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NEW FIXED POINT THEOREMS ON COMPLETE b-METRIC SPACE BY USING RUS CONTRACTION MAPPING

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ABSTRACT. This paper investigates a fixed point over a complete b-metric space for a family of contractive mappings. In this paper, we have discovered new results in the direction of the complete b-metric space by using Rus contraction. Furthermore, we establish a common fixed point theorem between two mappings over complete b-metric space. We also provide some non-trivial examples to display the authenticity of our established results.

1. Introduction

Fixed point theory is currently one of the most important tools in many scientific fields, such as engineering, computer science, applied science, and the development of nonlinear analysis.

One of the most useful tools in this area is the Banach contraction theorem, which states that suppose T is a mapping on a complete metric space (X, d) into itself satisfying

$$d(Tx, Ty) \leq Kd(x, y), \quad \text{where} \quad K \in [0, 1) \ \forall \ x, y \in X,$$

then T has a unique fixed point on X which was introduced by Banach [3] in 1922. Then, using various contractions and mappings in various metric spaces, this result was expanded by a large number of authors.

The Banach contraction principle's counterpart in b-metric space, proposed by Bakhtin [2] as a generalization of metric space, was established in 1989. Various problems of the convergence of measurable functions with respect

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to measure, Czerwik [6] first presented as a generalization of Banach fixed point theorem in b-metric spaces.

In 2009, Boriceanu et al. [5] investigated fixed point theory for multivalued generalized contraction on a set with two b-metrics. A class of contractive mappings for b-metric space was used by S. Agrawal, K. Qureshi and J. Nema [1] in 2016 to study the fixed point theorem. In 2006, I. A. Rus [12] primarily studied contractions in metric space with fixed point properties and first proposed a new contraction known as the Rus contraction.

Many researchers developed fixed point theory and applied it in the field of core mathematics. Recently, Dhanraj et al [8] studied "Solution to integral equation in an O-complete Branciari b-metric spaces", Senthil Kumar et al [16] studied "Fixed point for an OgF-c in O-complete b-metric-like spaces", Gnanaprakasam et al [9] studied "New fixed point results in orthogonal b--metric spaces with related applications", Mani et al [13] studied "Fixed point results in C*-algebra-valued partial b-metric spaces with related application", Mani et al [14] studied "Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space", Gholidahneh et al [10] studied "The Meir-Keeler type contractions in extended modular b-metric spaces with an application", etc.

In this paper, we apply Rus contraction right now and also introduce some new fixed point theorems that are also valid in b-metric space.

2. Mathematical Preliminary

In this section, we obtain some basic definitions and results for our work.

Definition 2.1 ([5])**.** Let X be a non-empty set and $p \geq 1$ be any real number. A mapping $\mathrm{D} : X \times X \to [0, \infty)$ is said to be b-metric with coefficient "p" if the following conditions hold:

(b1) $\mathrm{D}(\bar{\vartheta}, \bar{q}) \geq 0$;

(b2) $\mathrm{D}(\bar{\vartheta}, \bar{q}) = 0$ if and only if $\bar{\vartheta} = \bar{q}$;

(b3) $\mathrm{D}(\bar{\vartheta}, \bar{q}) = d(\bar{q}, \bar{\vartheta})$;

(b4) $d(\bar{\vartheta}, \bar{q}) \leq p[\mathrm{D}(\bar{\vartheta}, \bar{w}) + d(\bar{w}, \bar{q})]$ for all $\bar{\vartheta}, \bar{q}, \bar{w} \in X$.

The pair (X, D) is called a b-metric space.

Example 2.1 ([5]). Let $M = L^p[0,1]$ be the collections of all real functions $m(t)$ such that $\int_0^1 |m(t)|^p dt < \infty$, where $t \in [0,1]$ and $0 < p < 1$. For the function $\mathrm{D}\colon M \times M \to \mathbb{R}_0^+$ defined by $\mathrm{D}(m,n) := \left(\int_0^1 (|m(t) - n(t)|^p dt)\right)^{1/p}$ for each $m, n \in L^p[0,1]$, then the ordered pair (M, D) forms a b-metric space with $s = 2^{1/p}$.

Definition 2.2. Let $(X, \text{Ɗ})$ be a b-metric space and $\{\vartheta_n\}$ be a sequence in X. Then, the sequence $\{\vartheta_n\}$ is called a convergent sequence and converges to some ϑ in X if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\text{Ɗ}(\vartheta_n, \vartheta) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} \vartheta_n = \vartheta$.

Definition 2.3. Let $(X, \text{Ɗ})$ be a b-metric space and $\{\vartheta_n\}$ be a sequence in X. The sequence $\{\vartheta_n\}$ is called a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\text{Ɗ}(\vartheta_n, \vartheta_m) < \epsilon$ for all $m, n \geq N$.

Definition 2.4. Let $(X, \text{Ɗ})$ be a b-metric space and $\{\vartheta_n\}$ be a sequence in X. The b-metric space $(X, \text{Ɗ})$ is said to be complete b-metric space if every Cauchy sequence $\{\vartheta_n\}$ is convergent in X.

Definition 2.5. Let $(X, \text{Ɗ})$ be a b-metric space. Let $\vartheta \in X$ and $\epsilon > 0$:

(i) The open ball $B(\vartheta, \epsilon)$ is $B(\vartheta, \epsilon) = \{\bar{w} \in X : \text{Ɗ}(\vartheta, \bar{w}) < \epsilon\}$.

(ii) The mapping $\text{Þ} : X \to X$ is said to be continuous at $\vartheta \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\text{Þ}\big(B(\vartheta, \delta)\big) \subseteq B(\text{Þ}\vartheta, \delta)$.

Definition 2.6. Let us assume that there are two self mappings L and J defined on a metric space $(X, \text{Ɗ})$. Let $u \in \text{Ɗ}$ be present. If u is such that $u = L(u)$, then u is considered a fixed point of L. Moreover, if $u = L(u) = J(u)$, then u is said to be a common fixed point of the mappings L and J.

Definition 2.7 (Rus contraction [12])**.** A self-map Þ on a metric space $(X, \text{Ɗ})$ is called Rus contraction mapping if there exist non-negative numbers $a, b, c \geq 0$, where $0 \leq a + 2b + 2c < 1$, such that for each $\bar{\vartheta}, \bar{u} \in X$

$$\text{Ɗ}(\text{Þ}\bar{\vartheta}, \text{Þ}\bar{u}) \leq a\text{Ɗ}(\bar{\vartheta}, \bar{u}) + b[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\text{Þ}\bar{u}, \bar{u})] + c[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{u}) + \text{Ɗ}(\bar{\vartheta}, \text{Þ}\bar{u})]. \quad (1)$$

EXAMPLE 2.2. Let $X = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$. Let us consider the usual metric of $\mathbb{R}$ such that $\text{Ɗ}(x, y) = |x - y|$ where $x, y \in X$. Then, we easily verify that $(X, \text{Ɗ})$ is a complete b-metric space for any $p \geq 1$.

Let Þ be a self mapping on X defined by

$$\text{Þ}(x) = \begin{cases} \frac{1}{2} & \text{if} \quad x = 1, \\ 0 & \text{if} \quad 1 < x < 2, \\ -\frac{1}{2} & \text{if} \quad x = 2. \end{cases}$$

Let us choose $a \geq 0.14, b \geq 0.15, c \geq 0.15$ such that $a + 2b + 2c < 1$. Then, condition (1) is fulfilled for any choosen a, b, c. Hence, Þ is a Rus contraction mapping on X.

3. Main results

This section contains the established main outcomes of this work and we provide some examples to demonstrate those theorems.

Theorem 3.1. *Let $(X, \text{Ɗ})$ be a complete b-metric space. If the mapping $\text{Þ} : X \to X$ satisfies*

$$\text{Ɗ}(\text{Þ}\bar{\vartheta}, \text{Þ}\bar{u}) \leq \mu_1 \text{Ɗ}(\bar{\vartheta}, \bar{u}) + \mu_2[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\text{Þ}\bar{u}, \bar{u})] + \mu_3[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{u}) + \text{Ɗ}(\bar{\vartheta}, \text{Þ}\bar{u})], \quad (2)$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ such that

$$0 \leq \mu_1 + 2\mu_2 + 2p\mu_3 < \frac{1}{p}, \quad (3)$$

then Þ has a unique fixed point on X.

P r o o f. Let $\bar{\vartheta}_0 \in X$, then we take picard iteration such that

$$\bar{\vartheta}_1 = \text{Þ}\bar{\vartheta}_0,\ \bar{\vartheta}_2 = \text{Þ}\bar{\vartheta}_1 = \text{Þ}^2\bar{\vartheta}_0,\ \dots,\ \bar{\vartheta}_{n+1} = \text{Þ}^n\bar{\vartheta}_0 .$$

If $\bar{\vartheta}_n = \bar{\vartheta}_{n+1}$, then we can trivially say that $\bar{\vartheta}_n$ is a fixed point of Þ. Suppose that $\bar{\vartheta}_n \neq \bar{\vartheta}_{n+1}$ for all $n \geq 0$. Then, we first show that the sequence $\{\bar{\vartheta}_n\}$ is a Cauchy sequence.

Now, we have

$$\begin{aligned}
\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) &= \text{Ɗ}(\text{Þ}\bar{\vartheta}_n, \text{Þ}\bar{\vartheta}_{n-1}) \\
&\leq \mu_1 \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + \mu_2[\text{Ɗ}(\text{Þ}\bar{\vartheta}_n, \bar{\vartheta}_n) + \text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}_{n-1})] \\
&\quad + \mu_3[\text{Ɗ}(\text{Þ}\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + \text{Ɗ}(\bar{\vartheta}_n, \text{Þ}\bar{\vartheta}_{n-1})] \\
&= \mu_1 \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + \mu_2[\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) + \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1})] \\
&\quad + \mu_3[\text{Ɗ}(\vartheta_{n+1}, \vartheta_{n-1}) + \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_n)] \\
&= \mu_1 \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + \mu_2[\text{Ɗ}(\text{Þ}\bar{\vartheta}_n, \bar{\vartheta}_n) + \text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}_{n-1})] \\
&\quad + \mu_3[\text{Ɗ}(\vartheta_{n+1}, \vartheta_{n-1})] \\
&\qquad [\text{as} \quad \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_n) = 0 \quad \text{by (b2)}] \\
&\leq \mu_1 \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + \mu_2[\text{Ɗ}(\text{Þ}\bar{\vartheta}_n, \bar{\vartheta}_n) + \text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}_{n-1})] \\
&\quad + \mu_3 p[\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) + \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1})] \\
&= (\mu_1 + \mu_2 + \mu_3 p)\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) + (\mu_2 + \mu_3 p)\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) \\
&\Longrightarrow (1 - \mu_2 - \mu_3 p)\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) \leq (\mu_1 + \mu_2 + \mu_3 p)\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) \\
&\Longrightarrow \text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) \leq \frac{(\mu_1 + \mu_2 + \mu_3 p)}{(1 - \mu_2 - \mu_3 p)}\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}). \quad (4)
\end{aligned}$$

Now, let $\frac{(\mu_1+\mu_2+\mu_3 p)}{(1-\mu_2-\mu_3 p)} = k$. So, we claim that $k < 1$ as $(\mu_1 + 2\mu_2 + 2p\mu_3) < \frac{1}{p}$. So, from equation (4), it follows that:

$$\begin{aligned} \text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) &\le k\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1}) \\ &\vdots \\ \implies \text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_n) &\le k^n \text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0)\,. \end{aligned} \tag{5}$$

Now, using the triangle inequality and (5), for any positive integers m, n such that $m > n$, we have

$$\begin{aligned} \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_m) &\le p\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n+1}) + p^2\text{Ɗ}(\bar{\vartheta}_{n+1}, \bar{\vartheta}_{n+2}) + \cdots + p^{(m-n)}\text{Ɗ}(\bar{\vartheta}_{m-1}, \bar{\vartheta}_m) \\ &\le pk^n\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) + p^2k^{n+1}\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) + \cdots + p^{(m-n)}k^{(m-1)}\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) \\ &= \left(pk^n + p^2k^{n+1} + \cdots + p^{(m-n)}k^{(m-1)}\right)\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) \\ &= pk^n\left(1 + (pk) + (pk)^2 + \cdots + (pk)^{m-n-1}\right)\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) \\ &\le pk^n\left(\sum_{i=0}^{\infty}(pk)^i\right)\text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0), \quad \text{where} \quad pk < 1 \\ &= \left[\frac{pk^n}{1-pk}\right] \quad \text{Ɗ}(\bar{\vartheta}_1, \bar{\vartheta}_0) \to 0 \quad \text{as} \quad k^n \to 0 \quad \text{when} \quad n \to \infty \\ \implies \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_m) &\to 0 \;\text{ as }\; n, m \to \infty\,. \end{aligned}$$

Hence, $\bar{\vartheta}_n$ is a Cauchy sequence in X. Since X is a complete b-metric space, so there is a $\bar{\vartheta} \in X$ such that $\bar{\vartheta}_n \to \bar{\vartheta}$ $(as\ n \to \infty)$.

Again, we have

$$\begin{aligned} \text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) &\le p[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}_n) + \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta})] \\ &\le p[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \text{Þ}\bar{\vartheta}_{n-1}) + \text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta})] \\ &= p\text{Ɗ}(\text{Þ}\bar{\vartheta}, \text{Þ}\bar{\vartheta}_{n-1}) + p\text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}) \\ &\le p[\mu_1\text{Ɗ}(\bar{\vartheta}, \bar{\vartheta}_{n-1}) + \mu_2\{\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}_{n-1})\} \\ &\quad + \mu_3\{\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}_{n-1}) + \text{Ɗ}(\bar{\vartheta}, \text{Þ}\bar{\vartheta}_{n-1})\}] + p\text{Ɗ}(\text{Þ}\bar{\vartheta}_{n-1}, \bar{\vartheta}) \\ &\le p\big[\mu_1\text{Ɗ}(\bar{\vartheta}, \bar{\vartheta}_{n-1}) + \mu_2\{\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}_{n-1})\} \\ &\quad + \mu_3\{\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}_{n-1}) + \text{Ɗ}(\bar{\vartheta}, \bar{\vartheta}_n)\}\big] + p\text{Ɗ}(\bar{\vartheta}_n, \bar{\vartheta}) \\ &\quad \text{when} \quad n \to \infty, \quad \bar{\vartheta}_n \to \bar{\vartheta} \\ &\le p\big[\mu_1\text{Ɗ}(\bar{\vartheta}, \bar{\vartheta}) + \mu_2[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\bar{\vartheta}, \bar{\vartheta})] \\ &\quad + \mu_3[\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \text{Ɗ}(\bar{\vartheta}, \bar{\vartheta})]\big] + p\text{Ɗ}(\bar{\vartheta}, \bar{\vartheta}) \\ &= p[\mu_2\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta}) + \mu_3\text{Ɗ}(\text{Þ}\bar{\vartheta}, \bar{\vartheta})] \quad \text{[by using (b2)]} \end{aligned}$$

$$
\begin{aligned}
&= (p\mu_2 + p\mu_3)Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) \\
\implies &[1 - p(\mu_2 + \mu_3)]Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) \le 0\,. \qquad (6)
\end{aligned}
$$

Case 1. If $\mu_2 = \mu_3 = 0$, then from equation (6) we easily get

$$Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) = 0 \implies Þ\bar{\vartheta} = \bar{\vartheta} \quad \forall\ \bar{\vartheta} \in X.$$

Hence, $\bar{\vartheta}$ is a fixed point of Ɗ.

Case 2. Since $p(\mu_2 + \mu_3) < 1$, then it is obvious that $[1 - p(\mu_2 + \mu_3)] > 0$ and $Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) \ge 0$, so we get

$$[1 - p(\mu_2 + \mu_3)]Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) \ge 0\,. \qquad (7)$$

Therefore, from (6) and (7) we conclude that

$$[1 - p(\mu_2 + \mu_3)]Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) = 0 \implies Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) = 0 \implies Þ\bar{\vartheta} = \bar{\vartheta}.$$

Therefore, $\bar{\vartheta}$ is a fixed point of Þ. Hence, Þ has a fixed point on X.

Finally, we show that $\bar{\vartheta} \in X$ is a unique fixed point of Þ. Let us assume that there exists another fixed point $\bar{u} \in X$ such that $Ɗ(Þ\bar{u}, \bar{u}) = 0$ We get

$$
\begin{aligned}
Ɗ(\bar{\vartheta}, \bar{u}) &= Ɗ(Þ\bar{\vartheta}, Þ\bar{u}) \\
&\le \mu_1 Ɗ(\bar{\vartheta}, \bar{u}) + \mu_2[Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) + Ɗ(Þ\bar{u}, \bar{u})] + \mu_3[Ɗ(Þ\bar{\vartheta}, \bar{u}) + Ɗ(\bar{\vartheta}, Þ\bar{u})] \\
&= \mu_1 Ɗ(\bar{\vartheta}, \bar{u}) + \mu_3[Ɗ(Þ\bar{\vartheta}, \bar{u}) + Ɗ(\bar{\vartheta}, Þ\bar{u})] \\
&= \mu_1 Ɗ(Þ\bar{\vartheta}, Þ\bar{u}) + \mu_3[Ɗ(Þ\bar{\vartheta}, Þ\bar{u}) + Ɗ(Þ\bar{\vartheta}, Þ\bar{u})] \\
&= (\mu_1 + 2\mu_3)Ɗ(Þ\bar{\vartheta}, Þ\bar{u}) \\
&= (\mu_1 + 2\mu_3)Ɗ(\bar{\vartheta}, \bar{u}) \\
\implies &(1 - \mu_1 - 2\mu_3)Ɗ(\bar{\vartheta}, \bar{u}) \le 0\,. \qquad (8)
\end{aligned}
$$

Since $(1 - \mu_1 - 2\mu_3) > 0$, then from (8) we get $Ɗ(\bar{\vartheta}, \bar{u}) \le 0$, which contradicts the fact that

$$Ɗ(\bar{\vartheta}, \bar{u}) \not< 0 \quad \forall \bar{\vartheta}, \bar{u} \in X.$$

Therefore, the only possible condition is that

$$Ɗ(\bar{\vartheta}, \bar{u}) = 0 \implies \bar{\vartheta} = \bar{u}\,.$$

Hence, $\bar{\vartheta}$ is a unique fixed point of Þ. □

Example 3.1. Let

$$X = \{(x, 1) \in \mathbb{R}^2 : 1 \le x \le 2\} \cup \{(1, x) \in \mathbb{R}^2 : 1 \le x \le 2\}.$$

Let us define a mapping

$$Ɗ : X \times X \to \mathbb{R} \quad \text{by} \quad Ɗ(p,q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where

$$p = (x_1, y_1), \quad q = (x_2, y_2) \quad \text{and} \quad p, q \in X.$$

Then, $(X, Ɗ)$ is a complete b-metric space for any $p \geq 1$.

Let $Þ : X \to X$ be a function defined by $Þ(x,1) = (1, \sqrt{x})$ and $Þ(1,x) = (\sqrt{x}, 1)$. Then, for $\mu_1 = 0.4, \mu_2 = 0.005, \mu_3 = 0.009$ and $p = 2$ conditions (3) and (2) are also fulfilled. Here, the unique fixed point of Ɗ is $(1,1)$.

Theorem 3.2. *Let $(X, Ɗ)$ be a complete b-metric space with $p \geq 1$. Let the mapping $\{Þ_j\}$ be a sequence of self mappings of X satisfying*

$$Ɗ(Þ_j\bar{\vartheta}, Þ_j\bar{q}) \leq \mu_1 Ɗ(\bar{\vartheta}, \bar{q}) + \mu_2[Ɗ(Þ_j\bar{\vartheta}, \bar{\vartheta}) + Ɗ(Þ_j\bar{q}, \bar{q})] + \mu_3[Ɗ(Þ_j\bar{\vartheta}, \bar{q}) + Ɗ(\bar{\vartheta}, Þ_j\bar{q})], \quad (9)$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ with $0 \leq \mu_1 + 2\mu_2 + 2p\mu_3 < \frac{1}{p}$, and have unique fixed points $\bar{u}_j \in X$. Suppose $\{Þ_j\}$ converges uniformly to a mapping $Þ : X \to X$. Then, $\bar{u}_j$ converges to a unique fixed point of Þ.

Proof. Since $\{Þ_j\}$ converges uniformly to $Þ : X \to X$, Þ also satisfies

$$Ɗ(Þ\bar{\vartheta}, Þ\bar{q}) \leq \mu_1 Ɗ(\bar{\vartheta}, \bar{q}) + \mu_2[Ɗ(Þ\bar{\vartheta}, \bar{\vartheta}) + Ɗ(Þ\bar{q}, \bar{q})] + \mu_3[Ɗ(Þ\bar{\vartheta}, \bar{q}) + Ɗ(\bar{\vartheta}, \bar{q})]. \quad (10)$$

So, by theorem (3.1) we claim that Þ has a unique fixed point $\bar{u}$ on X.

Now, using triangle inequality and (9), we have

$$\begin{aligned}
Ɗ(\bar{u}_j, \bar{u}) &= Ɗ(Þ_j\bar{u}_j, Þ\bar{u}) \\
&\leq p\{Ɗ(Þ_j\bar{u}_j, Þ_j\bar{u}) + Ɗ(Þ_j\bar{u}, Þ\bar{u})\} \\
&\leq p\mu_1 Ɗ(\bar{u}_j, \bar{u}) + p\mu_2[Ɗ(Þ_j\bar{u}_j, \bar{u}_j) + Ɗ(Þ_j\bar{u}, \bar{u})] \\
&\quad + p\mu_3[Ɗ(Þ_j\bar{u}_j, \bar{u}) + Ɗ(\bar{u}_j, Þ\bar{u})] + pƊ(Þ_j\bar{u}, Þ\bar{u}). \\
&= p\mu_1 Ɗ(\bar{u}_j, \bar{u}) + p\mu_2[Ɗ(\bar{u}_j, \bar{u}_j) + Ɗ(Þ_j\bar{u}, Þ\bar{u})] \\
&\quad + p\mu_3[Ɗ(\bar{u}_j, \bar{u}) + Ɗ(\bar{u}_j, \bar{u})] + pƊ(Þ_j\bar{u}, Þ\bar{u}). \\
&\implies (1 - p\mu_1 - 2p\mu_3)Ɗ(\bar{u}_j, \bar{u}) \leq (p + p\mu_2)Ɗ(Þ_j\bar{u}, Þ\bar{u}) \\
&\implies Ɗ(\bar{u}_j, \bar{u}) \leq \frac{(p + p\mu_2)}{(1 - p\mu_1 - 2p\mu_3)} Ɗ(Þ_j\bar{u}, Þ\bar{u}).
\end{aligned}$$

So,

$$Ɗ(\bar{u}_j, \bar{u}) \to 0 \quad \text{as} \quad j \to \infty\,.$$

Therefore, we easily claim that $\bar{u}_j$ converges to a unique fixed point $\bar{u}$ of Þ. □

Theorem 3.3. *Let $(X, \eth)$ be a complete b-metric space with $p \geq 1$. If two continuous mappings $Þ, \Psi : X \to X$ satisfy*

$$\eth(Þ\bar{\vartheta}, \psi\bar{u}) \leq \mu_1\eth(\bar{\vartheta}, \bar{u}) + \mu_2[\eth(Þ\bar{\vartheta}, \bar{\vartheta}) + \eth(\psi\bar{u}, \bar{u})] + \mu_3[\eth(Þ\bar{\vartheta}, \bar{u}) + \eth(\bar{\vartheta}, \psi\bar{u})], \quad (11)$$

where $\mu_1, \mu_2, \mu_3 \geq 0$ such that

$$0 \leq \mu_1 + 2\mu_2 + 2p\mu_3 < \frac{1}{p}\ , \quad (12)$$

then Þ and ψ have a unique common fixed point on X.

P r o o f. Let $\bar{u}_0 \in X$. We set

$$\begin{aligned} \bar{u}_1 &= Þ(\bar{u}_0), & \bar{u}_2 &= \psi(\bar{u}_1), \\ \bar{u}_3 &= Þ(\bar{u}_2), & \bar{u}_4 &= \psi(\bar{u}_3), \\ &\vdots & &\vdots \\ \bar{u}_{2n+1} &= Þ(\bar{u}_{2n}), & \bar{u}_{2n+2} &= \psi(\bar{u}_{2n+1}). \end{aligned}$$

Then, we get

$$\begin{aligned} \eth(\bar{u}_{2n+1}, \bar{u}_{2n+2}) &= \eth(Þ\bar{u}_{2n}, \psi\bar{u}_{2n+1}) \\ &\leq \mu_1\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) + \mu_2[\eth(Þ\bar{u}_{2n}, \bar{u}_{2n}) + \eth(\bar{u}_{2n+1}, \psi\bar{u}_{2n+1})] \\ &\quad + \mu_3[\eth(Þ\bar{u}_{2n}, \bar{u}_{2n+1}) + \eth(\bar{u}_{2n}, \psi\bar{u}_{2n+1})] \\ &= \mu_1\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) + \mu_2[\eth(\bar{u}_{2n+1}, \bar{u}_{2n}) + \eth(\bar{u}_{2n+1}, \bar{u}_{2n+2})] \\ &\quad + \mu_3[\eth(\bar{u}_{2n+1}, \bar{u}_{2n+1}) + \eth(\bar{u}_{2n}, \bar{u}_{2n+2})] \\ &\leq \mu_1\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) + \mu_2[\eth(\bar{u}_{2n+1}, \bar{u}_{2n}) + \eth(\bar{u}_{2n+1}, \bar{u}_{2n+2})] \\ &\quad + \mu_3 p[\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) + \eth(\bar{u}_{2n+1}, \bar{u}_{2n+2})] \\ &\quad [\text{as } \eth(\bar{\vartheta}_{2n+1},\ \bar{\vartheta}_{2n+1}) = 0 \text{ by (b2)}] \\ &= (\mu_2 + p\mu_3)\eth(\bar{u}_{2n+1}, \bar{u}_{2n+2}) + (\mu_1 + \mu_2 + p\mu_3)\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) \\ &\Longrightarrow (1 - \mu_2 - p\mu_3)\eth(\bar{u}_{2n+1}, \bar{u}_{2n+2}) \\ &\leq (\mu_1 + \mu_2 + p\mu_3)\eth(\bar{u}_{2n}, \bar{u}_{2n+1}) \\ &\Longrightarrow \eth(\bar{u}_{2n+1}, \bar{u}_{2n+2}) \\ &\leq \frac{(\mu_1 + \mu_2 + p\mu_3)}{(1 - \mu_2 - p\mu_3)}\eth(\bar{u}_{2n}, \bar{u}_{2n+1}). \end{aligned} \quad (13)$$

Now, let

$$\frac{(\mu_1 + \mu_2 + p\mu_3)}{(1 - \mu_2 - p\mu_3)} = k'.$$

We can clearly assert that

$$k' < 1 \quad \text{as} \quad \mu_1 + 2\mu_2 + 2p\mu_3 < 1\,.$$

So, from equation (13) it follows that:

$$\begin{aligned} \mathfrak{D}(\bar{u}_{2n+1}, \bar{u}_{2n+2}) &\leq k'\mathfrak{D}(\bar{u}_{2n}, \bar{u}_{2n+1}) \\ &\vdots \\ \Longrightarrow \mathfrak{D}(\bar{u}_{2n+1}, \bar{u}_{2n+2}) &\leq (k')^{2n+1}\mathfrak{D}(\bar{u}_0, \bar{u}_1). \end{aligned}$$

So, we can write

$$\mathfrak{D}(\bar{u}_n, \bar{u}_{n+1}) \leq (k')^n\mathfrak{D}(\bar{u}_0, \bar{u}_1)\,. \tag{14}$$

Now, using triangle inequality and (14) for any positive integers m, n such that $m > n$, we have

$$\begin{aligned} \mathfrak{D}(\bar{u}_n, \bar{u}_m) &\leq p\mathfrak{D}(\bar{u}_n, \bar{u}_{n+1}) + p^2\mathfrak{D}(\bar{u}_{n+1}, \bar{u}_{n+2}) + \dots \\ &\quad \dots + p^{(m-n)}\mathfrak{D}(\bar{u}_{m-1}, \bar{u}_m) \\ &\leq p(k')^n\mathfrak{D}(\bar{u}_1, \bar{u}_0) + p^2(k')^{n+1}\mathfrak{D}(\bar{u}_1, \bar{u}_0) + \dots \\ &\quad \dots + p^{(m-n)}(k')^{(m-1)}\mathfrak{D}(\bar{u}_1, \bar{u}_0) \\ &= \left(p(k')^n + p^2(k')^{n+1} + \dots + p^{(m-n)}(k')^{(m-1)}\right)\mathfrak{D}(\bar{u}_1, \bar{u}_0) \\ &= pk'^n\left(1 + (pk') + (pk')^2 + \dots + (pk')^{m-n-1}\right)\mathfrak{D}(\bar{u}_1, \bar{u}_0) \\ &\leq pk'^n\left(\sum_{i=0}^{\infty}(pk')^i\right)\mathfrak{D}(\bar{u}_1, \bar{u}_0), \quad \text{where} \quad pk' < 1 \\ &\leq \left[\frac{pk'^n}{1 - pk'}\right]\mathfrak{D}(\bar{u}_1, \bar{u}_0) \to 0 \text{ as } n \to \infty, \;\; k'^n \to 0\,. \end{aligned}$$

$$\Longrightarrow \mathfrak{D}(\bar{u}_n, \bar{u}_m) \to 0 \text{ as } n, m \to \infty\,.$$

Hence, $\{\bar{u}_n\}$ is a Cauchy sequence in X. Since X is a complete b-metric space, so there exists a number

$$\bar{u} \in X \quad \text{such that} \quad \bar{u}_n \to \bar{u} \quad (\text{as } n \to \infty).$$

Again, we have

$$\begin{aligned}
\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) &\leq p[\text{Ɗ}(\text{Þ}\bar{u},\psi\bar{u}_{2n-1}) + \text{Ɗ}(\psi\bar{u}_{2n-1},\bar{u})] \\
&= p\text{Ɗ}(\text{Þ}\bar{u},\psi\bar{u}_{2n-1}) + p\text{Ɗ}(\psi\bar{u}_{2n-1},\bar{u}) \\
&\leq p[\mu_1\text{Ɗ}(\bar{u},\bar{u}_{2n-1}) + \mu_2\{\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) + \text{Ɗ}(\psi\bar{u}_{2n-1},\bar{u}_{2n-1})\} \\
&\quad + \mu_3\{\text{Ɗ}(\text{Þ}\bar{u},\bar{u}_{2n-1}) + \text{Ɗ}(\bar{u},\psi\bar{u}_{2n-1})\}] + p\text{Ɗ}(\psi\bar{u}_{2n-1},\bar{u}) \\
&\quad \text{when } \ n\to\infty, \quad \bar{u}_n \to \bar{u} \\
&\leq p\,[\mu_1\text{Ɗ}(\bar{u},\bar{u}) + \mu_2\{\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) + \text{Ɗ}(\psi\bar{u},\bar{u})\} \\
&\quad + \mu_3\{\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) + \text{Ɗ}(\bar{u},\psi\bar{u})\}] + p\text{Ɗ}(\psi\bar{u},\bar{u}) \\
&= (p\mu_2 + p\mu_3)\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) + (p + p\mu_2 + p\mu_3)\text{Ɗ}(\psi\bar{u},\bar{u}) \\
&\implies (1 - p\mu_2 - p\mu_3)\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) \leq (p + p\mu_2 + p\mu_3)\text{Ɗ}(\psi\bar{u},\bar{u}) \\
&\implies 0 \leq \text{Ɗ}(\text{Þ}\bar{u},\bar{u}) \leq \frac{(p + p\mu_2 + p\mu_3)}{(1 - p\mu_2 - p\mu_3)}\text{Ɗ}(\psi\bar{u},\bar{u})\,. \qquad (15)
\end{aligned}$$

Since

$$\frac{p(1+\mu_2+\mu_3)}{(1 - p\mu_2 - p\mu_3)} > 0\,,$$

then

$$\frac{p(1+\mu_2+\mu_3)}{(1 - p\mu_2 - p\mu_3)}\text{Ɗ}(\psi\bar{u},\bar{u}) \geq 0$$

that means that the only possibility of occurrence for equality is that

$$\text{Ɗ}(\psi\bar{u},\bar{u}) = 0 \implies \psi\bar{u} = \bar{u}\ \forall\ \bar{u}\in X. \qquad (16)$$

So, $\bar{u}$ is the fixed point of ψ. From (15) and (16) it follows that

$$\text{Ɗ}(\text{Þ}\bar{u},\bar{u}) = 0 \implies \text{Þ}\bar{u} = \bar{u}, \quad \text{as } \ \text{Ɗ}(\text{Þ}\bar{u},\bar{u}) \not< 0\ \forall\ \ \bar{u}\in X.$$

So, $\bar{u}$ is the fixed point of Þ. Therfore, $\bar{u}$ is the common fixed point of Þ and ψ and uniqueness of $\bar{u}$ is obvious. □

Example 3.2. In the example (3.1), we consider $\psi : X \to X$ is another mapping defined by

$$\psi(x,1) = \left(1, \frac{1}{\sqrt{x}}\right) \quad \text{and} \quad \psi(1,x) = \left(\frac{1}{\sqrt{x}}, 1\right).$$

Then, for the values of

$$\mu_1 = 0.4, \quad \mu_2 = 0.005, \quad \mu_3 = 0.009 \quad \text{and} \quad p = 2,$$

conditions (11) and (12) are fulfilled and $(1,1)$ is a common unique fixed point for Þ and ψ.

4. Conclusion

We updated the broad backdrop of b-metric space and demonstrated some well-known fixed points results in complete b-metric space in the aforementioned parts. We anticipate that our findings will aid future research in this area and demonstrate the singularity of fixed point utilizing various contraction conditions. In the future, if we employ the ideas presented in this paper and attempt to do similar types of research on alternative metric spaces, such as control b-metric space, cone b-metric space, rectangular b-metric space, etc., then we might uncover some insightful findings.

`Conflict of interest.` All the authors have equal contributions to the preparation of this article.

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REFERENCES

[1] AGRAWAL, S.—QURESHI, K.—NEMA, J.: *A fixed point theorem for b-metric space*, IJPAM **9** (2016), no. 1, 45–50.

[2] BAKHTIN, I. A.: *The contraction mapping principle in almost metric spaces*, Func. Anal. Gos. Ped. Inst. Ul'yanowsk **30** (1989), 26–37.

[3] BANACH, S.: *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales* [*On the operations in abstract sets and their application to integral equations*], Funf. Math. **3** (1922) 133–181.

[4] BOTA, M.—MOLNAR, A.—VARGA, C.: *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory **12** (2011), no. 2, 21–28.

[5] BORICEANU, M.: *Fixed point theory for multivalued generalized contraction on a set with two b-metric*, Stud. Univ. Babeş-Bolyai Math. **54** (2009), no. 3, 3–14.

[6] CZERWIK, S,: *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostraviensis **1** (1993), 5–11.

[7] CZERWIK, S,: *Non-linear set valued contraction mappings in b-metric spaces*, Atti Semin. mat. fis. Univ. Modena Reggio Emilia **46** (1998), no. 2, 263–276.

[8] DHANRAJ, M—GNANAPRAKASAM, A. J.—MANI, G.—EGE, O.—DE LA SEN, M.: *Solution to integral equation in an O-complete Branciari b-metric spaces*, Axioms **11** (2022), no. 12, Doi: 10.3390/axioms11120728
`https://www.mdpi.com/2075-1680/11/12/728`

[9] GNANAPRAKASAM, A. J.—MANI, G.—EGE, O.—ALOQAILY, A.—MLAIKI, N.: *New fixed point results in orthogonal b-metric spaces with related applications*, Mathematics **11** (2023), no. 3,
https://doi.org/10.3390/math11030677

[10] GHOLIDAHNEH, A.—SEDGHI, S.—EGE, O.— MITROVIC, Z. D.— DE LA SEN M.: *The Meir-Keeler type contractions in extended modular b-metric spaces with an application*, AIMS Math. **6** (2021), no. 2, 1781–1799.

[11] KIR, M.—KIZILTUNC, H.: *On some well-known fixed point theorems in b-metric space*, Turkish J. Anal. and Number Theory **1** (2013), no. 1, 13–16, http://pubs.sciepub.com/tjant/1/1/4

[12] RUS, I. A.: *Metric space with fixed point property with respect to contractions*, Studia Univ. Babeş-Bolyai **51** (2006), no. 3, 115–121.

[13] MANI, G.—GNANAPRAKASAM, A. J.—EGE, O.—ALOQAILY, A.—MLAIKI, N.: *Fixed point results in C*-algebra-valued partial b-metric spaces with related application* Mathematics **11** (2023), no. 5, 1–9. https://doi.org/10.3390/math11051158

[14] MANI, G.—GNANAPRAKASAM A. J.—EGE, O.—FATIMA, N.—MLAIKI, N.: *Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space* Symmetry **15** (2023), no. 2, 1–15, https://doi.org/10.3390/sym15020297

[15] PĂCURAR, M.: *Sequences of almost contractions and fixed points in b-metric spaces*, An. Univ. Vest Timiş., Ser. Mat.-Inform. **48** (2010) no. 3, 125–137.

[16] SENTHIL KUMAR, P.—ARUL JOSEPH, G.—EGE, O.—GUNASEELAN, M.—HAQUE, S.—MLAIKI, N.: *Fixed point for an OgF-c in O-complete b-metric-like spaces*, Aims Math. **8** (2022), 1022–1039.

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