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Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2024-0018
Tatra Mt. Math. Publ. **86** (2024), 175–184

STAR γ COVERS AND THEIR APPLICATIONS TO SELECTION PRINCIPLES

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ABSTRACT. By imposing the star operator on γ covers, we establish the notion of st-γ covers. The interrelationships among many recent variants of γ covers have been identified, and various counterexamples have been shown to distinguish them. An investigation has been conducted on the basic topological attributes of these covers. We have also examined the implications between various selection principles and topological games in the context of the class of st-γ coverings.

1. Introduction

A set of sets that collectively cover the entire space is referred to as a 'cover' in topology. There are various kinds of covers, and the idea is crucial when talking about characteristics like compactness, Lindelöfness. When exploring covers in topology, open sets are usually the sets under discussion. A set of open sets $\{U_\alpha\}$ such that their union covers the whole topological space X is called an open cover. Here, $\mathrm{X} = \cup_\alpha U_\alpha$ [8]. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is called γ cover if for every $x \in X, \{n \in \mathbb{N} : x \notin U_n\}$ is finite [12].

In recent literature, we have found some interesting variants of γ covers. Two of the most interesting variations are statistical γ covers [6] and star statistical γ covers [2], where the concepts of asymptotic density (also called natural density) and star operator have been utilized. Let $M \subseteq \mathbb{N}$, then the asymptotic density of M is denoted by $\delta(M)$ and defined as

$$\delta(M) = \lim_{n\to\infty} \frac{|\{k \le n : k \in M\}|}{n} \text{ [10,15].}$$

2020 Mathematics Subject Classification: 54D20, 54B20, 54C35.
Keywords: γ-covers, star operator, selection principles, topological game.
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In 1951, Fast [10] generalized the definition of convergence to include statistical convergent for sequences of real numbers. In 2008, Maio and Kočinac [14] defined statistical convergence in topological space. A sequence $\{a_n : n \in \mathbb{N}\}$ in a topological space X is said to be statistical convergence to $a \in X$, if for every neighborhood U of a, $\delta\left(\{n \in \mathbb{N} : a_n \notin U\}\right) = 0$. In the same paper, Maio and Kočinac [14] have generalized the concept of γ cover to statistical γ cover. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is called a statistical γ cover if for each $x \in X$, $\delta(\{n \in \mathbb{N} : x \notin U_n\}) = 0$.

In the year 1991, in [7], the concept of the star operator was introduced by van Douwen. If $A \subseteq X$ and $\mathcal{U}$ represents a collection of subsets of X, then

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}.$$

So, $\mathrm{St}(A,\mathcal{U})$ is the union of all open sets in U that contain a non-empty intersection with A. For a given point $x \in X$, $\mathrm{St}(x,\mathcal{U}) = \cup\{U \in \mathcal{U} : x \in U\}$. By applying the notion of St-operator, Kočinac expanded the idea of classical selection principles to star-selection principles [11, 13, 16]. Some recent applications of the star operator can be found in [1–5, 17]. In [2], Bal and Rakshit bring a variation to statistical γ cover by forcing the star operator which is called star statistical γ cover. A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a topological space (X, τ) is said to be a statistical star γ cover (in short, s-s-γ cover) if for each $x \in X$, the set $\{n \in \mathbb{N} : x \in \mathrm{st}(U_n,\mathcal{U})\}$ has natural density zero, i.e.,

$$\delta\big(\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\}\big) = 0.$$

In this paper, we investigate and develop a new class of γ covers in topological spaces by utilizing the star operator, which lies somewhere between the class of γ covers and star statistical γ covers. Additionally, we provide some context for the theory of selection principles in light of star γ covers.

2. Preliminaries

In this section, we mention some prerequisite concepts for readers to use easily.

Definition 2.1 ([2]). A countable cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a topological space (X, τ) is said to be a statistical star γ cover (in short, s-s-γ cover) if for each $x \in X$, the set $\{n \in \mathbb{N} : x \in \mathrm{st}(U_n,\mathcal{U})\}$ has natural density zero, i.e.,

$$\delta\big(\{n \in \mathbb{N} : x \notin \mathrm{St}(U_n,\mathcal{U})\}\big) = 0.$$

Definition 2.2 ([5]). A family of pairwise disjoint open sets in a topological space (X, τ) is called a cellular open family.

In this paper, a space X represents a topological space X. No specific separation axiom has been taken unless otherwise mentioned.

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of families of subsets of an infinite set X. Then,

Definition 2.3 ([16]). $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there exist a sequence $\{B_n : n \in \mathbb{N}\}$ such that $B_n \in A_n$, for each n and $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Definition 2.4 ([16]). denotes the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there exist a sequence $\{B_n : n \in \mathbb{N}\}$ of finite sets such that $B_n \subseteq A_n$ for each n, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of families of subsets of an infinite set X. Then,

Definition 2.5 ([13]). $\alpha_1(\mathcal{A}, \mathcal{B})$ **denotes** the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of infinite element of $\mathcal{A}$ there exists an element $B \in \mathcal{B}$ such that for every $n \in \mathbb{N}$, the set $A_n \setminus B$ is finite.

Definition 2.6 ([13]). $\alpha_2(\mathcal{A}, \mathcal{B})$ **denotes** the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of infinite element of $\mathcal{A}$ there exists an element $B \in \mathcal{B}$ such that for every $n \in \mathbb{N}$, the set $A_n \cap B$ has infinite elements.

Definition 2.7 ([13]). $\alpha_3(\mathcal{A}, \mathcal{B})$ **denotes** the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of infinite element of $\mathcal{A}$ there exists an element $B \in \mathcal{B}$ such that for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B$ has infinite elements.

Definition 2.8 ([13]). $\alpha_4(\mathcal{A}, \mathcal{B})$ **denotes** the selection hypothesis that for each sequence $\{A_n : n \in \mathbb{N}\}$ of infinite element of $\mathcal{A}$ there exists an element $B \in \mathcal{B}$ such that for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B \neq \emptyset$.

Definition 2.9 ([16]). Let ONE and TWO be two players who play the topological game $G_1(\mathcal{A}, \mathcal{B})$, where each plays a round for each positive integer. In the nth round, suppose player ONE chooses a set $A_n \in \mathcal{A}$ and player TWO responds by choosing an element $b_n \in A_n$. If $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$, then player TWO wins, otherwise player ONE wins.

3. Main Results

Extending the concept of γ cover with the help of the star operator, we introduce star γ cover. Star γ cover can offer new perspectives and methodologies to understand convergence in topological spaces when examining different types of convergence, such as statistical or weighted statistical convergence. This new idea will enable us to analyze topological properties related to covering features more thoroughly.

Definition 3.1. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ will be called a star γ cover if for every $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ is finite.

Theorem 3.2. *In a topological space (X, τ), every γ cover is a st-γ cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a γ cover of a topological space (X, τ). Then for every $x \in X$, the set $\{n \in \mathbb{N} : x \notin U_n\}$ is finite.

Let $p \in \{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ for some $x \in X$ implies that $x \notin \mathrm{st}(U_p, \mathcal{U})$. Also, $U_p \subseteq \mathrm{st}(U_p, \mathcal{U})$. Therefore, $x \notin U_p$. So, $p \in \{n \in \mathbb{N} : x \notin U_n\}$. Thus,

$$\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\} \subseteq \{n \in \mathbb{N} : x \notin U_n\} \quad \text{for each } \; x \in X.$$

Consequently, the set $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ is finite for each $x \in X$. Therefore, $\mathcal{U}$ is a st-γ cover. Hence the theorem. □

Example. However, the converse of the theorem need not be true.

Let, $X = B_1(a)$, an open ball of radius 1 whose center is at a and $\tau = \{B_r(a) : r \in [0, 1]\}$, be a topology on X.

Consider a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ such that $U_n = B_{\frac{1}{n}}(a)$. So, $\mathrm{st}(U_n, \mathcal{U}) = X$ for every $n \in \mathbb{N}$. Thus, for every $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\} = \{n \in \mathbb{N} : x \notin X\} = \emptyset$, i.e., $\mathcal{U}$ is a st-γ cover.

Let α be a point on the ball $B_{\frac{1}{2}}(a)$. So, $\{n \in \mathbb{N} : \alpha \notin U_n\} = \{2, 3, 4, \dots\}$ is not finite. Therefore, $\mathcal{U}$ is not a γ cover.

Theorem 3.3. *Every st-γ cover is an s-s-γ cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a st-γ cover of the topological space (X, τ). Then, for every $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ is finite which implies that $\delta(\{n \in \mathbb{N} : x \notin \mathrm{st}(V_n, \mathcal{U})\}) = 0$ for all $x \in X$.

Therefore, $\mathcal{U}$ is an s-s-γ cover. □

Example. However, the converse need not be true.

Let $X = \{p, q, r, s\}$ & $\tau = P(X)$ be a topology on X. Let us consider the cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ where

$$U_n = \begin{cases} \{p, q\} & \text{if} \quad n = m^2, \quad \text{for } m \in \mathbb{N}, \\ \{r, s\} & \text{if} \quad n = 2, \\ \{q, r\} & \text{otherwise.} \end{cases}$$

So

$$\mathrm{st}(U_n, \mathcal{U}) = \begin{cases} \{p, q, r\} & \text{if} \quad n = m^2, \quad \text{for } m \in \mathbb{N}, \\ \{q, r, s\} & \text{for} \quad n = 2, \\ X, & \text{otherwise.} \end{cases}$$

Here, $\delta(\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}) = 0$, for all $x \in X$, i.e., $\mathcal{U}$ is a st-s-γ cover.

But, the set $\{n\in\mathbb{N}:a\notin \mathrm{st}(U_n,\mathcal{U})\}$ is not finite. Therefore, $\mathcal{U}$ is not a st-γ cover.

In short, we have the following diagram:

$$\begin{tikzcd}
\boxed{\text{s-s dense subset}} \arrow[r, bend left] & \boxed{\text{s-dense subset}} \arrow[l, bend left, "/" marking] \arrow[r, bend left] & \boxed{\text{s-s dense subset}} \arrow[l, bend left, "/" marking]
\end{tikzcd}$$

FIGURE 1. Inter relation between variants of γ covers.

THEOREM 3.4. *A cellular* st-γ *cover is a* γ *cover.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a cellular st-γ cover of a topological space (X, τ), i.e., $U_i \cap U_j = \emptyset$ if $i \neq j$. Then, for every $n \in \mathbb{N}$, $\mathrm{st}(U_n,\mathcal{U}) = U_n$. So, for all $x \in X$, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\} = \{n \in \mathbb{N} : x \notin U_n\}$. Therefore, the set $\{n \in \mathbb{N} : x \notin U_n\}$ is finite.

Therefore, $\mathcal{U}$ is a γ cover of (X, τ). $\square$

THEOREM 3.5. *An open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of a topological space* (X, τ) *is a* st-γ *cover iff for each* $F \subseteq X$ *such that* $F \setminus \mathrm{st}(U_p,\mathcal{U})$ *is finite for each* $p \in \{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ *holds that the set* $\{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ *is finite.*

Proof. Let $\mathcal{U}$ be a st-γ cover of a topological space (X, τ) and $F \subseteq X$ such that $F \setminus \mathrm{st}(U_p,\mathcal{U})$ is finite for each $p \in \{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$.

Let $k \in \{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ be arbitrary which implies that $F \nsubseteq \mathrm{st}(U_k,\mathcal{U})$. So, there exists a $x \in F\setminus \mathrm{st}(U_k,\mathcal{U})$ such that $x \in F$ and $x \notin \mathrm{st}(U_k,\mathcal{U})$. Thus, $k \in \{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\}$ $\forall x \in F \setminus \mathrm{st}(U_k,\mathcal{U})$. Therefore, $\{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\} \subseteq \{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\}$ $\forall x \in F \setminus \mathrm{st}(U_k,\mathcal{U})$.

Since $\mathcal{U}$ is a st-γ cover, so $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\}$ is finite. Therefore, the set $\{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ is finite. [Since $\mathcal{U}$ is a s^{α}-s-γ cover.]

Conversely, let for each $F \subseteq X$ such that $F \setminus \mathrm{st}(U_p,\mathcal{U})$ is finite for each $p \in \{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ hold that $\{n \in \mathbb{N} : F \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ is finite.

Now, for arbitrary $x \in X$, $\{x\} \subseteq X$ such that $\{x\} \setminus \mathrm{st}(U_p,\mathcal{U})$ is finite for each $p \in \mathbb{N}$ implies that $\{n \in \mathbb{N} : \{x\} \nsubseteq \mathrm{st}(U_n,\mathcal{U})\}$ is finite for all $x \in X$.

Thus, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\}$ is finite for all $x \in X$

Therefore, $\mathcal{U}$ is a st-γ cover. $\square$

THEOREM 3.6. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be two* st-γ *covers of a topological space* (X, τ). *Then,* $\mathcal{U} \sqcup \mathcal{V} = \{U_i \cup V_i : U_i \in \mathcal{U}, V_i \in \mathcal{V}$ *and* $i \in \mathbb{N}\}$ *is also a* st-γ *cover of* (X, τ).

P r o o f. Let $\mathcal{U}$ and $\mathcal{V}$ be two st-γ covers of a topological space (X, τ). Then, for every $x \in X$, the sets $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ and $\{n \in \mathbb{N} : x \notin \mathrm{st}(V_n, \mathcal{V})\}$ are finite. Then, $\{n \in \mathbb{N} : x \notin U_n\}$ is finite. Let $p \in \{n \in \mathbb{N} : x \notin \mathrm{st}(U_n \cup V_n, \mathcal{U} \sqcup \mathcal{V})\}$ for some $x \in X$ which implies that $x \notin \mathrm{st}(U_p \cup V_p, \mathcal{U} \sqcup \mathcal{V})$. Thus, $x \notin U_p \cup V_p$. [Since $U_p \cup V_p \subseteq \mathrm{st}(U_p \cup V_p, \mathcal{U} \sqcup \mathcal{V})$.] So, $x \notin U_p$ & $x \notin V_p$ which implies that $p \in \{n \in \mathbb{N} : x \notin U_n\}$. Consequently,

$$\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n \cup V_n, \mathcal{U} \sqcup \mathcal{V})\} \subseteq \{n \in \mathbb{N} : x \notin U_n\}.$$

Therefore, the set $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n \cup V_n, \mathcal{U} \sqcup \mathcal{V})\}$ is finite.

Hence, $(\mathcal{U} \sqcup \mathcal{V})$ is a st-γ cover. □

EXAMPLE. If $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is st-γ cover of a topological space (X, τ)&$A \subset X$, then $\mathcal{U}_A = \{A \cap U_n : U_n \in \mathcal{U}$&$n \in \mathbb{N}\}$ need not be a st-γ cover for (A, τ_A) where τ_A is the subspace topology.

Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = P(X)$. Also, $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where

$$U_n = \begin{cases} \{a, b\} & \text{if} \quad n = 2k, \text{ for } k \in \mathbb{N}, \\ \{b, c\} & \text{for} \quad n = 2k + 1 \end{cases}$$

is a cover for (X, τ). Also, for every $n \in \mathbb{N}$, $\mathrm{st}(U_n, \mathcal{U}) = X$. Thus, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\} = \{n \in \mathbb{N} : x \notin U_n\} = \emptyset$. So, for every $x \in X$, the set $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n, \mathcal{U})\}$ is finite, i.e., $\mathcal{U}$ is a st-γ cover

Let $A = \{a, c\} \subset X$. Then, $\mathcal{U}_{\mathcal{A}} = \{V_n = A \cap U_n : U_n \in \mathcal{U}$ & $n \in \mathbb{N}\}$, where

$$V_n = \begin{cases} \{a\} & \text{if} \quad n = 2k, \text{ for } k \in \mathbb{N}, \\ \{c\} & \text{for} \quad n = 2k + 1. \end{cases}$$

Then,

$$\mathrm{st}(V_n, \mathcal{U}_{\mathcal{A}}) = \begin{cases} \{a\} & \text{if} \quad n = 2k, \text{ for } k \in \mathbb{N}, \\ \{c\} & \text{for} \quad n = 2k + 1. \end{cases}$$

So, for every $x \in A$, $\{n \in \mathbb{N} : x \notin \mathrm{st}(V_n, \mathcal{U}_{\mathcal{A}})\}$ is not finite.

Therefore, $\mathcal{U}_{\mathcal{A}}$ is not a st-γ cover for (A, τ_A).

THEOREM 3.7. *Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of a topological space (X, τ) and (X_i, τ_{X_i}) are the subspaces of (X, τ) for $i = 1, 2, 3, \ldots, n$ such that $X = \bigcup_{i=1}^{n} X_i$. If $U_{X_i} = \{X_i \cap U_n : n \in \mathbb{N}\}$ are s^{α}-s-γ covers of (X_i, τ_{X_i}) for $i = 1, 2, 3, \ldots, n$, then $\mathcal{U}$ is also a* st-γ *cover for (X, τ).*

P r o o f. Let $p \in X = \bigcup_{i=1}^{n} X_i$. So, there exists at least one $j \in \{1, 2, 3, \ldots\}$ such that $p \in X_j$. Since U_{X_j} is a st-γ cover of (X_j, τ_{X_j}). Therefore, the set $\{n \in \mathbb{N} : p \notin \mathrm{st}(X_j \cap U_n, \mathcal{U}_{X_j})\}$ is finite. However, $\{n \in \mathbb{N} : p \notin \mathrm{st}(X_j \cap U_n, \mathcal{U}_{X_j})\} \supseteq \{n \in \mathbb{N} : p \notin \mathrm{st}(U_n, \mathcal{U})\}$. [Since $\mathrm{st}(X_j \cap U_n, \mathcal{U}_{X_j}) \subseteq \mathrm{st}(U_n, \mathcal{U})$.] Thus, $\{n \in \mathbb{N} : p \notin \mathrm{st}(U_n, \mathcal{U})\}$ is also finite for all $p \in X$.

Therefore, $\mathcal{U}$ is a st-γ cover for (X,τ). $\square$

Example. An infinite subset of a st-γ cover need not be st-γ cover.

Proof. Let $X = \{\alpha,\beta,\gamma\}$ and $\tau = \big\{\emptyset,\{\alpha\},\{\alpha,\beta\},X\big\}$ be a topology on X. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be such that

$$U_n = \begin{cases} \{\alpha\} & \text{if} \quad n = 2k, \quad \text{for } k \in \mathbb{N}, \\ X & \text{for} \quad n = 2k+1. \end{cases}$$

Then,

$$\mathrm{st}(U_n,\mathcal{U}) = \begin{cases} X & \text{if} \quad n = 2k, \quad \text{for } k \in \mathbb{N}, \\ X & \text{for} \quad n = 2k+1. \end{cases}$$

So, $\{n \in \mathbb{N} : x \notin \mathrm{st}(U_n,\mathcal{U})\} = \emptyset$ for all $x \in X$. Therefore, $\mathcal{U}$ is a st-γ cover.

Consider the subcover $\mathcal{V} = \{V_n = U_{2n} : n \in \mathbb{N}\}$ of $\mathcal{U}$. However, although $\mathcal{V}$ is an infinite subset of $\mathcal{U}$, $\mathcal{V}$ is not a cover of X. So, $\mathcal{V}$ is not a st-γ cover.

Hence, an infinite subset of a st-γ cover need not be a st-γ cover. $\square$

Lemma 3.8. *Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ and $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ be a sequence of countable st-γ covers of X. Then, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ defined by $\mathcal{V}_n = \{U_{1,m} \cap U_{2,m} \cap \dots \cdots \cap U_{n,m} : m \in \mathbb{N}\} \setminus \{\emptyset\}$ is also a sequence of st-γ covers.*

Proof. Let $p \in X$ and let n be fixed. Then, for each $i \le n$, we have put

$$A_i = \{m \in \mathbb{N} : p \notin \mathrm{st}(U_{i,m},\mathcal{U}_n)\}.$$

Clearly, for each $i \le n$, $\delta(A_i) = 0$ [since $\mathcal{U}_n$ is st-γ cover]. Moreover, $V_{n,m} = U_{1,m} \cap U_{2,m} \cap \cdots \cap U_{n,m}$. Then, $\{m \in \mathbb{N} : p \notin \mathrm{st}(V_{n,m},\mathcal{V}_n)\} \subset \bigcup_{i=1}^{n} A_i$. Since $\delta(\bigcup_{i=1}^{n} A_i) = 0$, so $\delta\big(\{m \in \mathbb{N} : p \notin \mathrm{st}(V_{n,m},\mathcal{V}_n)\}\big) = 0$. Therefore, $\mathcal{V}_n$ is a st-γ cover of X. $\square$

We consider the following notions:

- Γ: collection of all γ covers.
- st-Γ: collection of all star γ covers.
- s-s-Γ: collection of all star statistical γ covers.

In virtue of Theorem 3.2 and Theorem 3.3, we have $\Gamma \subseteq$ st-$\Gamma \subseteq \mathrm{s-s-}\Gamma$.

It is also clear that the selection principle $S_1(\mathcal{A},\mathcal{B})$ implies the selection principle $S_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$. Since $S_1(\mathcal{A},\mathcal{B})$ and $S_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ are monotonic in the first collection and anti-monotonic in the second collection, we have the implication diagram Figure 2.

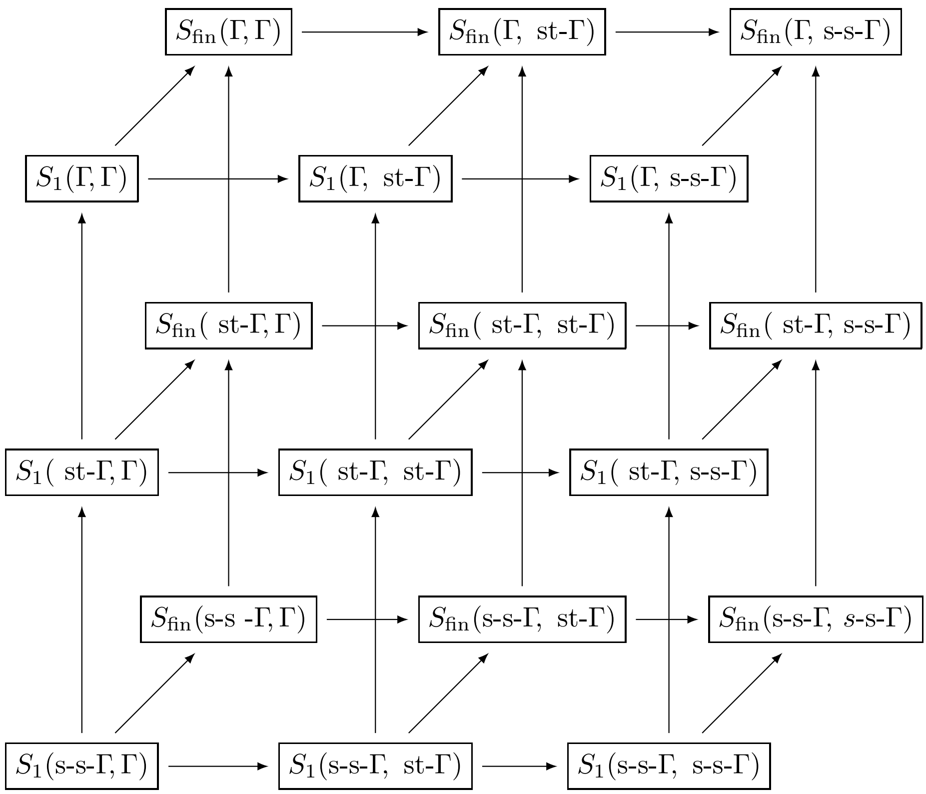

FIGURE 2. Classical selection principles under the variation of γ covers.

THEOREM 3.9. *For a space X, (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (1) holds where*

(1) *X satisfies $\alpha_2(\text{st-}\Gamma, \Gamma)$;*

(2) *X satisfies $\alpha_3(\text{st-}\Gamma, \Gamma)$;*

(3) *X satisfies $\alpha_4(\text{st-}\Gamma, \Gamma)$;*

(4) *X satisfies $S_1(\text{st-}\Gamma, \Gamma)$;*

(5) *One has no winning strategy in the game $G_1(st\text{-}\Gamma, \Gamma)$ on X.*

Proof.
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (4). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence such that $\mathcal{U}_n \in \text{st-}\Gamma$ for all $n \in \mathbb{N}$. Suppose $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. We construct $V_{n,m} = \bigcap_{i=1}^{n} U_{i,m}$.

Then, by Lemma 3.8, $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a st-γ cover. Therefore, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{V}_n \in$ st-Γ for all $n \in \mathbb{N}$. Now, selection principle α_4(st-Γ, Γ) holds. Then, $\exists$ a $\beta \in \Gamma$ such that $\beta \cap \mathcal{V}_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Thus, there exists an increasing sequence $\{n_1 < n_2 < \dots\}$ in $\mathbb{N}$ such that $\beta \cap \mathcal{V}_{n_i} \neq \emptyset$ for all $i \in \mathbb{N}$. So, there exists at least one element $V_{n_i,m_i} \in \beta \cap \mathcal{V}_{n_i}$. Now, $\mathcal{V} = \{V_{n_i,m_i} : i \in \mathbb{N}\}$ is an infinite subset of β. However, $\beta \in \Gamma$, and an infinite subset of an γ-cover is an γ- cover. Therefore, $\{V_{n_i,m_i} : i \in \mathbb{N}\} \in \Gamma$, and also, $V_{n_i,m_i} \in \mathcal{V}_{n_i}$ for all $i \in \mathbb{N}$.

However, $V_{n_i,m_i} = \bigcap_{j=1}^{n_i} U_{j,m_i}$. Let, $n_0 = 0$. For each i ≥ 0, $n_i < n \leq n_{i+1}$ and $O_n = U_{k,m_i}$. Now, for each $n \in \mathbb{N}, O_n \in \mathcal{U}_n$. Also, $\{O_n : n \in \mathbb{N}\}$ is refined by $\mathcal{V} \in \Gamma$. Therefore, $\{O_n : n \in \mathbb{N}\} \in \Gamma$.

Hence, the selection principle α_2(st-Γ, Γ) holds.

(5) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence such that $\mathcal{U}_n \in$ st-Γ $\forall n \in \mathbb{N}$.

Suppose $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}, n \in \mathbb{N}$. From the sequence $\mathcal{U}_n$, form a sequence of new st-γ covers of X in accordance with the strategy σ for ONE defined in this way. ONE's first move is $\sigma(\emptyset) = \mathcal{U}_k$. Let $U_{k,m_i} \in \mathcal{U}_k$ be TWO's response. Then, ONE replies by choosing $\sigma(U_{k,m_i}) = \{U_{k,m} : m > m_i\}$. After that, TWO chooses a set $U_{k,m_{i_2}} \in \sigma(U_{k,m_{i_1}})$. Then again, ONE plays $\sigma(U_{k,m_{i_1}}, U_{k,m_{i_2}}) = \{U_{k,m} : m > m_{i_2}\}$. Then, TWO chooses a set $U_{k,m_{i_3}} \in \sigma\big(U_{k,m_{i_1}}, U_{k,m_{i_2}}\big)$, and so on. Proceeding in this way, $\forall n \geq 2$. Since σ is not a winning strategy for ONE, the σ-play

$$\sigma(\emptyset), U_{k,m_i}; \sigma(U_{k,m_i}), U_{k,m_{i_2}}; \sigma(U_{k,m_{i_1}}, U_{k,m_{i_2}}), U_{k,m_{i_3}}; \dots$$

is lost by ONE, i.e., TWO's moves $U_{k,m_{i_1}}, U_{k,m_{i_2}}, U_{k,m_{i_3}}, \dots$ form a sequence that is a γ cover of X. As the union of γ covers is also a γ cover, so $B = \bigcup_{k\in\mathbb{N}}\{U_{k,m_{i_1}}, U_{k,m_{i_2}}, U_{k,m_{i_3}}, \dots\} \in \Gamma$. Also, $\mathcal{U}_n \cap B$ is infinite for all $n \in \mathbb{N}$.

So, this sequence shows that α_2 (st-Γ, Γ) holds. □

Problem 1. Does the selection principle S_1(st-Γ, Γ) imply that ONE does not have any winning strategy in the game G_1(st-Γ, Γ) ?

Conclusion. The inclusion of the star γ cover allows us to investigate and comprehend different topological features in a more comprehensive and precise way. There are numerous uses for this specific kind of γ cover in topological games and selection principles. Also, it is preserved under closed subspace.

Acknowledgement. The authors thank the referee and the editor for their insightful comments that greatly improved the quality and representation of the paper.

`Declaration on data availability and conflict of interest.`

In this article, no dataset has been generated or analyzed. So, data sharing is not applicable here. We declare that all of the images/graphics included in this article are the authors' own works. There are no conflicts of interest in any of the topics covered in this article.

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Received March 12, 2024
Revised May 8, 2024
Accepted June 8, 2024
Publ. online September 30, 2024

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