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VARIATIONAL MCSHANE AND PETTIS INTEGRALS OF MULTIFUNCTIONS

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ABSTRACT. In this paper, we present full characterizations of variationally McShane and Pettis integrable multifunctions in terms of the cubic derivative and the variational McShane measure of additive interval multifunctions.

1. Introduction and Preliminaries

In the paper [14], full characterizations of a strongly McShane integrable (variationally McShane integrable) function $f : W \subset \mathbb{R}^m \to X$ are given in terms of the cubic derivative of additive interval functions, where X is a Banach space and W is a compact non-degenerate subinterval of $\mathbb{R}^m$. In the monograph [24], Štefan Schwabik and Ye Guoju have proved a full characterization of the strongly McShane integrable functions defined on a compact non-degenerate subinterval of $\mathbb{R}$ and taking values in a Banach space, Theorem 7.4.14. There are also full descriptive characterizations of the variational McShane integral of Banach-space valued functions in [28] and [15]. In the paper [18], Valeria Marraffa has proved some characterizations of the strong McShane integral of functions taking values in locally convex spaces.

In this paper, we first define the cubic derivative and the variational McShane measure of additive interval multifunctions. Then, we present characterizations of variational McShane and Pettis integrals of multifunctions $\Gamma : W \to bcc(X)$

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($cwk(X)$) in terms of the cubic derivative and the variational McShane measure of additive interval multifunctions.

Throughout this paper, X is an arbitrary Banach space with its dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. We denote by 2^X the family of all non-empty subsets of X and by $bcc(X)$ ($cwk(X)$) we denote the subfamily of 2^X of all bounded closed convex (convex weakly compact) subsets of X. We consider on $bcc(X)$ the Minkowski addition

$$A \oplus B = \overline{\{a + b : a \in A, b \in B\}},$$

and the standard multiplication by scalars (in particular, $A \oplus B = \{a + b : a \in A, b \in B\}$ for every $A, B \in cwk(X)$). We denote by $\mathcal{H}$ the subfamily of 2^X of all bounded closed subsets of X. By [4, Theorem II.3], $\mathcal{H}$ is a complete metric space with the Hausdorff distance, given by

$$d_{\mathcal{H}}(A, B) = \max\{e(A, B), e(B, A)\},$$

where

$$e(A, B) = \sup_{a \in A} \operatorname{dist}(a, B), \quad \operatorname{dist}(a, B) = \inf\{||a - b|| : b \in B\}.$$

It is easy to see that $bcc(X)$ ($cwk(X)$) is a closed subspace of the complete metric space $(\mathcal{H}, d_{\mathcal{H}})$. For each $C \in 2^X$ the *support* of C is defined by equality

$$\sigma(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}.$$

Let $\alpha = (a_1, \ldots, a_m)$ and $\beta = (b_1, \ldots, b_m)$ with $-\infty < a_j < b_j < +\infty$, for $j = 1, \ldots, m$. The set $[\alpha, \beta] = \prod_{j=1}^{m} [a_j, b_j]$ is called a *closed non-degenerate interval* in $\mathbb{R}^m$. If $b_1 - a_1 = \ldots = b_m - a_m$, then $I = [\alpha, \beta]$ is called a *cube* and we set $l_I = b_1 - a_1$. We denote by W a compact non-degenerate subinterval of the m-dimensional Euclidean space $\mathbb{R}^m$ and by $\mathcal{I}$ the family of all closed non-degenerate subintervals of W is denoted. The Euclidean space $\mathbb{R}^m$ is equipped with the maximum norm. $B_m(t, r)$ is the open ball in $\mathbb{R}^m$ with center t and radius $r > 0$. ∂B and B^o are *boundary* and *interior* of a subset $B \subset \mathbb{R}^m$. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of W and by $\mathscr{B}$ is denoted the family of all Borel subsets of W. $|E|$ is the Lebesgue measure of a Lebesgue measurable set $E \in \mathcal{L}$. Thus, if I is a cube, then $|I| = (l_I)^m$. The word "at almost all" is always referred to the Lebesgue measure λ on W.

A map $\Gamma : W \to 2^X$ is called a *multifunction* and a map $\Phi : \mathcal{I} \to 2^X$ is said to be an *interval multifunction*. A function $f : W \to X$ is said to be a *selection* of a multifunction $\Gamma : W \to 2^X$ if $f(t) \in \Gamma(t)$ for each $t \in W$. We say that an interval multifunction $\Phi : \mathcal{I} \to bcc(X)$ is an *additive interval multifunction*, if for each two non-overlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$ we have $\Phi(I \cup J) = \Phi(I) \oplus \Phi(J)$. Two intervals I and J are said to be *non-overlapping* if $I^o \cap J^o = \emptyset$. An additive interval function $\varphi : \mathcal{I} \to X$ is said to be a *selection* of an additive interval multifunction $\Phi : \mathcal{I} \to bcc(X)$ if $\varphi(I) \in \Phi(I)$ for every $I \in \mathcal{I}$.

We denote by $\mathcal{S}_\Phi$ the family of all selections of Φ. Thanks to [2, Theorem 3.6.1], the following lemma can be proved in a very similar way to [11, Proposition 2.1].

Lemma 1.1. *If $\Phi : \mathcal{I} \to cwk(X)$ is an additive interval multifunction, then $\mathcal{S}_\Phi \neq \emptyset$.*

The following embedding result will be useful to us (see [4, Theorems II. 18 and II. 19]).

Theorem 1.2. *Let $\ell_\infty(B_{X^*})$ be the Banach space of all bounded real valued functions defined on B_{X^*} endowed with the supremum norm $||\cdot||_\infty$. Then, the map*

$$i : bcc(X) \to \ell_\infty(B_{X^*}), \quad i(C) = \sigma(\cdot, C)$$

satisfies the following properties:

(i) $i(A \oplus B) = i(A) + i(B)$ *for every* $A, B \in bcc(X)$,

(ii) $i(\alpha A) = \alpha . i(A)$ *for every* $\alpha \geq 0$ *and every* $A \in bcc(X)$,

(iii) $d_{\mathcal{H}}(A, B) = ||i(A) - i(B)||_\infty$ *for every* $A, B \in bcc(X)$,

(iv) $i(bcc(X))$ *is closed in* $\ell_\infty(B_{X^*})$.

Definition 1.3. We say that an additive interval multifunction $\Phi : \mathcal{I} \to bcc(X)$ is sAC, if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that for every finite collection π of pairwise non-overlapping intervals in $\mathcal{I}$, we have

$$\sum_{I \in \pi} |I| < \eta_\varepsilon \Rightarrow \sum_{I \in \pi} d_{\mathcal{H}}\big(\Phi(I), \{\theta\}\big) < \varepsilon,$$

where θ is the zero vector in X. Replacing the last inequality with

$$d_{\mathcal{H}}\left(\bigoplus_{I \in \pi} \Phi(I), \{\theta\}\right) < \varepsilon,$$

we obtain the notion AC for Φ.

Definition 1.4. Given a point $t \in W$, we set

$$\mathcal{I}(t) = \{I \in \mathcal{I} : t \in I, I \text{ is a cube}\}.$$

We say that an additive interval function $\varphi : \mathcal{I} \to X$ has the *cubic derivative* at the point t, if there exists a vector $\varphi'_c(t) \in X$ such that

$$\lim_{\substack{|I| \to 0 \\ I \in \mathcal{I}(t)}} ||\Delta\varphi(t, I) - \varphi'_c(t)|| = 0, \quad \left(\Delta\varphi(t, I) = \frac{\varphi(I)}{|I|}\right).$$

$\varphi'_c(t)$ is said to be the *cubic derivative* of φ at t.

We say that the additive interval multifunction $\Phi : \mathcal{I} \to bcc(X)$ has the *cubic derivative* in $bcc(X)$ $(cwk(X))$ at the point t, if there exists $\Phi'_c(t) \in bcc(X)$ $(cwk(X))$ such that

$$\lim_{\substack{|I|\to 0 \\ I\in\mathcal{I}(t)}} d_{\mathcal{H}}\big(\Delta\Phi(t,I), \Phi'_c(t)\big) = 0, \quad \left(\Delta\Phi(t,I) = \frac{\Phi(I)}{|I|}\right).$$

Given a sequence (B_n) of non-empty subsets of X, we write $\sum_n B_n$ to denote the set of all elements of X which can be written as the sum of an unconditionally convergent series $\sum_n x_n$, where $x_n \in B_n$ for every $n \in \mathbb{N}$. It is known that if $B_n \in cwk(X)$ for all $n \in \mathbb{N}$, then $\sum_n B_n$ is unconditionally convergent if and only if the series $\sum_n i(B_n)$ is unconditionally convergent in the Banach space $\ell_\infty(B_{X^*})$ (in this case, $i\left(\sum_n B_n\right) = \sum_n i(B_n)$), cf. [5, Lemma 2.3].

Definition 1.5. A mapping $M : \mathcal{L} \to 2^X$ is said to be a *strong multimeasure* if:

(i) $M(\emptyset) = \{\theta\}$,

(ii) for each sequence (E_n) of pairwise disjoint members of $\mathcal{L}$, we have

$$M\left(\bigcup_n E_n\right) = \sum_n M(E_n).$$

The mapping $M : \mathcal{L} \to 2^X$ is said to be a *weak multimeasure* or simple *multimeasure*, if for each $x^* \in X^*$, the function $\sigma\big(x^*, M(\cdot)\big)$ is a signed measure on $\mathbb{R}\cup\{+\infty\}$. If $M : \mathcal{L} \to bcc(X)$ is countably additive in the Hausdorff distance, then it is called an *h-multimeasure*. Thanks to [12, Proposition 4.7, p. 851], if $M : \mathcal{L} \to bcc(X)$ is a strong multimeasure, then M is an h-multimeasure and, if M is an h-multimeasure, then M is a multimeasure. The three definitions are equivalent in the case of $cwk(X)$-valued multimeasures (see [10, Proposition 3] and [12, Proposition 4.10, p. 852]). A multimeasure $M : \mathcal{L} \to bcc(X)$ is said to be λ-*continuous*, if $M(Z) = \{\theta\}$ whenever $Z \subset W$ satisfies $|Z| = 0$.

Definition 1.6. A multifunction $\Gamma : W \to bcc(X)$ is called *Pettis integrable* in $bcc(X)$ $(cwk(X))$ if:

(i) $\sigma\big(x^*, \Gamma(\cdot)\big)$ is Lebesgue integrable for every $x^* \in X^*$,

(ii) for each $E \in \mathcal{L}$, there is $C_E \in bcc(X)$ $(cwk(X))$ such that

$$\sigma\left(x^*, C_E\right) = \int_E \sigma\big(x^*, \Gamma(t)\big) d\lambda \quad \text{for every } x^* \in X^*.$$

We call C_E the *Pettis integral* of Γ over E and set $(P)\int_E \Gamma(t) d\lambda = C_E$.

The Pettis integral for multifunctions was first considered by Castaing and Valadier [4, Chapter V] and has been widely studied in papers [6, 20, 21] and [9].

The notion of Pettis integrable function $f : W \to X$ as can be found in the literature (see [7], [19], [24] and [25]) corresponds to Definition 1.6 for $\Gamma(t) = \{f(t)\}$ when the integral $(P) \int_E \Gamma(t)d\lambda$ is a singleton. For the definition and the properties of Bochner integral, we refer to [7] and [24].

A pair (I, t) of an interval $I \in \mathcal{I}$ and a point $t \in W$ is called an *$\mathcal{M}$-tagged interval* in W. A finite collection $\{(I_i, t_i) : i = 1, \dots, p\}$ of $\mathcal{M}$-tagged intervals in W is called an *$\mathcal{M}$-partition* in W, if $\{I_i \in \mathcal{I} : i = 1, \dots, p\}$ is a collection of pairwise non-overlapping intervals. A positive function $\delta : E \subset W \to (0, +\infty)$ is called a *gauge* on E. We say that an $\mathcal{M}$-partition π in W is

- a partition of W if $\bigcup_{(I,t)\in\pi} I = W$,
- E-tagged if $(I, t) \in \pi$ implies $t \in E$,
- δ-fine if for each $(I, t) \in \pi$, we have $I \subset B_m\big(t, \delta(t)\big)$.

Definition 1.7. A multifunction $\Gamma : W \to bcc(X)$ is said to be McShane integrable in $bcc(X)$ ($cwk(X)$) if there exists a set $\Phi_\Gamma \in bcc(X)$ ($cwk(X)$) with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$d_{\mathcal{H}} \left(\Phi_\Gamma, \bigoplus_{(I,t)\in\pi} \Gamma(t)|I| \right) < \varepsilon.$$

We then write $(M) \int_W \Gamma(t)d\lambda = \Phi_\Gamma$. The multifunction Γ is said to be McShane integrable in $bcc(X)$ $\big(cwk(X)\big)$ over $E \in \mathcal{L}$, if the multifunction $\Gamma \mathbb{1}_E$ is McShane integrable in $bcc(X)$ ($cwk(X)$), where $\mathbb{1}_E$ is the characteristic function of E. In this case, we write

$$(M) \int_E \Gamma(t)d\lambda = (M) \int_W \Gamma(t)\mathbb{1}_E(t)d\lambda.$$

Definition 1.8. A multifunction $\Gamma : W \to bcc(X)$ is said to be variationally McShane integrable in $bcc(X)$ ($cwk(X)$) if there is an additive interval multifunction $\Phi : \mathcal{I} \to bcc(X)$ ($\Phi : \mathcal{I} \to cwk(X)$) with the following property: for every $\varepsilon > 0$ there exists a gauge δ on W such that for every δ-fine $\mathcal{M}$-partition π of W we have

$$\sum_{(I,t)\in\pi} d_{\mathcal{H}}(\Phi(I), \Gamma(t)|I|) < \varepsilon.$$

In this case, the additive interval multifunction Φ is called the variational McShane primitive of Γ.

When a multifunction is a function, then the above two definitions coincide with McShane integrability and variationally McShane integrability (or strongly McShane integrability) for vector valued functions, cf. [24, Definition 3.2.1 and Definition 3.6.2]. A function $f : W \to X$ is variationally McShane integrable with the primitive φ if and only if f is Bochner integrable (cf. [24, Theorem 5.1.4]). In this case,

$$\varphi(I) = (M)\int_I f(t)d\lambda = (B)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Since $bcc(X)$ $(cwk(X))$ is a closed subset of $(\mathcal{H}, d_H)$, we obtain by embedding theorem (Theorem 1.2) that a multifunction $\Gamma : W \to bcc(X)$ is McShane integrable (variationally McShane integrable) in $bcc(X)$ $(cwk(X))$ if and only if the function $\Gamma^\infty = i \circ \Gamma$ is McShane integrable (variationally McShane integrable).

The following result follows immediately from definitions.

Lemma 1.9. *Let $\Gamma : W \to bcc(X)$ be a multifunction which is variationally McShane integrable in $bcc(X)$ $(cwk(X))$ with the primitive Φ. Then, for each $I \in \mathcal{I}$ the multifunction Γ is McShane integrable in $bcc(X)$ $(cwk(X))$ over I and*

$$\Phi(I) = (M)\int_I \Gamma(t)d\lambda.$$

Definition 1.10. Given an additive interval function $\varphi : \mathcal{I} \to X$, a subset $E \subset W$ and a gauge δ on E, we write

$$V(\varphi, E, \delta) = \sup\left\{\sum_{(I,t)\in\pi} ||\varphi(I)|| : \pi \text{ is an } E\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W\right\},$$

and then,

$$V_\varphi(E) = \inf\{V(\varphi, E, \delta) : \ \delta \text{ is a gauge on } E\}.$$

The set function V_φ is said to be the *McShane variational measure* generated by additive interval function φ. The notions of variational measures are useful tools to study the primitives of real valued or, more in general, vector valued integrable functions (cf. [1], [8], [26] and [27]).

We now extend the notion of McShane variational measure of an additive interval function to an additive interval multifunction $\Phi : \mathcal{I} \to bcc(X)$. Given a subset $E \subset W$ and a gauge δ on E, we write

$$V(\Phi, E, \delta) = \sup\left\{\sum_{(I,t)\in\pi} d_{\mathcal{H}}(\Phi(I), \{\theta\}) : \pi \text{ is an } E\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W\right\},$$

and then,

$$V_\Phi(E) = \inf\{V(\Phi, E, \delta) : \ \delta \text{ is a gauge on } E\}.$$

The set function V_Φ is called the *McShane variational measure* generated by additive interval multifunction Φ. We say that V_Φ is absolutely continuous with respect to λ and we write $V_\Phi \ll \lambda$, if for every $Z \in \mathcal{L}$ we have

$$|Z| = 0 \Rightarrow V_\Phi(Z) = 0.$$

By embedding theorem (Theorem 1.2), it is easy to see that

$$V_{i\circ\Phi}(E) = V_\Phi(E) \quad \text{for each subset } \ E \subset W.$$

Definition 1.11. Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $t \in W^o$. We set $\mathcal{I}^o(t) = \{I \in \mathcal{I}(t) : t \in I^o\}$. If $I \in \mathcal{I}^o(t)$, then we write

$$\mathcal{I}^o(t, I) = \{J \in \mathcal{I}^o(t) : J \subset I\}$$

and define a partial ordering $\preceq_t$ on $\mathcal{I}^o(t)$ by saying that $I' \preceq_t I''$ if and only if $I' \supset I''$. Then, $(\mathcal{I}^o(t), \preceq_t)$ is a directed set. For the concepts of nets and subnets, we refer to [16]. We now define

$$L_\varphi(t) = \bigcap_{I\in\mathcal{I}^o(t)} \overline{L_\varphi(t, I)}^{\sigma(X,X^*)}, \tag{1.1}$$

where $L_\varphi(t, I) = \{\Delta\varphi(t, J) \in X : J \in \mathcal{I}^o(t, I)\}$ and $\overline{L_\varphi(t, I)}^{\sigma(X,X^*)}$ is the closure of $L_\varphi(t, I)$ with respect to the weak topology $\sigma(X, X^*)$. By [16, Theorem 7, p. 72], it follows that $L_\varphi(t)$ is the set of all weak limit points of the net $\big(\Delta\varphi(t, I)\big)_{I\in\mathcal{I}^o(t)}$.

2. The main results

We present full characterizations of variationally McShane and Pettis integrable multifunctions in terms of the cubic derivative and the variational McShane measure of additive interval multifunctions, see Theorem 2.4 and Theorem 2.6. Let us start with a few auxiliary lemmas.

Lemma 2.1. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $C \in \mathcal{I}(t)$. Assume that*

- *φ is sAC,*
- *$C \subset W^o$.*

Then, given $0 < \varepsilon < 1$, there exists $C_\varepsilon \in \mathcal{I}^o(t)$ with $C_\varepsilon \supset C$ such that

$$||\Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon)|| < \varepsilon.$$

Proof. Let us consider the case when t is a boundary point of C, since if $C \in \mathcal{I}^o(t)$, then $C_\varepsilon = C$. Since $C \subset W^o$ is a cube, there exist $a = (a_1, \dots, a_m) \in W^o$ and $r > 0$ such that $C = \prod_{i=1}^m [a_i - r, a_i + r]$. Thus, for each $s > 1$ we have $C(s) = \prod_{i=1}^m [a_i - r.s, a_i + r.s] \supset C$ and t is the interior point of $C(s)$.

Then, there exists $s_0 > 1$ with the following property: for each $1 < s < s_0$ there exists a finite collection $\pi(s) \subset \mathcal{I}$ of pairwise non-overlapping intervals with $J_{\pi(s)} = \bigcup_{J\in\pi(s)} J$ such that $C(s) = C \cup J_{\pi(s)}$ and $C^o \cap (J_{\pi(s)})^o = \emptyset$. Since φ is sAC, we obtain

$$\lim_{\substack{s\to 1\\ s>1}} \big\|\varphi\big(C(s)\big) - \varphi(C)\big\| = \lim_{\substack{s\to 1\\ s>1}} \left\| \sum_{J\in\pi(s)} \varphi(J) \right\| = 0,$$

and consequently,

$$\lim_{\substack{s\to 1\\ s>1}} \big\|\Delta\varphi\big(t, C(s)\big) - \Delta\varphi(t, C)\big\| = \lim_{\substack{s\to 1\\ s>1}} \left\| \frac{\varphi(C(s))}{|C(s)|} - \frac{\varphi(C)}{|C|} \right\| = 0.$$

The last result yields that there exists $s_\varepsilon > 1$ with $C \subset C(s_\varepsilon) \subset W$ such that

$$\big\|\Delta\varphi\big(t, C(s_\varepsilon)\big) - \Delta\varphi(t, C)\big\| < \varepsilon.$$

This means that $C_\varepsilon = C(s_\varepsilon)$ is the required cube and the proof is complete. □

The next lemma characterizes the cubic derivative in terms of a convergent net.

Lemma 2.2. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $t \in W^o$. Then, the following statements are equivalent:*

(i) *φ has the cubic derivative $\varphi'_c(t) = z$,*

(ii) *the net $\big(\Delta\varphi(t, I)\big)_{I\in\mathcal{I}^o(t)}$ converges to z.*

Proof.

(i)⇒(ii): Assume that (i) holds and let $\varepsilon > 0$. Then, there exists $\eta_\varepsilon > 0$ such that for each $I \in \mathcal{I}(t)$ we have

$$|I| < \eta_\varepsilon \Rightarrow ||\Delta\varphi(t, I) - z|| < \varepsilon.$$

Since $t \in W^o$, there exists $I_{\eta_\varepsilon} \in \mathcal{I}^o(t)$ such that $|I_{\eta_\varepsilon}| < \eta_\varepsilon$. Hence, for each $I \in \mathcal{I}^o(t) \subset \mathcal{I}(t)$ we have

$$I \subset I_{\eta_\varepsilon} \Rightarrow |I| \le |I_{\eta_\varepsilon}| < \eta_\varepsilon \Rightarrow ||\Delta\varphi(t, I) - z|| < \varepsilon.$$

This means that the net $\big(\Delta\varphi(t, I)\big)_{I\in\mathcal{I}^o(t)}$ converges to z.

(ii)⇒(i): Assume that (ii) holds and let $0 < \varepsilon < 1$. Then, there exists $I_0 \in \mathcal{I}^o(t)$ such that for each $I \in \mathcal{I}^o(t)$, we have

$$I_0 \preceq_t I \Rightarrow ||\Delta\varphi(t, I) - z|| < \frac{\varepsilon}{2}. \tag{2.1}$$

Since $t = (t_1, \ldots, t_m)$ is the interior point of I_0, there exists $r > 0$ such that $B_m(t, r) = \prod_{i=1}^m (t_i - r, t_i + r) \subset I_0$. Choose $0 < \eta_\varepsilon < r^m$ and fix an arbitrary cube $C \in \mathcal{I}(t)$ with $|C| < \eta_\varepsilon$. Since $l_C < r$, it follows that $C \subset B_m(t, r)$.

If $C \in \mathcal{I}^o(t)$, then $I_0 \preceq_t C$, and consequently, we obtain by (2.1) that

$$||\Delta\varphi(t, C) - z|| < \frac{\varepsilon}{2} < \varepsilon. \tag{2.2}$$

It remains to consider the case when t is a boundary point of C. Applying Lemma 2.1 with C and $\prod_{i=1}^m [t_i - r, t_i + r]$ instead of W, there exists $C_\varepsilon \in \mathcal{I}^o(t)$ with $C \subset C_\varepsilon \subset \prod_{i=1}^m [t_i - r, t_i + r]$ such that

$$||\Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon)|| < \frac{\varepsilon}{2},$$

and since $I_0 \preceq_t C_\varepsilon$, we obtain by (2.1) that

$$\begin{aligned} ||\Delta\varphi(t, C) - z|| &\leq ||\Delta\varphi(t, C) - \Delta\varphi(t, C_\varepsilon)|| + ||\Delta\varphi(t, C_\varepsilon) - z|| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since C was arbitrary, the last result together with (2.2) yields that for each $C \in \mathcal{I}(t)$, we have

$$|C| < \eta_\varepsilon \Rightarrow ||\Delta\varphi(t, C) - z|| < \varepsilon.$$

Then,

$$\lim_{\substack{|I| \to 0 \\ I \in \mathcal{I}(t)}} ||\Delta\varphi(t, I) - z|| = 0.$$

This means that $\varphi'_c(t)$ exists and $\varphi'_c(t) = z$, and this completes the proof. □

Lemma 2.3. *Let $\varphi : \mathcal{I} \to X$ be an additive interval function and let $f : W \to X$ be a function. Assume that*

- *φ is sAC,*
- *$f(t) \in L_\varphi(t)$ at almost all $t \in W^o$, where $L_\varphi(t)$ is defined by (1.1).*

Then, f is variationally McShane integrable with

$$\varphi(I) = (M)\int_I f(t)d\lambda \quad \textit{for every } I \in \mathcal{I}. \tag{2.3}$$

P r o o f. By hypothesis, there exists $Z \subset W$ with $|Z| = 0$ such that $f(t) \in L_\varphi(t)$ for all $t \in W^o \setminus Z$. We first prove that f is Pettis integrable. To see this, fix an arbitrary $x^* \in X^*$. Since $\langle x^*, \varphi\rangle$ is sAC, by [14, Lemma 2.4], there exists a Lebesgue integrable function $g : W \to \mathbb{R}$ such that

$$\langle x^*, \varphi(I)\rangle = \int_I g(t)d\lambda \quad \text{for every } I \in \mathcal{I}$$

and there exists $Z^{x^*} \subset W$ with $|Z^{x^*}| = 0$ such that

$$\lim_{\substack{|I| \to 0 \\ I \in \mathcal{I}(t)}} |\langle x^*, \Delta\varphi(t, I)\rangle - g(t)| = 0 \quad \text{for every } t \in W \setminus Z^{x^*}. \tag{2.4}$$

Hence, by Lemma 2.2 we obtain that the net $(\langle x^*, \Delta\varphi(t,I)\rangle)_{I\in\mathcal{I}^o(t)}$ converges to $g(t)$ for every $t \in W^o \setminus Z^{x^*}$, i.e.,

$$\lim_{I\in\mathcal{I}^o(t)} \langle x^*, \Delta\varphi(t,I)\rangle = g(t) \quad \text{for every } \ t \in W^o \setminus Z^{x^*}.$$

This means that

$$L_{\langle x^*,\varphi\rangle}(t) = \{g(t)\} \quad \text{for every } \ t \in W^o \setminus Z^{x^*},$$

and since $\langle x^*, L_\varphi(t,I)\rangle = L_{\langle x^*,\varphi\rangle}(t,I)$ for all $I \in \mathcal{I}^o(t)$, it follows that

$$\langle x^*, f(t)\rangle \in \langle x^*, L_\varphi(t)\rangle \subset L_{\langle x^*,\varphi\rangle}(t) = \{g(t)\} \quad \text{for all } \ t \in W^o \setminus (Z \cup Z^{x^*}).$$

The last result together with (2.4) yields

$$\lim_{\substack{|I|\to 0\\ I\in\mathcal{I}(t)}} \langle x^*, \Delta\varphi(t,I)\rangle = \langle x^*, f(t)\rangle \quad \text{for all } \ t \in W^o \setminus (Z \cup Z^{x^*}).$$

Since x^* was arbitrary and φ is sAC, by [14, Lemma 2.5] we obtain that f is Pettis integrable with

$$\varphi(I) = (P)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}. \tag{2.5}$$

By [14, Lemma 2.3] (see also [17, Theorem 2]), there exists a unique countable additive vector measure $m_\varphi : \mathcal{L} \to X$ such that m_φ is λ-continuous of bounded variation and $m_\varphi(I) = \varphi(I)$ for all $I \in \mathcal{I}$. Thus,

$$m_\varphi(E) = (P)\int_E f(t)d\lambda \quad \text{for every } \ E \in \mathcal{L}.$$

Thanks to [14, Lemma 2.2], the set $\varphi(\mathcal{I}) = \{\varphi(I) : I \in \mathcal{I}\}$ is a separable subset of X. If Y is the closed linear subspace spanned by $\varphi(\mathcal{I})$, then Y is also a separable subset of X. Note that by [22, Proposition 8, p. 34] or [23, Corollary 2, p. 65], we have $Y = \overline{Y} = \overline{Y}^{\sigma(X,X^*)}$, and since $\Delta\varphi(t,I) = \frac{\varphi(I)}{|I|} \in Y$ for all $I \in \mathcal{I}^o(t)$, we obtain that $f(t) \in Y$ at almost all $t \in W$. Thus, f is λ-essentially separably valued, and since $\langle x^*, f\rangle$ is measurable for all $x^* \in X^*$, by Pettis's Measurability Theorem (cf. [7, Theorem II.1.2, p. 42] it follows that f is measurable. Hence, using Remark 4.1 in [21], we obtain that

$$|m_\varphi|(W) = \int_W ||f(t)||d\lambda < +\infty.$$

Thus, the function $||f(\cdot)||$ is Lebesgue integrable. Therefore, by [7, Theorem II.2.2], the function f is Bochner integrable. Further, by [24, Proposition 2.3.1] and (2.5) we obtain

$$\varphi(I) = (B)\int_I f(t)d\lambda \quad \text{for every } \ I \in \mathcal{I}.$$

By [24, Theorem 5.1.4] we infer that f is variationally McShane integrable with the primitive φ satisfying (2.3) and the proof is complete. □

The first theorem characterize variational McShane integral of multifunctions taking values in the hyperspace $bcc(X)$.

Theorem 2.4. *Let $\Gamma : W \to bcc(X)$ be a multifunction and let $\Phi : \mathcal{I} \to bcc(X)$ be an additive interval multifunction. Then, the following statements are equivalent*:

(i) Γ *is variationally McShane integrable in* $bcc(X)$ *with the primitive* Φ,

(ii) Φ *is sAC*, $\Phi'_c(t)$ *exists and* $\Phi'_c(t) = \Gamma(t)$ *at almost all* $t \in W$,

(iii) $V_\Phi \ll \lambda$, $\Phi'_c(t)$ *exists and* $\Phi'_c(t) = \Gamma(t)$ *at almost all* $t \in W$.

Proof.

(i)⇒(ii): Assume that Γ is variationally McShane integrable in $bcc(X)$ with the primitive Φ. Then, $\Gamma^\infty = i \circ \Gamma$ is variationally McShane integrable with the primitive $\Phi^\infty = i \circ \Phi$. Hence, by [14, Theorem 2.8], the additive interval function Φ^∞ is sAC, $(\Phi^\infty)'_c(t)$ exists and $(\Phi^\infty)'_c(t) = \Gamma^\infty(t)$ at almost all $t \in W$. Since Φ^∞ is sAC and

$$||\Phi^\infty(I)||_\infty = \left\| i\big(\Phi(I)\big) - i(\{\theta\}) \right\|_\infty = d_{\mathcal{H}}\big(\Phi(I), \{\theta\}\big),$$

we obtain that Φ is sAC, and since $(\Phi^\infty)'_c(t) = \Gamma^\infty(t)$ at almost all $t \in W$, there exists a Lebesgue measurable set $Z \subset \mathcal{L}$ with $|Z| = 0$ such that $(\Phi^\infty)'_c(t) = \Gamma^\infty(t)$ for every $t \in W \setminus Z$. Hence, for each $t \in W \setminus Z$ there exists a unique set $D(t) \in bcc(X)$ such that $i\big(D(t)\big) = (\Phi^\infty)'_c(t) = \Gamma^\infty(t)$. Thus, for each $t \in W \setminus Z$ we have $D(t) = \Gamma(t)$ and

$$\lim_{\substack{|I| \to 0 \\ I \in \mathcal{I}(t)}} d_{\mathcal{H}}\left(\frac{\Phi(I)}{|I|}, D(t)\right) = \lim_{\substack{|I| \to 0 \\ I \in \mathcal{I}(t)}} \left\| \frac{\Phi^\infty(I)}{|I|} - (\Phi^\infty)'_c(t) \right\|_\infty = 0.$$

This means that the cubic derivative $\Phi'_c(t)$ of Φ in $cwk(X)$ exists and $\Phi'_c(t) = D(t)$ at all $t \in W \setminus Z$.

(ii)⇒(iii): Assume that (ii) holds. Then, the additive interval function Φ^∞ is sAC and there exists $Z \subset \mathcal{L}$ with $|Z| = 0$ such that $(\Phi^\infty)'_c(t)$ exists and $(\Phi^\infty)'_c(t) = \Gamma^\infty(t)$ for every $t \in W \setminus Z$. Hence, by [14, Theorem 2.8], we obtain $V_{\Phi^\infty} \ll \lambda$, and since $V_{\Phi^\infty} = V_\Phi$, it follows that $V_\Phi \ll \lambda$.

(iii) ⇒ (i): Assume that (iii) holds. Then, $V_{\Phi^\infty} \ll \lambda$, $(\Phi^\infty)'_c(t)$ exists and $(\Phi^\infty)'_c(t) = \Gamma^\infty(t)$ at almost all $t \in W$. Hence, by [14, Theorem 2.8], we obtain that Γ^∞ is variationally McShane integrable with the primitive Φ^∞. Thus, Γ is variationally McShane integrable in $bcc(X)$ with the primitive Φ and the proof is completed. □

The problem of existence of at least one variationally McShane integrable selection of a variationally McShane integrable $cwk(X)$-valued multifunction for Banach spaces has been solved by D. Candeloro, L. Di Piazza, K. Musiał and A. R. Sambucini, see [3, Theorem 3.9]. Here, we prove the existence a variationally McShane integrable selection in terms of density of an additive interval selection $\varphi \in \mathcal{S}_\Phi$.

Corollary 2.5. *If a multifunction $\Gamma : W \to cwk(X)$ is variationally McShane integrable in $cwk(X)$ with the primitive $\Phi : \mathcal{I} \to cwk(X)$ and $\varphi \in \mathcal{S}_\Phi$, then there exists a variationally McShane integrable selection f of Γ such that*

$$\varphi(I) = (M)\int_I f(t)d\lambda \quad \text{for every } I \in \mathcal{I}. \tag{2.6}$$

P r o o f. By Theorem 2.4, Φ is sAC and there exists $Z \subset W$ with $|Z| = 0$ such that $\Phi'_c(t)$ exists and $\Phi'_c(t) = \Gamma(t)$ for all $t \in W \setminus Z$. Hence, φ is also sAC and for each $t \in W \setminus Z$ we have

$$\lim_{\substack{|I|\to 0\\ I\in\mathcal{I}(t)}} d_{\mathcal{H}}\big(\Delta\Phi(t, I), \Gamma(t)\big) = 0.$$

Fix an arbitrary $t \in W^o \setminus Z$. Since for each $I \in \mathcal{I}^o(t)$ we have

$$0 \le \operatorname{dist}\big(\Delta\varphi(t, I), \Gamma(t)\big) \le d_{\mathcal{H}}\big(\Delta\Phi(t, I), \Gamma(t)\big),$$

it follows that

$$\lim_{I\in\mathcal{I}^o(t)} \operatorname{dist}\big(\Delta\varphi(t, I), \Gamma(t)\big) = 0. \tag{2.7}$$

Since $\Gamma(t)$ is a closed subset of X, for each $I \in \mathcal{I}^o(t)$ there exists $x_I \in \Gamma(t)$ such that

$$\|\Delta\varphi(t, I) - x_I\| \le 2 \cdot \operatorname{dist}\big(\Delta\varphi(t, I), \Gamma(t)\big),$$

and since the net $(x_I)_{I\in\mathcal{I}^o(t)}$ is with terms in $\Gamma(t) \in cwk(X)$, it follows that $(x_I)_{I\in\mathcal{I}^o(t)}$ has a weak limit point $x_t \in \Gamma(t)$. The last result together with (2.7) yields

$$\lim_{I\in\mathcal{I}^o(t)} \|\Delta\varphi(t, I) - x_I\| = 0 \tag{2.8}$$

and there exists a subnet $(y_d)_{d\in D(t)}$ of $(x_I)_{I\in\mathcal{I}^o(t)}$ converging to x_t with respect to the weak topology. Let $N : D(t) \to \mathcal{I}^o(t)$ be a function satisfying the subnet definition, cf. [16, p. 70]. Then, $y_d = x_{N(d)}$ for all $D \in D(t)$ and the net $(x_{N(d)})$ converges weakly to x_t, i.e.,

$$\lim_{d\in D(t)} \langle x^*, x_{N(d)}\rangle = \langle x^*, x_t\rangle \quad \text{for all } x^* \in X^*.$$

By (2.8) we also have

$$\lim_{d\in D(t)}\big|\big\langle x^*,\, \Delta\varphi\big(t,N(d)\big)\big\rangle - \langle x^*, x_{N(d)}\rangle\big| = 0 \quad \text{for every } \ x^*\in X^*$$

and since

$$\big|\big\langle x^*,\, \Delta\varphi\big(t,N(d)\big)\big\rangle - \langle x^*, x_t\rangle\big| \le \\ \big|\big\langle x^*,\, \Delta\varphi\big(t,N(d)\big)\big\rangle - \langle x^*, x_{N(d)}\rangle\big| + \big|\langle x^*, x_{N(d)}\rangle - \langle x^*, x_t\rangle\big|,$$

it follows that

$$\lim_{d\in D(t)}\big\langle x^*,\, \Delta\varphi\big(t,N(d)\big)\big\rangle = \langle x^*, x_t\rangle \quad \text{for all } \ x^*\in X^*.$$

Since $\big(\Delta\varphi(t,N(d))\big)_{d\in D(t)}$ is a subnet of $\big(\Delta\varphi(t,I)\big)_{I\in\mathcal{I}^o(t)}$, the last result means that x_t is a weak limit point of $\big(\Delta\varphi(t,I)\big)_{I\in\mathcal{I}^o(t)}$, and therefore, $x_t \in L_\varphi(t)$. Since t was arbitrary, the last result yields that for each $t\in W^o\setminus Z$ there exists $x_t\in\Gamma(t)\cap L_\varphi(t)$. Hence, if for each $t\in Z\cup\partial W$ we choose $z_t\in\Gamma(t)$, then the function

$$f(t)=\begin{cases} x_t, & t\in W^o\setminus Z,\\ z_t, & t\in Z\cup\partial W\end{cases}$$

is a selection of Γ. Thus, we have that φ is sAC and $f(t)\in L_\varphi(t)$ at almost all $t\in W^o$. Therefore, by Lemma 2.3, the function f is variational McShane integrable satisfying (2.6) and this completes the proof. □

We say that a multifunction $\Gamma : W \to bcc(X)$ is a *scalar derivative* of an additive interval multifunction $\Phi:\mathcal{I}\to bcc(X)$ if for each $x^*\in X^*$, the additive interval function $\sigma\big(x^*,\Phi(\cdot)\big)$ has the cubic derivative $\sigma(x^*,\Phi)'_c(t)$ and

$$\sigma(x^*,\Phi)'_c(t)=\sigma\big(x^*,\Gamma(t)\big) \quad \text{at almost all } \ t\in W,$$

where the exceptional set may vary with x^*. We are now ready to present the last result.

Theorem 2.6. *Let $\Gamma : W \to bcc(X)$ be a multifunction and let $\Phi:\mathcal{I}\to bcc(X)$ be an additive interval multifunction. Then, the following statements are equivalent*:

(i) Γ *is Pettis integrable in* $bcc(X)$ *with*

$$\Phi(I)=(P)\int_I \Gamma(t)d\lambda \quad \textit{for every } \ I\in\mathcal{I},$$

(ii) Φ *is* AC *and* Γ *is a scalar derivative of* Φ.

P r o o f.

(i)⇒(ii): Assume that Γ is Pettis integrable in $bcc(X)$ with

$$\Phi(I) = (P)\int_I \Gamma(t)d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Then, the mapping

$$M : \mathcal{L} \to bcc(X), \quad M(E) = (P)\int_E \Gamma(t)d\lambda$$

is a λ-continuous strong multimeasure. Hence, $M^\infty = i \circ M$ is a λ-continuous countable additive vector measure, and since λ is a finite measure on $\mathcal{L}$, we obtain by [7, Theorem I.2.1, p. 10] that Φ^∞ is AC, and consequently, Φ is also AC.

Fix an arbitrary $x^* \in X^*$. Then, the function $\sigma\big(x^*, \Gamma(\cdot)\big)$ is Lebesgue integrable and

$$\sigma\big(x^*, \Phi(I)\big) = \int_I \sigma\big(x^*, \Gamma(s)\big)d\lambda \quad \text{for every} \quad I \in \mathcal{I}.$$

Hence, by [14, Lemma 2.4], there exists $Z^{x^*} \in \mathcal{L}$ with $|Z^{x^*}| = 0$ such that

$$\sigma(x^*, \Phi)'_c(t) = \sigma\big(x^*, \Gamma(t)\big) \quad \text{for each} \quad t \in W \setminus Z^{x^*}.$$

(ii)⇒(i): Assume that (ii) holds. Let $\mathcal{A}$ be algebra generated by $\mathcal{I}$ and let $\Phi_{\mathcal{A}} : \mathcal{A} \to bcc(X)$ be an extension of Φ from $\mathcal{I}$ to $\mathcal{A}$ such that $\Phi_{\mathcal{A}}(A) = \bigoplus_{j=1}^m \Phi(I_j)$, where $A = \bigcup_{j=1}^m I_j$ and $I_1, \ldots, I_m$ are pairwise non-overlapping intervals in $\mathcal{I}$. Then, each function $\sigma\big(x^*, \Gamma(\cdot)\big)$ is Lebesgue integrable with

$$\sigma\big(x^*, \Phi_{\mathcal{A}}(A)\big) = \int_A \sigma\big(x^*, \Gamma(t)\big)d\lambda \quad \text{for all } A \in \mathcal{A}.$$

Hence, for each $x^* \in X^*$ the function $\sigma\big(x^*, \Phi_{\mathcal{A}}(\cdot)\big)$ is λ-continuous and countable additive on $\mathcal{A}$. Therefore, by [13, Proposition 2.5], the multifunction $\Phi_{\mathcal{A}}$ can be extended to a weak multimeasure $\Phi_{\mathscr{B}} : \mathscr{B} \to bcc(X)$, where $\mathscr{B} = \sigma(\mathcal{A})$ is the Borel σ-algebra generated by $\mathcal{A}$.

It is easy to see that the family

$$\mathcal{C} = \left\{ B \in \mathscr{B} : (\forall x^* \in X^*) \left[\sigma\big(x^*, \Phi_{\mathscr{B}}(B)\big) = \int_B \sigma\big(x^*, \Gamma(t)\big)d\lambda \right] \right\}$$

is a σ-algebra, and since $\mathcal{A} \subset \mathcal{C} \subset \mathscr{B}$, by equality $\mathscr{B} = \sigma(\mathcal{A})$ it follows that $\mathcal{C} = \mathscr{B}$. Thus, for each $B \in \mathscr{B}$, we have

$$\sigma\big(x^*, \Phi_{\mathscr{B}}(B)\big) = \int_B \sigma\big(x^*, \Gamma(t)\big) d\lambda \quad \text{for all } \ x^* \in X^*,$$

and the weak multimeasure $\Phi_{\mathscr{B}}$ is λ-continuous. The last result together with the fact that the Lebesgue σ-algebra $\mathcal{L}$ is a completion of the Borel σ-algebra $\mathscr{B}$ yields that there exists the weak multimeasure $H : \mathcal{L} \to bcc(X)$ such that for each $E \in \mathcal{L}$, we have

$$\sigma\big(x^*, H(E)\big) = \int_E \sigma\big(x^*, \Gamma(t)\big) d\lambda \quad \text{for every } \ x^* \in X^*.$$

This means that Γ is Pettis integrable with

$$H(E) = (P)\int_E \Gamma(t) d\lambda \quad \text{for all } \ E \in \mathcal{L},$$

and this completes the proof. □

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