} \in \mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mu\mathcal{H}$-Lindelöf.

(3)$\to$(1) : It is clear that $X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Int}_\mu\,\mathrm{Cl}_\mu(U_{\lambda_{x_k}}) \subseteq X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Cl}_\mu((M_{x_k}) \in \mathcal{H}$. Thus, $X\backslash\bigcup_{k\in\mathbb{N}}\mathrm{Int}_\mu\,\mathrm{Cl}_\mu(U_{\lambda_{x_k}}) \in \mathcal{H}$. That means, $(X_\mu,\mathcal{H})$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf. □

Theorem 2.5. *Let X_μ be a μ-space. Then, the following are equivalent:*

1. *$(X_\mu,\mathcal{H})$ is nearly $\mu\mathcal{H}$-Lindelöf;*
2. *For any set $\{U_\lambda : \lambda\in\Lambda\}$ of μ-closed subsets of X such that $\bigcap_{\lambda\in\Lambda} U_\lambda = \emptyset$, there exists countable sub-set $\{U_\lambda : \lambda\in\Lambda_0\in\Lambda\}$ such that*
$$\bigcap_{\lambda\in\Lambda_0}\mathop{\mathrm{Cl}}_\mu\mathop{\mathrm{Int}}_\mu(U_\lambda)\in\mathcal{H};$$
3. *For any collection $\{U_\lambda : \lambda\in\Lambda\}$ of μ-regular closed subsets of X such that $\bigcap_{\alpha\in\Lambda} U_\lambda = \emptyset$, there exists countable sub-collection $\{U_\lambda : \lambda\in\Lambda_0\in\Lambda\}$ such that*
$$\bigcap_{\lambda\in\Lambda_0}\mathop{\mathrm{Cl}}_\mu\mathop{\mathrm{Int}}_\mu(U_\lambda)\in\mathcal{H}.$$

Proof.

(1) $\Rightarrow$ (2) : Let $\{U_\lambda : \lambda\in\Lambda\}$ be a collection of μ-closed sets of X such that $\bigcap_{\alpha\in\Lambda} U_\lambda = \emptyset$. Then, $\{X\backslash U_\lambda : \lambda\in\Lambda\}$ is a μ-open cover of X. Since $(X_\mu,\mathcal{H})$ is nearly $\mu\mathcal{H}$-Lindelöf, there exists a countable sub-collection $\{U_\lambda : \lambda\in\Lambda_0\in\Lambda\}$ such that $X\backslash\bigcup_{\lambda\in\Lambda_0}\mathrm{Int}_\mu\,\mathrm{Cl}_\mu\,(X\backslash U_\lambda)\in\mathcal{H}$. So, we get the following

$$\begin{aligned}
X\Big\backslash\bigcup_{\lambda\in\Lambda_0}\mathop{\mathrm{Int}}_\mu\mathop{\mathrm{Cl}}_\mu(X\backslash U_\lambda) &= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash\mathop{\mathrm{Int}}_\mu\mathop{\mathrm{Cl}}_\mu(X\backslash U_\lambda)\\
&= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash\mathop{\mathrm{Int}}_\mu\Big(X\Big\backslash\mathop{\mathrm{Int}}_\mu(U_\lambda)\Big)\\
&= \bigcap_{\lambda\in\Lambda_0}\mathop{\mathrm{Cl}}_\mu\mathop{\mathrm{Int}}_\mu(U_\lambda)\in\mathcal{H}
\end{aligned}$$

(1) $\Rightarrow$ (3) : Let $\{U_\lambda : \lambda\in\Lambda\}$ be a collection of μ-regular closed sets of X such that $\bigcap_{\alpha\in\Lambda} U_\lambda = \emptyset$. Then,
$$\Big\{X\Big\backslash\mathop{\mathrm{Int}}_\mu\mathop{\mathrm{Cl}}_\mu(U_\lambda) : \lambda\in\Lambda\Big\}$$

is a μ-regular open cover of X. Since $(X_\mu, \mathcal{H})$ is nearly $\mu\mathcal{H}$-Lindelöf, there exists a countable sub-collection

$$\left\{X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) : \lambda \in \Lambda_0 \in \Lambda\right\}$$

such that

$$X \Big\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu \left(X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\right) \in \mathcal{H}.$$

Then, we get

$$\begin{aligned}
X \Big\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu \left(X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\right) &= \bigcap_{\lambda\in\Lambda_0} X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu \left(X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\right) \\
&= \bigcap_{\lambda\in\Lambda_0} X\backslash X\backslash \operatorname{Cl}\left(X \Big\backslash \operatorname*{Cl}_\mu\Big(X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\Big)\right) \\
&= \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu(X \Big\backslash \operatorname*{Cl}_\mu\left(X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\right) \\
&= \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu(X\backslash X \Big\backslash \operatorname*{Int}_\mu \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) \\
&= \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) \\
&= \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) \in \mathcal{H}.
\end{aligned}$$

However,

$$\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(U_\lambda) \subseteq \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda).$$

Hence,

$$\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(U_\lambda) \in \mathcal{H}.$$

(3) $\Rightarrow$ (1): Let $\{\operatorname{Int}_\mu \operatorname{Cl}_\mu(U_\lambda) : \lambda \in \Lambda\}$ be a cover of X by μ-regular open sets. Now, the set $\{X\backslash \operatorname{Int}_\mu \operatorname{Cl}_\mu (U_\lambda) : \lambda \in \Lambda\}$ is a collection of μ-regular closed sets and $\bigcap_{\lambda\in\Lambda} X\backslash \operatorname{Int}_\mu \operatorname{Cl}_\mu (U_\lambda) = \bigcap_{\lambda\in\Lambda} \operatorname{Cl}_\mu \operatorname{Int}_\mu (X\backslash U_\lambda) = \emptyset$. Thus by the above assumption, there exists a countable subset $\{\operatorname{Int}_\mu \operatorname{Cl}_\mu (U_\lambda)) : \lambda \in \Lambda\}$ such that $\bigcap_{\lambda\in\Lambda_0} \operatorname{Cl}_\mu \operatorname{Int}_\mu (\operatorname{Cl}_\mu \operatorname{Int}_\mu (X\backslash U_\alpha)) \in \mathcal{H}$. Thus, we have the following

$$\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu (X\backslash U_\lambda) \subseteq \bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu \left(\operatorname*{Cl}_\mu \operatorname*{Int}_\mu (X\backslash U_\lambda)\right),$$

and hence, we get

$$\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu (X\backslash U_\lambda) = X \Big\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu (U_\lambda) \in \mathcal{H},$$

which means that $\mathcal{N}\mu\mathcal{H}$-Lindelöf.

(2) $\Leftrightarrow$ (3): It is obvious since μ-regular closed is μ-closed.

(2) $\Rightarrow$ (1): It is similar to (3) $\Rightarrow$ (1) : since μ-regular closed is μ-closed. $\square$

Theorem 2.6. *Let $(X_\mu, \mathcal{H})$ be a μ-space with respect to $\mathcal{H}$. The pair $(X_\mu, \mathcal{H})$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf space if and only if for any collection $\{U_\lambda : \lambda \in \Lambda\}$ of μ-regular closed sets of X having the property that $\bigcap_{\lambda\in\Lambda_0} \mathrm{Cl}_\mu \mathrm{Int}_\mu(U_\lambda) \notin \mathcal{H}$ for every countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$, then $\bigcap_{\lambda\in\Lambda} U_\lambda \neq \emptyset$.*

P r o o f. Assume that $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf space and $\{U_\lambda : \lambda \in \Lambda\}$ is any collection of μ-closed sets of X having the property that $\bigcap_{\lambda\in\Lambda_0} \mathrm{Cl}_\mu \mathrm{Int}_\mu(U_\lambda) \notin \mathcal{H}$. Now, if $\bigcap_{\lambda\in\Lambda} U_\lambda = \emptyset$, then $\{X\backslash U_\lambda : \lambda \in \Lambda\}$ is a μ-open cover of X. Since $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf space, then $X\backslash \bigcup_{\lambda\in\Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(X\backslash U_\lambda) \in \mathcal{H}$. Then,

$$\begin{aligned} X\Big\backslash & \bigcup_{\lambda\in\Lambda_0} \mathop{\mathrm{Int}}_\mu \mathop{\mathrm{Cl}}_\mu(X\backslash U_\lambda) \\ &= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash \mathop{\mathrm{Int}}_\mu \mathop{\mathrm{Cl}}_\mu(X\backslash U_\lambda) \\ &= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash \mathop{\mathrm{Int}}_\mu\Big(X\Big\backslash \mathop{\mathrm{Int}}_\mu(U_\lambda)\Big) \\ &= \bigcap_{\lambda\in\Lambda_0} \mathop{\mathrm{Cl}}_\mu \mathop{\mathrm{Int}}_\mu(U_\lambda) \in \mathcal{H}, \quad \text{but} \quad \bigcap_{\lambda\in\Lambda_0} \mathop{\mathrm{Cl}}_\mu \mathop{\mathrm{Int}}_\mu(U_\lambda) \notin \mathcal{H}, \end{aligned}$$

which contradicts the assumption. Thus,

$$\bigcap_{\lambda\in\Lambda} U_\lambda \neq \emptyset.$$

Conversely, let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ-open cover of X. Assume that for any countable sub-collection $\{U_\lambda : \lambda\in\Lambda_0 \in \Lambda\}$ we have $X\backslash \bigcup_{\lambda\in\Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(U_\lambda) \notin \mathcal{H}$. Then,

$$\begin{aligned} X\Big\backslash \bigcup_{\lambda\in\Lambda_0} \mathop{\mathrm{Int}}_\mu \mathop{\mathrm{Cl}}_\mu(U_\lambda) &= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash \mathop{\mathrm{Int}}_\mu \mathop{\mathrm{Cl}}_\mu(U_\lambda) \\ &= \bigcap_{\lambda\in\Lambda_0} X\Big\backslash \mathop{\mathrm{Int}}_\mu\Big(X\Big\backslash \mathop{\mathrm{Int}}_\mu(X\backslash U_\lambda)\Big) \\ &= \bigcap_{\lambda\in\Lambda_0} \mathop{\mathrm{Cl}}_\mu \mathop{\mathrm{Int}}_\mu(X\backslash U_\lambda) \notin \mathcal{H}. \end{aligned}$$

However, $\{X\backslash U_\lambda : \lambda \in \Lambda\}$ is a collection of μ-closed subsets of X, and by the assumption $\{X\backslash U_\lambda : \lambda \in \Lambda\} \neq \emptyset$, this is a contradiction to the fact that $\{U_\lambda : \lambda \in \Lambda\}$ is a cover of X. Thus, $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-lindelöf space. $\square$

Theorem 2.7. *Let A be a subset of X_μ. The following statements are equivalent:*

(1) *A is $\mathcal{N}\mu\mathcal{H}$-Lindelöf;*

(2) *For any collection $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$ of a μ-closed subset of X such that $[\bigcap\{U_\lambda : \lambda \in \Lambda\}] \cap A = \emptyset$, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ of $\mathcal{F}$ such that*

$$\left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(U_\lambda)\right] \cap A \in \mathcal{H};$$

(3) *For any collection $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$ of μ-regular closed subsets of X such that $[\bigcap\{U_\lambda : \lambda \in \Lambda\}] \cap A = \emptyset$, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ of $\mathcal{F}$ such that*

$$\left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(U_\lambda)\right] \bigcap A \in \mathcal{H}.$$

Proof. (1) $\Rightarrow$ (2) : Suppose A is $\mathcal{N}\mu\mathcal{H}$-Lindelöf set and $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$ is a μ-closed collection of X such that $\bigcap\{U_\lambda : \lambda \in \Lambda\} \cap A = \emptyset$. Then, $A \subseteq X\backslash \cap \mathcal{F} = \bigcup X\backslash\mathcal{F}$. Since A is $\mathcal{N}\mu\mathcal{H}$-Lindelöf, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ cover of A such that $\{X\backslash U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$. Thus, $A\backslash \bigcup_{\lambda\in\Lambda_0} \mathrm{Int}_\mu \mathrm{Cl}_\mu(X\backslash U_\lambda)$. Hence,

$$\begin{aligned} A\Big\backslash \bigcup_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda) &= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda) \\ &= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}\Big(X\Big\backslash \underset{\mu}{\mathrm{Int}}(U_\lambda)\Big) \\ &= \bigcap_{\lambda\in\Lambda_0} A\backslash X\Big\backslash \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,(U_\lambda) \in \mathcal{H}. \end{aligned}$$

It is clear that

$$\left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(U_\lambda)\right] \cap A = \bigcap_{\lambda\in\Lambda_0} A\backslash X\Big\backslash \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,(U_\lambda) \in \mathcal{H}.$$

(2) $\Rightarrow$ (1): Suppose $A \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$, where $U_\lambda \in \mu$ for all $\lambda \in \Lambda$ and Λ is index set. Thus, $\{X\backslash U_\lambda : \lambda \in \Lambda\}$ is a μ-closed subset of X. By the assumption that $X\backslash \bigcup_{\lambda\in\Lambda}(U_\lambda) \cap A = \bigcap_{\lambda\in\Lambda}(X\backslash U_\lambda) \cap A = \emptyset$, so, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$ of $\mathcal{F}$ such that

$$\underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\left(\bigcap_{\lambda\in\Lambda_0} (X\backslash U_\lambda)\right) \cap A \in \mathcal{H}.$$

Hence,

$$\left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\right]\cap A = \bigcap_{\lambda\in\Lambda_0} A\backslash X\Big\backslash \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,(X\backslash U_\lambda)$$
$$= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}\Big(X\Big\backslash \underset{\mu}{\mathrm{Int}}(U_\lambda)\Big)$$
$$= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda)$$
$$= A\Big\backslash \bigcup_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda)\in\mathcal{H}.$$

Therefore, A is $\mathcal{N}\mu\mathcal{H}$-Lindelöf set.

(3) $\Rightarrow$ (1): Suppose $A\subseteq\bigcup_{\lambda\in\Lambda}U_\lambda$, where $U_\lambda\in\mu$ for all $\lambda\in\Lambda$ and Λ is index set, so, $U_\lambda\subseteq\mathrm{Int}_\mu\big(\mathrm{Cl}_\mu(U_\lambda)\big)$. Thus, $\{X\backslash\,\mathrm{Int}_\mu\,\mathrm{Cl}_\mu(U_\lambda):\lambda\in\Lambda\}$ is a μ-regular closed subset of X. From the assumption, we have the following

$$X\Big\backslash \bigcup_{\lambda\in\Lambda} \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}(U_\lambda)\cap A = \bigcap_{\lambda\in\Lambda} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\cap A=\emptyset,$$

so, there exists a countable sub-collection $\{U_\lambda:\lambda\in\Lambda_0\subseteq\Lambda\}$ of $\mathcal{F}$ such that

$$\left[\underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\left(\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\right)\right]\cap A = \left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\right]\cap A\in\mathcal{H}.$$

Hence,

$$\left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\right]\cap A \supseteq \left[\bigcap_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}(X\backslash U_\lambda)\right]\cap A$$
$$= \bigcap_{\lambda\in\Lambda_0} A\backslash X\Big\backslash \underset{\mu}{\mathrm{Cl}}\,\underset{\mu}{\mathrm{Int}}\,(X\backslash U_\lambda)$$
$$= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}(X\Big\backslash \underset{\mu}{\mathrm{Int}}(U_\lambda)$$
$$= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda)$$
$$= A\Big\backslash \bigcup_{\lambda\in\Lambda_0} \underset{\mu}{\mathrm{Int}}\,\underset{\mu}{\mathrm{Cl}}\,(X\backslash U_\lambda)\in\mathcal{H}.$$

It is clear that A is $\mathcal{N}\mu\mathcal{H}$-Lindelöf set.

(2) $\Leftrightarrow$ (3): It is obvious, since μ-regular closed is μ-closed.

(1) $\Rightarrow$ (3): It is similar to (1) $\Rightarrow$ (2), since μ-regular closed is μ-closed. $\square$

Theorem 2.8. *Let $(X_\mu, \mathcal{H})$ be a μ-space with respect to $\mathcal{H}$. The pair $(X_\mu, \mathcal{H})$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf space if and only if for any family $\{U_\lambda : \lambda \in \Lambda\}$ of μ-regular closed sets of X having the property that*

$$\Big(\bigcap_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\Big) \cap A \notin \mathcal{H}$$

for every countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \subseteq \Lambda\}$, then $(\bigcap_{\lambda\in\Lambda} U_\lambda) \cap A \neq \emptyset$.

Proof. Assume that A is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf set and $\{U_\lambda : \lambda \in \Lambda\}$ is any collection of μ-closed sets of X having the property that

$$\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(U_\lambda) \notin \mathcal{H}.$$

Now, if $\bigcap_{\lambda\in\Lambda} U_\lambda = \emptyset$, then $\{X\backslash U_\lambda : \lambda \in \Lambda\}$ is a μ-open cover of X. Since A is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf set, then $A\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname{Int}_\mu \operatorname{Cl}_\mu(X\backslash U_\lambda) \in \mathcal{H}$. Then, we get

$$\begin{aligned} A\Big\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(X\backslash U_\lambda) &= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(X\backslash U_\lambda)\\ &= \bigcap_{\lambda\in\Lambda_0} A\backslash \operatorname*{Int}_\mu\Big(X\Big\backslash \operatorname*{Int}_\mu(U_\lambda)\Big)\\ &= \Big[\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(U_\lambda)\Big] \cap A \in \mathcal{H}, \text{ but } \Big[\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(U_\lambda)\Big] \cap A \notin \mathcal{H}, \end{aligned}$$

which contradicts the assumption. Thus,

$$\bigcap_{\lambda\in\Lambda} U_\lambda \neq \emptyset.$$

Conversely, let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ-open cover of A. Assume that for any countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$ we have $A\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname{Int}_\mu \operatorname{Cl}_\mu(U_\lambda) \notin \mathcal{H}$. Then,

$$\begin{aligned} A\Big\backslash \bigcup_{\lambda\in\Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) &= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda)\\ &= \bigcap_{\lambda\in\Lambda_0} A\Big\backslash \operatorname*{Int}_\mu\Big(X\Big\backslash \operatorname*{Int}_\mu(X\backslash U_\lambda)\Big)\\ &= \Big[\bigcap_{\lambda\in\Lambda_0} \operatorname*{Cl}_\mu \operatorname*{Int}_\mu(X\backslash U_\lambda)\Big] \cap A \notin \mathcal{H}. \end{aligned}$$

However, $\{X \backslash U_\lambda : \lambda \in \Lambda\}$ is a collection of μ-closed subsets of X, and by the assumption $\{X \backslash U_\lambda : \lambda \in \Lambda\} \neq \emptyset$, this is a contradiction to the fact that $\{U_\lambda : \lambda \in \Lambda\}$ is a cover of A. Thus, A is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf set. $\square$

Theorem 2.9. *If a μ-space X_μ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then for every cover of X by μ_θ-open sets $\{U_\lambda : \lambda \in \Lambda\}$ there exists a countable sub-collection*

$$\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$$

such that

$$X \Big\backslash \bigcup_{\lambda \in \Lambda_0} U_\lambda \in \mathcal{H}.$$

Proof. Suppose $(X_\mu, \mathcal{H})$ is $\mathcal{N}\mu\mathcal{H}$-Lindelöf and let $\mathcal{F} = \{U_\lambda : \lambda \in \Lambda\}$ be μ_θ-open cover of X. Then for each $x \in X$, there exists $\lambda_x \in \Lambda$ such that $x \in U_{\lambda_x}$. Thus, there exists a μ-open set M_x such that

$$x \in M_x \subset \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(M_x) \subseteq \operatorname*{Cl}_\mu(M_x) \subset U_{\lambda_x}.$$

Then, the sub-collection $\{M_{x_n} : x \in X\}$ is a countable μ-open cover of X. Since, X is $\mathcal{N}\mu\mathcal{H}$-Lindelöf

$$X \Big\backslash \bigcup_{k \in \mathbb{N}} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(M_{x_n}) \in \mathcal{H}.$$

However,

$$X \Big\backslash \bigcup_{k \in \mathbb{N}} (U_{\lambda_x}) \subseteq X \Big\backslash \bigcup_{k \in \mathbb{N}} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(M_x) \in \mathcal{H}.$$

Hence,

$$X \Big\backslash \bigcup_{k \in \mathbb{N}} (U_{\lambda_x}) \in \mathcal{H}.$$ $\square$

Theorem 2.10. *Let μ-spaceX_μ be $\mathcal{N}\mu$-Lindelöf if and only if $(X, \mu, \mathcal{H}_c)$ is $\mathcal{N}\mu\mathcal{H}_c$-Lindelöf.*

Proof.

$\Rightarrow$ It is straightforward and therefore omitted.

$\Leftarrow$ Suppose $(X, \mu, \mathcal{H}_c)$ is $\mathcal{N}\mu\mathcal{H}_c$-Lindelöf. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a μ-open cover of X. Then, there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$ such that

$$X \Big\backslash \bigcup_{\lambda \in \Lambda_0} \operatorname*{Int}_\mu \operatorname*{Cl}_\mu(U_\lambda) \in \mathcal{H}_c.$$

Suppose $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$ such that $X \setminus \bigcup \lambda \in \Lambda_0 \operatorname{Int}_\mu \operatorname{Cl}_\mu(U_{\lambda_k}) = \{x_i : i \in \mathbb{N}\}$, choose U_{λ_i} such that $x_i \in U_{\lambda_i}$. Thus,

$$X = \operatorname*{Int}_{\mu} \operatorname*{Cl}_{\mu}(U_\lambda) \cup \left(\bigcup_{\lambda \in \Lambda_0} \operatorname*{Int}_{\mu} \operatorname*{Cl}_{\mu}(U_{\lambda_i}) \right).$$

It is clear that X is $\mathcal{N}\mu$-Lindelöf. □

The following figure is a diagram showing the relationship between most types of generalizations of spaces regarding Lindelöfness in generalized topology. Moreover, these are some counterexamples to the diagram $\mu\mathcal{H}$-Lindelöf $\not\to$ μ-Lindelöf [16, Ex 3.5], $\mathcal{N}\mu$-Lindelöf $\not\to$ μ-Lindelöf [24, Ex 2.1], $\mathcal{N}\mu\mathcal{H}$-Lindelöf $\not\to$ $\mu\mathcal{H}$-Lindelöf Ex 2.2, $\mathcal{W}\mu\mathcal{H}$-Lindelöf $\not\to$ $\mathcal{N}\mu\mathcal{H}$-Lindelöf Ex 2.1, and $\mathcal{W}\mu$-Lindelöf $\not\to$ $\mathcal{N}\mu$-Lindelöf [23, Ex 3.5].

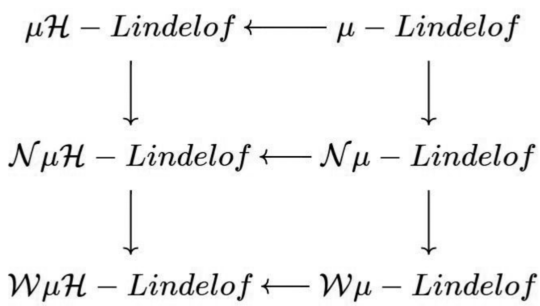

Figure 1. The relationship between most types of generalization of μ-Lindelöf spaces.

3. Function properties on $\mathcal{N}\mu$-Lindelöfness

Theorem 3.1. *Let $f : (X_\mu, \mathcal{H}) \to Y_\beta$ be a (μ, β)-continuous surjective, and $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then Y_β is $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf.*

Proof. Suppose $f(X) = \bigcup_{\lambda \in \Lambda} V_\lambda$, where $V_\lambda \in \beta$ for all $\lambda \in \Lambda$, and Λ is index set. Since f is (μ, β)-continuous, $X = \bigcup_{\lambda \in \Lambda} f^{-1}(V_\lambda)$, where $f^{-1}(V_\lambda) \in \mu$ for all $\lambda \in \Lambda$ and Λ is index and X is $\mathcal{N}\mu\mathcal{H}$-Lindelöf. Thus, there exist $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \in \Lambda$, where $X \setminus \bigcup_{k \in \mathbb{N}} \operatorname{Int}_\mu \operatorname{Cl}_\mu\big(f^{-1}(V_{\lambda_k})\big) \in \mathcal{H}$. Since f is (μ, β)-continuous and the $\operatorname{Int}_\mu\big(\operatorname{Cl}_\mu(f^{-1}(B))\big) \subset f^{-1}\big(\operatorname{Int}_\beta(\operatorname{Cl}_\beta(B))\big)$ for all $B \subseteq Y$, then we have that

$$X \Big\backslash \bigcup_{k \in \mathbb{N}} \Big(f^{-1}\Big(\operatorname*{Int}_{\beta}\big(\operatorname*{Cl}_{\beta}(V_{\lambda_k})\big)\Big) \subset X \Big\backslash \bigcup_{k \in \mathbb{N}} \operatorname*{Int}_{\mu}\Big(\operatorname*{Cl}_{\mu}\big(f^{-1}(V_k)\big)\Big)\Big) \in \mathcal{H}.$$

Since,

$$f\Big(\operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\big(f^{-1}(V_{\lambda_k})\big)\Big) \subset \operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta} f\big(f^{-1}(V_{\lambda_k})\big) \subset \operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k}).$$

Thus,

$$f(X)\setminus \bigcup_{k\in\mathbb{N}} \operatorname*{Int}_{\beta}\Big(\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big) \in f(\mathcal{H}).$$

Since f is surjective, then $f(X) = Y$. This means, Y is $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf. □

Theorem 3.2. *Let $f : (Y_\beta, \mathcal{H}) \to X_\mu$ be a (μ, β)-open bijective function and $\mathcal{N}\beta\mathcal{H}$-Lindelöf, then X_μ is $\mathcal{N}\mu f^{-1}(\mathcal{H})$-Lindelöf.*

Proof. Since $f : X_\mu \to (Y_\beta, \mathcal{H})$ is a (μ, β)-open bijective, then $f^{-1} : (Y_\beta, \mathcal{H}) \to X_\mu$ is a (β, μ)-continuous surjective. So, $(Y_\beta, \mathcal{H})$ is a $\mathcal{N}\beta\mathcal{H}$-Lindelöf, then X_μ is $\mathcal{N}\mu f^{-1}(\mathcal{H})$-Lindelöf. □

Theorem 3.3. *Let $f : (X_\mu, \mathcal{H}) \to Y_\beta$ be strongly $\emptyset(\mu, \beta)$-continuous surjective, and $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then Y_β is also $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf.*

Proof. Suppose $Y = \bigcup_{\lambda\in\Lambda} V_\lambda$, where $V_\lambda \in \beta$ for all $\lambda \in \Lambda$ and Λ is index set. Since f is a strongly $\emptyset(\mu, \beta)$-continuous, then $f^{-1}\big(\mathrm{Cl}_\beta(V_\lambda)\big) \in \mu$. Thus,

$$X = \bigcup_{\lambda\in\Lambda} f^{-1}\big(\operatorname*{Cl}_{\beta}(V_\lambda)\big) \quad \text{for all} \quad \lambda \in \Lambda$$

is the index set, then there exist

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \Lambda \quad \text{where} \quad X\Big\backslash \operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\Bigg(\bigcup_{k\in\mathbb{N}} f^{-1}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Bigg) \in \mathcal{H}.$$

Since

$$\operatorname*{Cl}_{\mu} f^{-1}(V_{\lambda_k}) \subset f^{-1}\operatorname*{Cl}_{\beta}(V_{\lambda_k}),$$

then

$$X\Big\backslash \bigcup_{k\in\mathbb{N}} f^{-1}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big) \subseteq X\Big\backslash \operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\Bigg(f^{-1}\Bigg(\bigcup_{k\in\mathbb{N}}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Bigg)\Bigg) \in \mathcal{H}.$$

Thus,

$$X\Big\backslash \bigcup_{k\in\mathbb{N}} f^{-1}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big) \in \mathcal{H},$$

it is clear that

$$f\left(X\Big\backslash\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\right)\right)\right)\right)$$

$$= f(X)\Big\backslash\left(f\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\right)\right)\right)\right)$$

$$= f(X)\Big\backslash\left(\bigcup_{k\in\mathbb{N}}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\right)\right) \in f(\mathcal{H}).$$

Hence, Y is a $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf. □

Theorem 3.4. *Let $f : (X_\mu, \mathcal{H}) \to Y_\beta$ be (δ, δ')-continuous surjective, and $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then Y_β is also $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf.*

Theorem 3.5. *Let $f : (X_\mu, \mathcal{H}) \to Y_\beta$ be super (μ, β)- continuous surjective, and $(X_\mu, \mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then Y_β is also $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf.*

Proof. Suppose $Y = \bigcup_{\lambda\in\Lambda} V_\lambda$, where $V_\lambda \in \beta$ for all $\lambda \in \Lambda$ and Λ is the index set. Since f is an super (μ, β)-continuous, then $f^{-1}(V_\lambda) \in \mu$. Thus,

$$X = \bigcup_{\lambda\in\Lambda} f^{-1}(V_\lambda) \quad \text{for all} \quad \lambda \in \Lambda$$

is an index set, then there exist values of lambdas

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \Lambda, \quad \text{where} \quad X\Big\backslash \operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\left(\bigcup_{k\in\mathbb{N}} f^{-1}(V_{\lambda_k})\right) \in \mathcal{H}.$$

Since

$$\operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\big(f^{-1}(V_{\lambda_k})\big) \subset \Big(f^{-1}\big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\big)\Big),$$

then we have the following

$$X\Big\backslash\bigcup_{k\in\mathbb{N}} f^{-1}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}\left(\bigcup_{k\in\mathbb{N}}(V_{\lambda_k})\right)\right) \subseteq X\Big\backslash \operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}(V_{\lambda_k})\right)\right) \in \mathcal{H}.$$

Thus,

$$X\Big\backslash\bigcup_{k\in\mathbb{N}} f^{-1}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}\big((V_{\lambda_k})\big)\right) \in \mathcal{H}.$$

It is clear that

$$f\left(X\Big\backslash\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big)\right)\right)\right)=f(X)\Big\backslash\left(f\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big)\right)\right)\right)$$

$$=f(X)\Big\backslash\bigcup_{k\in\mathbb{N}}\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\in f(\mathcal{H}).$$

Hence, Y is a $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf. □

Theorem 3.6. *Let $f:(X_\mu,\mathcal{H})\to Y_\beta$ be almost (μ,β)- continuous surjective, and $(X_\mu,\mathcal{H})$ is a $\mathcal{N}\mu\mathcal{H}$-Lindelöf, then Y_β is also $\mathcal{N}\beta f(\mathcal{H})$-Lindelöf.*

Proof. Suppose $Y=\bigcup_{\lambda\in\Lambda}V_\lambda$, where $V_\lambda\in\beta$ for all $\lambda\in\Lambda$ and Λ is the index set. Since f is an almost (μ,β)-continuous, then $f^{-1}\big(\mathrm{Int}_\beta\,\mathrm{Cl}_\beta(V_\lambda)\big)\in\mu$. Thus,

$$X=\bigcup_{\lambda\in\Lambda}f^{-1}\big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_\lambda)\big)\quad\text{for all}\quad\lambda\in\Lambda$$

is a index set, then there exist

$$\lambda_1,\lambda_2,\dots,\lambda_n,\dots\in\Lambda\quad\text{where}\quad X\Big\backslash\operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\left(\bigcup_{k\in\mathbb{N}}f^{-1}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\right)\right)\in\mathcal{H}.$$

Since

$$\operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\big(f^{-1}(V_{\lambda_k})\big)\subset\left(f^{-1}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big)\right),$$

then

$$X\Big\backslash\bigcup_{k\in\mathbb{N}}f^{-1}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}\left(\bigcup_{k\in\mathbb{N}}(V_{\lambda_k})\right)\right)\subseteq X\Big\backslash\operatorname*{Int}_{\mu}\operatorname*{Cl}_{\mu}\left(f^{-1}\left(\bigcup_{k\in\mathbb{N}}\Big(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}(V_{\lambda_k})\Big)\right)\right)\in\mathcal{H}.$$

Thus,

$$X\Big\backslash\bigcup_{k\in\mathbb{N}}f^{-1}\left(\operatorname*{Int}_{\beta}\operatorname*{Cl}_{\beta}\big((V_{\lambda_k})\big)\right)\in\mathcal{H},$$ □

4. Applications in soft set theory

As an application in soft set theory, the concepts soft μ-Lindelöf, soft $\mathcal{N}\mu$- -Lindelöf and soft $\mathcal{W}\mu$-Lindelöf have been defined as more general setting of soft generalized topological spaces.

Definition 4.1. [25] Let G be a set of all parameters and X be a universal set. The $\mathcal{S}_{\mathcal{A}}$ belongs to the set of all soft sets $\mathcal{S}(X)$, is defined by $\mathcal{S}_{\mathcal{A}} = \{(t, f_{\mathcal{A}}(t)) : t \in G, f_{\mathcal{A}}(t) \in 2^X\}$ where $A \subseteq G$ and $f_{\mathcal{A}}(t) = \emptyset$ if $t \notin \mathcal{A}$.

Definition 4.2. Let $\mathcal{S}_{\mathcal{A}} \in \mathcal{S}(X)$.

(1) If $f_{\mathcal{A}}(t) = X$ for each $t \in G$, then $\mathcal{S}_{\mathcal{A}}$ is said to be an A-universal soft set, denoted by $\mathcal{S}_{\hat{\mathcal{A}}}$. If $\mathcal{A} = G$, then $\mathcal{S}_{\hat{\mathcal{A}}}$ is said to be a universal soft set, denoted by $\mathcal{S}_{\hat{G}}$ [26].

(2) If $f_{\mathcal{A}}(t) = \emptyset$ for each $t \in G$, then $\mathcal{S}_{\mathcal{A}}$ is said to be an empty soft set, defined by $\mathcal{S}_{\emptyset}$ [26].

(3) The soft complement of $\mathcal{S}_{\mathcal{A}}$, denoted by $X \backslash \mathcal{S}_{\mathcal{A}}$, is defined by the approximate function $f_{X \backslash \mathcal{A}}(t) = X \backslash f_{\mathcal{A}}(t)$, where $X \backslash f_{\mathcal{A}}(t)$ is the complement of the set $f_{\mathcal{A}}(t)$ for all $t \in G$ [27].

Definition 4.3. Let $\mathcal{S}_{\mathcal{A}}$, $\mathcal{S}_{\mathcal{B}} \in \mathcal{S}(X)$.

(1) $\mathcal{S}_{\mathcal{B}}$ is a soft subset of $\mathcal{S}_{\mathcal{A}}$, denoted by $\mathcal{S}_{\mathcal{B}} \subseteq \mathcal{S}_{\mathcal{A}}$, if $f_{\mathcal{A}}(t) \subseteq f_{\mathcal{B}}(t)$ for all $t \in G$ [28].

(2) The soft union of $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$, denoted by $\mathcal{S}_{\mathcal{A}} \cup \mathcal{S}_{\mathcal{B}}$, is defined by the approximate function $f_{\mathcal{A} \cup \mathcal{B}}(t) = f_{\mathcal{A}}(t) \cup f_{\mathcal{B}}(t)$ [26].

(3) The soft intersection of $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{B}}$, denoted by $\mathcal{S}_{\mathcal{A}} \cap \mathcal{S}_{\mathcal{B}}$, is defined by the approximate function $f_{\mathcal{A} \cap \mathcal{B}}(t) = f_{\mathcal{A}}(t) \cap f_{\mathcal{B}}(t)$ [27].

Definition 4.4 ([29])**.** Let $\mathcal{S}_{\mathcal{A}} \in \mathcal{S}(X)$. A soft generalized topology (sGT, briefly) on S_A, denoted by $\mathcal{S}_{\mathcal{A}\mu}$, is a family of soft subsets of $\mathcal{S}_{\mathcal{A}}$ such that $S_{\emptyset} \in \mu$ and if a family $\{\mathcal{S}_{\mathcal{A}_\rangle} : \mathcal{S}_{\mathcal{A}i} \subseteq \mathcal{S}_{\mathcal{A}}, i \in J \subseteq \mathbb{N}\} \subseteq \mu$, then $\bigcup_{i \in J}(\mathcal{S}_{\mathcal{A}i}) \in \mu$.

Definition 4.5 ([29])**.** Let (S_A, μ) be an sGTS. Every element of μ is called a soft μ-open set. The $S_{\emptyset}$ is a soft μ-open set. If $\mathcal{S}_{\mathcal{B}}$ is a soft set, then $\mathcal{S}_{\mathcal{B}}$ is called soft μ-closed if its soft complement $X \backslash \mathcal{S}_{\mathcal{B}}$ is a soft μ-open.

Definition 4.6 ([29])**.** Let (S_A, μ) be an sGTS and $\mathcal{S}_{\mathcal{B}} \subseteq \mathcal{S}_{\mathcal{A}}$, then

(a) the soft union of all soft μ-open subsets of $\mathcal{S}_{\mathcal{B}}$ is said to be soft μ-interior of $\mathcal{S}_{\mathcal{B}}$ and is denoted by $\mathrm{Int}_{\mathcal{S}_{\mathcal{A}}\mu}\, \mathcal{S}_{\mathcal{B}}$.

(b) the soft intersection of all soft μ-closed super sets of $\mathcal{S}_{\mathcal{B}}$ is said to be soft μ-closure of $\mathcal{S}_{\mathcal{B}}$ and is denoted by $\mathrm{Cl}_{\mathcal{S}_{\mathcal{A}}\mu}\, \mathcal{S}_{\mathcal{B}}$.

Definition 4.7. Let (S_A, μ) be an sGTS and $\mathcal{S_B} \subseteq \mathcal{S_A}$, then

(1) the soft μ-regular open set if and only if $\mathcal{S_B} = \mathrm{Int}_{\mathcal{S_A}\mu} \mathrm{Cl}_{\mathcal{S_A}\mu}(\mathcal{S_B})$.

(2) the soft μ-regular closed set if and only if $\mathcal{S_B} = \mathrm{Cl}_{\mathcal{S_A}\mu} \mathrm{Int}_{\mathcal{S_A}\mu}(\mathcal{S_B})$.

Definition 4.8. An sGTS $(\mathcal{S_A}, \mu)$ is called soft μ-Lindelof whenever $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_{\mathcal{A}_\lambda}$, where $\mathcal{S}_{\mathcal{A}_\lambda}$ is soft μ-open for all $\lambda \in \Lambda$ and Λ is the index set, then there is a countable sub-collection $\{\mathcal{S}_{\mathcal{A}_\lambda} : \lambda \in \Lambda_0 \subseteq \Lambda\}$(such that $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda_0} \mathcal{S}_{\mathcal{A}_\lambda}$.

Definition 4.9. An sGTS $(\mathcal{S_A}, \mu)$ is called soft nearly μ-Lindelöf whenever $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_{\mathcal{A}_\lambda}$, where $\mathcal{S}_{\mathcal{A}_\lambda}$ is soft μ-open for all $\lambda \in \Lambda$ and Λ is the index set, then there is a countable sub-collection $\{S_{A_\lambda} : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda_0} \mathrm{Int}_{\mathcal{S_A}\mu} \mathrm{Cl}_{\mathcal{S_A}\mu}(\mathcal{S}_{\mathcal{A}_\lambda})$.

Definition 4.10. An sGTS $(\mathcal{S_A}, \mu)$ is called soft weakly μ-Lindelöf whenever $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_{\mathcal{A}_\lambda}$, where $\mathcal{S}_{\mathcal{A}_\lambda}$ is soft μ-open for all $\lambda \in \Lambda$ and Λ is the index set, then there is a countable sub-collection $\{\mathcal{S}_{\mathcal{A}_\lambda} : \lambda \in \Lambda_0 \subseteq \Lambda\}$ such that $\mathcal{S_A} = \bigcup_{\lambda\in\Lambda_0} \mathrm{Cl}_{\mathcal{S_A}\mu}(\mathcal{S}_{\mathcal{A}_\lambda})$.

Corollary 4.11. *Every soft μ-compact space is soft μ-Lindelöf space.*

Proof. It is straightforward and therefore omitted. □

The converse of Corollary 4.11 is not true as presented in Example 4.1

Example 4.1. Let $X = \mathbb{N}, G = \mathcal{A} = \{t_1, t_2\}$, $\mathcal{S}_{\widehat{G}} = \{(t_i, X) : t_i \in G\}$, and $\mathcal{S}_{\mathcal{A}_\mu}$ be a discrete soft generalized topology, then (S_A, μ) is soft μ-Lindelöf, but is not soft μ-compact.

Corollary 4.12. *Every soft μ-Lindelöf space is soft $\mathcal{N}\mu$-Lindelöf space.*

Proof. It is straightforward and therefore omitted. □

Corollary 4.13. *Every soft μ-Lindelöf space is soft $\mathcal{W}\mu$-Lindelöf space.*

Proof. It is straightforward and therefore omitted. □

The converse of Corollary 4.12 and 4.13 are not true as presented in Example 4.2.

Example 4.2. Let $X = \mathbb{R}, G = \mathcal{A} = \{t_1, t_2\}$, $\mathcal{S}_{\widehat{G}} = \{(t_i, X) : t_i \in G\}$, and $\mathcal{S}_{\mathcal{A}_\mu} = \{S_\emptyset, (H, F) \in \mathcal{S}_{\widehat{G}}$ such that $1 \in (H, F)\}$, then (S_A, μ) is soft $\mathcal{N}\mu$-Lindelöf, but is not soft μ-Lindelöf.

Corollary 4.14. *Every soft $\mathcal{N}\mu$-Lindelöf space is soft $\mathcal{W}\mu$-Lindelöf space.*

P r o o f. It is straightforward and therefore omitted. □

5. Conclusions

We have presented the definition of $\mathcal{N}\mu\mathcal{H}$-Lindelöfness as generalization of $\mu\mathcal{H}$-Lindelöfness. Example 2.7 shows that not every $\mathcal{N}\mu\mathcal{H}$-Lindelöf is $\mu\mathcal{H}$--Lindelöf. It is proved that $\mu\mathcal{H}$-lindelöf is $\mathcal{N}\mu\mathcal{H}$-Lindelöf. In order to add some conditions to make the converse true, it is shown that if X is $\mathcal{N}\mu\mathcal{H}$-Lindelöf, and if every cover of X by μ_θ-open sets, then there exists a countable sub-collection $\{U_\lambda : \lambda \in \Lambda_0 \in \Lambda\}$ such that

$$X \setminus \bigcup_{\lambda \in \Lambda_0} U_\lambda \in \mathcal{H}.$$

Moreover, it is proved that being μ-regular space is the necessary condition for $\mathcal{N}\mu\mathcal{H}$-Lindelöf and $\mu\mathcal{H}$-Lindelöf to coincide, see Theorem 2.8 By using μ-closed subsets and μ-regular closed subsets, some characterizations of $\mathcal{N}\mu\mathcal{H}$--Lindelöf have been established. Also, the new definitions are preserved under the image of some kinds of generalized continuity in section three.

Finally, more generalizations of soft generalized topological spaces have been defined. One example has been presented to verify concepts on soft set theory in the new last section.

In future, we can extend our work to have more results of $\mathcal{N}\mu\mathcal{H}$-Lindelöf as application in soft set theory. Further, preservation of soft $\mathcal{N}\mu\mathcal{H}$-Lindelöf can be studied in the sense of generalized continuity.

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