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ON I-STATISTICALLY CAUCHY SEQUENCES & THE CLASS OF I-STATISTICALLY CONCURRENT SEQUENCES

Debjani Rakshit[1] — Prasenjit Bal[1] — Shyamal Debnath[2]

[1]Department of Mathematics, The ICFAI University Tripura, Kamalghat, INDIA-799210

[2]Department of Mathematics, Tripura University, Agartala, INDIA

ABSTRACT. In this article, we present the ideas of I-statistically Cauchy criteria and I^*-statistically Cauchy criteria, which are the generalizations of I-Cauchy and I^*-Cauchy criterion, respectively. To grasp the differences, we compare this I-statistically Cauchy criterion with a few other Cauchy criteria. Also, we investigate a few characteristics of I^*-statistically Cauchy sequences and I-statistically Cauchy sequences and demonstrate their equivalence under the condition that the ideal I satisfies the property (AP). Furthermore, a relation is defined on the set S_X of all sequences in a metric space, which comes out to be an equivalence relation. Finally, we show that if two sequences belong to the same equivalence class, then either both of them are I-statistically Cauchy or none of them are.

1. Introduction

Let $M \subset \mathbb{N}$, then the natural density of M is denoted by $D(M)$ and defined as follows
$$D(M) = \lim_{m\to\infty} \frac{1}{m} |\{k \in M : k \leq m\}|.$$
Sometimes, natural density is also called asymptotic density. In 1951, utilizing natural density as a measurement, Fast [7] was the first to discuss statistical convergence (Steinhaus [19] also independently introduced the same concept

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in the same year). A sequence $\{z_n\}$ is considered to be statistically convergent in a metric space (X,ρ), if there exists $z_0 \in X$ such that for every $\varepsilon > 0$,

$$D\big(M(\varepsilon) = \{k \in \mathbb{N} : \rho(z_k, z_0) \geq \varepsilon\}\big) = 0.$$

To analyse the sequence space, numerous mathematicians such as Fridy [8, 9], Salat [16], Rath and Tripathy [15], Nuray and Ruckle [14] have used this convergence criterion. In a metric space setting, Kostyrko et al. [11] first suggested the notion of I-convergence in the year 2000. Although there have been a lot of extensions of convergence theory, we found I-convergence the most interesting one, where I is an ideal.

Let X be a non empty set and $I \subseteq \mathcal{P}(X)$. I is an ideal on X if and only if

(i) $\emptyset \in I$;

(ii) $M_1, M_2 \in I \Rightarrow M_1 \cup M_2 \in I$;

(iii) $M_1 \in I$ and $M_2 \subset M_1 \Rightarrow M_2 \in I$.

On the other hand, $\mathcal{F} \subset \mathcal{P}(X)$ is a filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) $M_1, M_2 \in \mathcal{F} \Rightarrow M_1 \cap M_2 \in \mathcal{F}$;

(iii) $M_1 \in \mathcal{F}$ and $M_2 \supset M_1 \Rightarrow M_2 \in \mathcal{F}$.

If $I \neq \emptyset$ and $X \notin I$, then I is a non-trivial ideal. The filter

$$\mathcal{F} = \mathcal{F}(I) = \{X \setminus M : M \in I\}$$

is taken to be the dual filter of the ideal I. An admissible ideal is one that encompasses all singleton sets and is non-trivial. A sequence $\{z_n\}$ is considered to be I-convergent in a metric space (X,ρ) if we can find a $z_0 \in X$ such that for every $\varepsilon > 0$,

$$M(\varepsilon) = \{k \in \mathbb{N} : \rho(z_k, z_0) \geq \varepsilon\} \in I.$$

Also, an analogous known as filter convergence was investigated earlier by Katetov [10]. Some recent applications of I-convergence can be found in [1–3, 17]. Dems [6] introduced and explored the I-Cauchy condition, which drew many mathematicians such as Nabiev et al. [13], Das and Ghosal [4] to this area of study. The notion of I-statistical convergence was incorporated into the theory by Savas and Das in [18]. Many authors, including Mursaleen et al. [12], Debnath and Rakshit [5], afterwards conducted extensive research on it.

Continuing the work of Savas and Das, we extend several I-Cauchy parameters to I-statistical Cauchy parameters. We offer a few counter-instances in order to differentiate between various convergence and Cauchy criteria. We also examine several characteristics of newly introduced Cauchy sequences and demonstrate their equivalence under some given conditions. A general question arose as we started our initial research about when two sequences will have the same nature by means of the I-statistical Cauchy property. In order to provide a positive response to that, we introduce the I-statistically concurrent relation on the collection S_X of all sequences on a metric space (X, ρ), comes out to be an equivalence relation. Lastly, we have shown that the sequences belonging to the same equivalence class will have the same nature in terms of the I-statistically Cauchy property.

2. Preliminaries

A space X represents a linear metric space equipped with the corresponding metric ρ otherwise mentioned, and I represents an admissible ideal of $\mathbb{N}$ throughout the paper.

Definition 2.1 ([8]). A sequence $\{z_n\}$ is assumed to satisfy the statistical Cauchy criteria in a space X if we can find a $p\big(= p(\varepsilon > 0)\big) \in \mathbb{N}$ for which

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : \rho(z_k, z_p) \geq \varepsilon\}| = 0.$$

Definition 2.2 ([13]). A sequence $\{z_n\}$ is considered to satisfy I-Cauchy criteria in a space X if we can find a $p\big(= p(\varepsilon > 0)\big) \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : \rho(z_k, z_p) \geq \varepsilon\} \in I.$$

Definition 2.3 ([13]). In a space X, a sequence $\{z_n\}$ is considered to satisfy I^*-Cauchy criteria if we can find a set $P = \{p_1 < p_2 < \cdots < p_k < \cdots\} \subset \mathbb{N}$ and $P \in \mathcal{F}(I)$ such that $\{z_{p_k}\}$ is a Cauchy sequence. In other words, if

$$\lim_{k,q\to\infty} \rho\left(z_{p_k}, z_{p_q}\right) = 0.$$

Definition 2.4 ([11]). For an admissible ideal I, if for any sequence $\{A_n : n \in \mathbb{N}\}$, where $A_n \in I$ and $A_n \cap A_m = \emptyset$ for $m \neq n$, there exists a sequence $\{B_n : n \in \mathbb{N}\}$ such that $A_n \Delta B_n (n \in \mathbb{N})$ is finite and $\bigcup_{n=1}^{\infty} B_n \in I$, then we say that I has the property (AP).

Lemma 2.5 ([13]). *Let I be an admissible ideal having the property (AP), $\mathcal{F}(I)$ be the dual filter of I, and $\{P_i\}_{i=1}^{\infty}$ be such that $P_i \subset \mathbb{N}$ and $P_i \in \mathcal{F}(I)$ for all i. Then we can find a $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(I)$ and $P \setminus P_i$ is finite for all i.*

Definition 2.6 ([18]). In a space X, a sequence $\{z_n\}$ is considered to I--statistically converge to z_0 if for every possible $\varepsilon, \delta > 0$, $\big\{m \in \mathbb{N} : \frac{1}{m} |\{k \leq m : \rho(z_k, z_0) \geq \varepsilon\}| \geq \delta\big\} \in I$.

3. On I-statistically Cauchy sequences

Definition 3.1. In a space (X, ρ), a sequence $\{z_n\}$ in X is considered to be an I-statistically Cauchy sequence in X if for every possible $\varepsilon, \delta > 0$, there exists a $p \in \mathbb{N}$ such that

$$\left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(z_k, z_p) \geq \varepsilon\}| \geq \delta\right\} \in I.$$

Theorem 3.2. *Every I-Cauchy sequence is an I-statistically Cauchy sequence.*

P r o o f. The definition promptly leads to the result. Hence, the proof is omitted. □

Example. The converse implication of the Theorem 3.2 may not hold.

We consider the space $(\mathbb{R}, d)$, where d is the Euclidean metric, and a sequence $\{x_n\}$, where

$$x_n = \begin{cases} 0, & \text{for} \quad n = k^2, \ \ k \in \mathbb{N} \\ 1, & \text{if} \quad n \neq k^2 \ \ \text{for any} \ \ k \in \mathbb{N}. \end{cases}$$

We also consider $I = I_{fin}$, the collection of all finite subsets of $\mathbb{N}$. For any $\varepsilon > 0$, there exists $x_2 \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{1}{n}|\{k \leq n : d(x_k, x_2) \geq \varepsilon\}| = 0.$$

Therefore, for any $\delta > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(x_k, x_2) \geq \varepsilon\}| \geq \delta\right\}$ must be a finite set. So, $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(x_k, x_2) \geq \varepsilon\}| \geq \delta\right\} \in I$, i.e., the sequence $\{x_n\}$ is an I-statistically Cauchy sequence. But for any $m \in \mathbb{N}$, $\{n \in \mathbb{N} : d(x_n, x_m) \geq \varepsilon\}$ is an infinite set. So, $\{n \in \mathbb{N} : d(x_n, x_m) \geq \varepsilon\} \notin I$. Therefore, $\{x_n\}$ is not an I Cauchy sequence.

Definition 3.3 ([13]). In a linear metric space (X, ρ), a sequence $\{z_n\}$ in X is considered to be an I^*-statistical Cauchy sequence if there exists a set $P = \{p_1 < p_2 < \cdots < p_k < \cdots\} \subset \mathbb{N}$ and $P \in \mathcal{F}(I)$ such that $\{z_{p_k}\}$ is a statistically Cauchy sequence. In other words, if there exists $l\big(= l(\varepsilon > 0)\big) \in \mathbb{N}$ such that

$$\lim_{m \to \infty} \frac{1}{m}|\,\{p_k \leq m : \rho(z_{p_k}, x_{p_l}) \geq \varepsilon\}\,| = 0.$$

Lemma 3.4. *In a space (X, ρ), for every sequence $\{x_n\}$,*

$$\{n \in \mathbb{N} : \rho\,(x_n, x_m) \geq \varepsilon\} \subseteq \left\{n \in \mathbb{N} : \rho\,(x_n, x_0) \geq \tfrac{\varepsilon}{2}\right\} \ \textit{whenever}\ \rho\,(x_m, x_0) < \tfrac{\varepsilon}{2}.$$

P r o o f. For any $x_n, x_m, x_0 \in X$, we have, $\rho\,(x_n, x_m) \leq \rho\,(x_n, x_0) + \rho\,(x_m, x_0)$. Now let $n_0 \in \{n \in \mathbb{N} : \rho\,(x_n, x_m) \geq \varepsilon\}$, i.e., $\varepsilon \leq \rho\,(x_{n_0}, x_m) \leq \rho\,(x_{n_0}, x_0) + \rho\,(x_m, x_0)$. So $\rho\,(x_{n_0}, x_0) > \frac{\varepsilon}{2}$. Therefore, $n_0 \in \left\{n \in \mathbb{N} : \rho\,(x_n, x_0) \geq \frac{\varepsilon}{2}\right\}$. Hence the theorem. □

Theorem 3.5. *If $\{x_n\}$ and $\{y_n\}$ are two I-statistically Cauchy sequences in a linear metric space (X, d). Then*

(i) *For any $a \in \mathbb{R}$, $\{ax_n\}$ is also an I-statistically Cauchy sequence.*

(ii) *$\{x_n + y_n\}$ is also an I-statistically Cauchy sequence.*

Proof.

(i) If $a = 0$, then $\{ax_n\}$ is a constant sequence and hence an I-statistically Cauchy sequence. So we take that a not to be 0.

Here, $\frac{1}{n}|\{i \le n : d(ax_i, ax_m) \ge \varepsilon\}| = \frac{1}{n}|\{i \le n : |a|d(x_i, x_m) \ge \varepsilon\}| \le \frac{1}{n}|\{i \le n : d(x_i, x_m) \ge \frac{\varepsilon}{|a|}\}| < \delta$. (Since $\{x_n\}$ is an I-statistically Cauchy sequence.) Therefore, $\big\{n \in \mathbb{N} : \frac{1}{n}|\,\{i \le n : d(ax_i, ax_m) \ge \varepsilon\}\,| < \delta\big\} \in \mathcal{F}(I)$, i.e., $\{ax_n\}$ is also an I-statistically Cauchy sequence.

(ii) Let $\{x_n\}$ and $\{y_n\}$ be two I-statistically Cauchy sequences. Therefore, for every $\varepsilon, \delta > 0$ there exits $m, k \in \mathbb{N}$ such that

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{i \le n : d(x_i, x_m) \ge \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$$

and

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{i \le n : d(y_i, y_k) \ge \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I).$$

Since $(A_1 \cap A_2) \in \mathcal{F}(I)$ and $\phi \notin \mathcal{F}(I)$, therefore $(A_1 \cap A_2) \ne \phi$ and for all $n \in (A_1 \cap A_2)$, we have

$$\begin{aligned} &\frac{1}{n}\left|\{i \le n : d\left(x_i + y_i, x_m + y_k\right) \ge \varepsilon\}\right| \\ &\le \frac{1}{n}\left|\left\{i \le n : d\left(x_i, x_m\right) \ge \frac{\varepsilon}{|2|}\right\}\right| \\ &\quad + \frac{1}{n}\left|\left\{i \le n : d\left(y_i, y_k\right) \ge \frac{\varepsilon}{|2|}\right\}\right| < \delta. \end{aligned}$$

Therefore, $\{n \in \mathbb{N} : \frac{1}{n}|\,\{i \le n : d(x_i + y_i, x_m + y_k) \ge \varepsilon\}\,| < \delta\} \in \mathcal{F}(I)$, i.e., $\{x_n + y_n\}$ is also an I-statistically Cauchy sequence. □

Theorem 3.6. *If $\{x_n\}$ is a sequence which satisfies I^*-statistically Cauchy criteria, then it is I-statistically Cauchy sequence.*

Proof. Let $\{x_n\}$ be a sequence that satisfies I^*-statistically Cauchy criteria. Therefore, there exists a set $P = \{p_1 < p_2 < \cdots < p_k < \cdots\} \subset \mathbb{N}$ and $P \in \mathcal{F}(I)$ and $p_q = \big(p_q(\varepsilon > 0)\big) \in P$ such that

$$\lim_{n\to\infty} \frac{1}{n}|\{p_k \le n : d(x_{p_k}, x_{p_q}) \ge \varepsilon\}| = 0.$$

Therefore, $\left\{n \in \mathbb{N} : \frac{1}{n}|\{p_k \leq n : d(x_{p_k}, x_{p_q}) \geq \varepsilon\}| \geq \delta\right\} \in I$, for every $\delta > 0$ where I is an admissible ideal. Now if $L = \mathbb{N} \setminus P$, then $L \in I$ and

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(x_k, x_{p_q}) \geq \varepsilon\}| \geq \delta\right\} \subset$$
$$L \cup \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{p_k \leq n : d\left(x_{p_k}, x_{p_q}\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \in I,$$

i.e., $\{x_n\}$ is I-statistically Cauchy sequence. □

Example. The converse implication of the Theorem 3.6 may not hold.

Consider the real number space $\mathbb{R}$ with usual metric d. Suppose $\mathbb{N} = \cup_{j=1}^{\infty} \triangle_j$ is a decomposition of $\mathbb{N}$ such that each $\triangle_j \subseteq \mathbb{N}$ is infinite and $\triangle_i \cap \triangle_j = \emptyset$ whenever $i \neq j$. Let

$$I = \left\{B \subset \mathbb{N} : \{\Delta_i : B \cap \Delta_i \neq \emptyset\} \quad \text{is finite}\right\}.$$

Then I comes out to be an admissible ideal of $\mathbb{N}$. In the mentioned space, the sequence $\{z_n = \frac{1}{n}\}_{n\in\mathbb{N}}$ satisfies the Cauchy criteria. We construct a sequence $\{(a_n\}$ where $a_n = \frac{1}{j}$ if $n \in \triangle_j$. For every $\varepsilon > 0$, there must exist a $k \in \mathbb{N}$ such that $d\left(\frac{1}{n}, \frac{1}{m}\right) < \varepsilon$ whenever $n, m \geq k$, i.e.,

$$\{n \in \mathbb{N} : d(a_n, a_m) \geq \varepsilon\} \subset \triangle_1 \cup \cdots \cup \triangle_k \in I.$$

Now,

$$\frac{1}{n}|\{k \leq n : d(a_k, a_m) \geq \varepsilon\}| \leq |\{n \in \mathbb{N} : d(a_n, a_m) \geq \varepsilon\}|.$$

For any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(a_k, a_m) \geq \varepsilon\}| \geq \delta\right\} \subseteq \{n \in \mathbb{N} : d(a_n, a_m) \geq \varepsilon\} \in I.$$

Therefore, $\{a_n\}$ is I-statistically Cauchy sequence.

If acceptable, assume that $\{a_n\}$ satisfies I^*-statistically Cauchy criteria. Then there is a $B \in \mathcal{F}(I)$ such that $\{a_n\}_{n\in B}$ is statistically Cauchy. Since $\mathbb{N} \setminus B \in I$, there exists a $c \in \mathbb{N}$ such that $\mathbb{N} \setminus B \subset \triangle_1 \cup \ldots \cup \triangle_c$. Therefore, $\triangle_{c+1}, \triangle_{c+2} \subset B$. Now if we take $m_k \in \triangle_{c+1}$ and $m_p \in \triangle_{c+2}$, then $m_k, m_p \in B$ and

$$\lim_{n\to\infty} \frac{1}{n}|\{m_k \leq n : d(a_{m_k}, a_{m_p}) \geq \varepsilon_0\}| = 2^{-(c+1)} > 0,$$

where

$$\varepsilon_0 = \frac{1}{3\,(c+1)\,(c+2)} > 0.$$

This contradicts the statistical Cauchyness of $\{a_n\}_{n\in B}$.

Theorem 3.7. *For an admissible ideal I having the property (AP), the notions of I^*-statistically Cauchy and I-statistically Cauchy sequences are synonymous.*

P r o o f. According to Theorem 3.6, I^*-statistical Cauchyness of a sequence implies its I-statistical Cauchyness. So we have to show that $\{a_n\}$ is I^*-statistically Cauchy sequence if we take it to be an I-statistically Cauchy sequence.

Suppose $\{a_n\}$ is an I-statistical Cauchy sequence. Then for every ε, $\delta > 0$ there exists a c such that $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(a_k, a_c) \geq \varepsilon\}| \geq \delta\right\} \in I$. We take $\delta_i = \frac{1}{i}$ for $i \in \mathbb{N}$ and let $T_i = \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(a_k, a_c) \geq \varepsilon\}| < \delta_i\right\}$, $i = 1, 2, \dots$ Now

$$\begin{aligned} T_i^c &= \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(a_k, a_c) \geq \varepsilon\}| < \delta_i\right\}^c \\ &= \left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(a_k, a_c) \geq \varepsilon\}| \geq \delta_i\right\} \in I, \end{aligned}$$

for all $i \in \mathbb{N}$. So, $T_i \in F(I)$ for $i = 1, 2, \dots$ Since I satisfies the (AP) criteria, by Lemma 2.5, there exists a set $T \subset \mathbb{N}$ and $T \setminus T_i$ is finite for all i. If $n \in T$, then $T \setminus T_j$ is finite, so there exists $n_0 = n_0(j)$ such that $n \in T_j$ and for all $n > n_0$, $\frac{1}{n}|\{k \leq n : d(a_k, a_{c_j}) \geq \varepsilon\}| < \frac{1}{j}$. Therefore,

$$\frac{1}{n}|\{m_k \leq n : d(a_{m_k}, a_{c_j}) \geq \varepsilon\}| \leq \frac{1}{n}|\{k \leq n : d(a_k, a_{c_j}) \geq \varepsilon\}| < \frac{1}{j}.$$

Now for a large j, we have,

$$\lim_{n\to\infty} \frac{1}{n}|\{m_k \leq n : d(a_{m_k}, a_{c_j}) \geq \varepsilon\}| = 0,$$

i.e., $\{a_n\}_{n \in T}$ is a statistically Cauchy sequence. Hence the theorem. □

Theorem 3.8. *In a metric space, every I-statistically convergent sequence satisfies the I-statistically Cauchy criteria.*

P r o o f. Let $\{a_n\}$ be I-statistically convergent to a_0 in a space X. So for any $\varepsilon, \delta > 0$, $\left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(a_k, a_0) \geq \varepsilon\}| \geq \delta\right\} \in I$. Let $m_0 \in \left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(a_k, a_p) \geq \varepsilon\}| \geq \delta\right\}$. Therefore, $\frac{1}{m_0}|\{k \leq m_0 : \rho(a_k, a_m) \geq \varepsilon\}| \geq \delta$. Using Lemma 3.4 we get,

$$\left\{k \leq m_0 : \rho(a_k, a_0) \geq \frac{\varepsilon}{2}\right\} \supseteq \{k \leq m_0 : \rho(a_k, a_m) \geq \varepsilon\}.$$

This implies that

$$\frac{1}{m_0}\left|\left\{k \leq m_0 : \rho(a_k, a_0) \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{1}{m_0}|\{k \leq m_0 : \rho(a_k, a_m) \geq \varepsilon\}| \geq \delta,$$

i.e.,

$$m_0 \in \left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(a_k, a_0) \geq \varepsilon\}| \geq \delta\right\},$$

i.e.,

$$\begin{aligned} &\left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(a_k, a_m) \geq \varepsilon\}| \geq \delta\right\} \\ &\qquad \subseteq \left\{m \in \mathbb{N} : \frac{1}{m}|\{k \leq m : \rho(a_k, a_0) \geq \varepsilon\}| \geq \delta\right\} \in I. \end{aligned}$$

Hence $\{a_n\}$ is I-statistically Cauchy sequence. □

Example. The converse implication of the Theorem 3.8 might not hold.

Consider $\{x_n\}$ where

$$x_n = \begin{cases} 0.9 & \text{for} \quad n = k^2, \ k \in \mathbb{N}, \\ 1/n & \text{for} \quad n \neq k^2 \ \text{ for any } \ k \in \mathbb{N}. \end{cases}$$

in the metric space (X, d), where $X = (0, 1)$ and $d(x, y) = |x-y|$. Also, let $I = I_\delta$, the collection of all subsets of $\mathbb{N}$ with zero density. Here $\{x_{n_p} : n_p \neq k^2, k \in \mathbb{N}\}$ forms a sub-sequence of $\{x_n\}$ with the set $\{n_p : n_p \neq k^2, k \in \mathbb{N}\} \in \mathcal{F}(I)$. Now, for given $\epsilon > 0$, there must be an $\eta \in \mathbb{N}$ such that $\frac{\eta}{2} < \epsilon$. Therefore, for $m_k, m_p > \eta$, $|\frac{1}{m_k} - \frac{1}{m_p}| \leq \frac{1}{m_k} + \frac{1}{m_p} < \frac{\eta}{2} < \epsilon$, i.e., $\{m_k \in \mathbb{N} : d(x_{m_k}, x_{m_p}) \geq \epsilon\}$ has density zero being a finite set. So

$$\left\{x_{n_p} : n_p \neq k^2, k \in \mathbb{N}\right\}$$

satisfies the statistically Cauchy criteria, i.e., $\{x_n\}$ is an I^*-statistically Cauchy sequence. Therefore, by Theorem 3.6 $\{x_n\}$ is an I-statistically Cauchy sequence.

Again, $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(x_k, l) \geq \varepsilon\}| \geq \delta\right\} \in I$ if and only if $l = 0$. But $0 \notin X$. So there does not exist any $l \in X$ to which $\{x_n\}$ can converge I-statistically.

From [6, 8, 11] and [5], we have the following knowledge:

(1) Every statistically convergent sequence satisfies the statistically Cauchy criteria. The converse implication might not hold.

(2) Every statistically convergent sequence is an I-convergent sequence. The converse implication might not hold.

(3) Every statistically Cauchy sequence is an I-Cauchy sequence. The converse implication might not hold.

(4) Every I-convergent sequence satisfies I-Cauchy criteria. The converse implication might not hold.

(5) Every I-convergent sequence is an I-statistically convergent sequence. The converse implication might not hold.

Now, based on our study, we can have the following relationship diagram shown in Fig. 1.

4. On I-statistically concurrent sequences

Theorem 4.1. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences that satisfy the I-statistically Cauchy criteria in a space (X, d). Then $\{z_n = d(x_n, y_n)\}$ also satisfies the I--statistically Cauchy criteria in a space (X, d_1) where $d_1(a, b) = |a - b|$.*

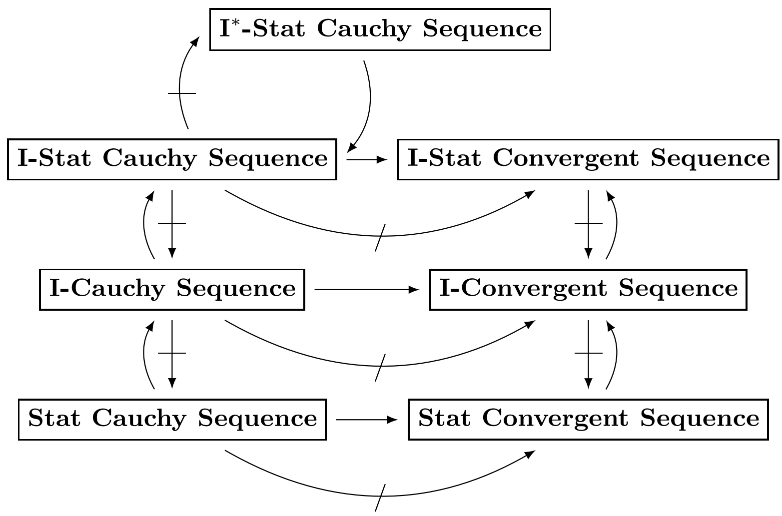

FIGURE 1. Relation diagram.

P r o o f. Let $\{x_n\}$ and $\{y_n\}$ be two I-statistically Cauchy sequences, therefore,

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_m) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$$

and

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(y_k, y_m) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I).$$

Since $(A_1 \cap A_2) \in \mathcal{F}(I)$ and $\phi \notin \mathcal{F}(I)$, therefore $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$ we have

$$\begin{aligned} &\frac{1}{n}|\{k \leq n : |z_k - z_m| \geq \varepsilon\}| \\ &\leq \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_m) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{n}\left|\left\{k \leq n : d(y_k, y_m) \geq \frac{\varepsilon}{2}\right\}\right| < \delta, \end{aligned}$$

i.e., $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |z_k - z_m| \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$.

Therefore, $\{z_n\}$ satisfies an I-statistically Cauchy criteria in the space (X, d_1). □

DEFINITION 4.2. $\{x_n\}$ and $\{y_n\}$ are said to be I-statistically concurrent to each other if the sequence $\{z_n\} = \{d(x_n, y_n)\}$ is I-statistically convergent to zero, i.e., $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |z_k| \geq \varepsilon\}| \geq \delta\right\} \in I$.

Theorem 4.3. *Let $\{x_n\}$ and $\{y_n\}$ be two I-statistically concurrent sequences. Then*

(i) *If $\{x_n\}$ is an I-statistically Cauchy sequence, $\{y_n\}$ is also an I-statistically Cauchy sequence.*

(ii) *Both are I-statistically convergent to the same limit.*

Proof.

(i) Since $\{x_n\}$ and $\{y_n\}$ are I-statistically concurrent, therefore,

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(x_k, y_k) \geq \frac{\varepsilon}{3}\right\}\right| < \frac{\delta}{3}\right\} \in \mathcal{F}(I).$$

Also, $\{x_n\}$ is an I-statistically Cauchy sequence, therefore,

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_m) \geq \frac{\varepsilon}{3}\right\}\right| < \frac{\delta}{3}\right\} \in \mathcal{F}(I).$$

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$, we have

$$\frac{1}{n}|\{k \leq n : d(y_k, y_m) \geq \varepsilon\}| \leq \frac{1}{n}\left|\left\{k \leq n : d(y_k, x_k) \geq \frac{\varepsilon}{3}\right\}\right|$$
$$+ \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_m) \geq \frac{\varepsilon}{3}\right\}\right| + \frac{1}{n}\left|\left\{m \leq n : d(x_m, y_m) \geq \frac{\varepsilon}{3}\right\}\right| < \delta,$$

i.e., $\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(y_k, y_m) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$ as $(A_1 \cap A_2) \in \mathcal{F}(I)$. Therefore, $\{y_n\}$ is also a I-statistically Cauchy sequence.

(ii) Since $\{x_n\}$ and $\{y_n\}$ are I-statistically concurrent, therefore,

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(x_k, y_k) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I).$$

Also, let $\{x_n\}$ be I-statistically convergent to the limit l, therefore,

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : d(x_k, x_m) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \in \mathcal{F}(I).$$

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$, we have

$$\frac{1}{n}|\{k \leq n : d(y_k, l) \geq \varepsilon\}|$$
$$\leq \frac{1}{n}\left|\left\{k \leq n : d(y_k, x_k) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{n}\left|\left\{k \leq n : d(x_k, l) \geq \frac{\varepsilon}{2}\right\}\right| < \delta,$$

i.e.,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : d(y_k, l) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I) \text{ as } (A_1 \cap A_2) \in \mathcal{F}(I).$$

Therefore, $\{y_n\}$ is also an I-statistically convergent to the same limit l. □

Theorem 4.4. *Let S_X be a collection of all sequences of a space (X, d). Then the I-statistically concurrent relation $(\approx_d)$ forms an equivalence relation on S_X.*

Proof. Since for any $\{x_n\} \in S_X$, $d(x_n, x_n) = 0$, for all $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_n)\}$ is I-statistically convergent to 0. So every sequence is I-statistically concurrent to itself, i.e., the I-statistically concurrent relation $(\approx_d)$ is a reflexive relation on S_X. Since for any $\{x_n\}, \{y_n\} \in S_X$, $d(x_n, y_n) = d(y_n, x_n)$, for all $n \in \mathbb{N}$. Therefore, if $\{x_n\}$ is I-statistically concurrent to $\{y_n\}$ then $\{y_n\}$, is also I-statistically concurrent to $\{x_n\}$, i.e., the I-statistically concurrent relation $(\approx_d)$ is a symmetric relation on S_X.

Since for any $\{x_n\}, \{y_n\}, \{v_n\} \in S_X$, $d(x_n, v_n) \leq d(x_n, y_n) + d(y_n, v_n)$, for all $n \in \mathbb{N}$. Therefore, if $\{d(x_n, y_n)\}$ and $\{d(y_n, v_n)\}$ is I-statistically convergent to 0, then $\{d(x_n, v_n)\}$ is also I-statistically convergent to 0. So if $\{x_n\}$ is I-statistically concurrent to $\{y_n\}$ and $\{y_n\}$ is I-statistically concurrent to $\{v_n\}$, then $\{x_n\}$ is also I-statistically concurrent to $\{v_n\}$, i.e., the I-statistically concurrent relation $(\approx_d)$ is a transitive relation on S_X. Therefore, the I-statistically concurrent relation $(\approx_d)$ forms an equivalence relation on S_X. □

Corollary 4.5. *The collection S_X of all sequences of space (X, d) splits into disjoint equivalent classes so that all the sequences of one class are:*

(i) *Either I-statistically convergent to the same limit or has no limit at all.*

(ii) *Either I-statistically Cauchy sequences or none of them are Cauchy.*

Proof. This can be easily verified from Theorems 4.3 and 4.4. □

5. Conclusion

In this paper, we introduce I-statistically Cauchy and I^*-statistically Cauchy sequences and examine various aspects of them. Also, included a relationship diagram based on the findings. We also define the I-statistically concurrent relation, which forms an equivalence relation. By generating the separate equivalent classes, this relation will simplify the investigation of the nature of I-statistically Cauchy sequences.

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Debjani Rakshit
Prasenjit Bal
Department of Mathematics
Faculty of Science & Technology
ICFAI University Tripura
Kamalghat, Agartala INDIA-799210
E-mail: debjanirakshit@gmail.com
balprasenjit177@gmail.com

Shyamal Debnath
Department of Mathematics
Tripura University
Suryamaninagar, Agartala INDIA-799022
E-mail: shyamalnitamath@gmail.com