VERSITA

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/v10127-011-0019-z
Tatra Mt. Math. Publ. **49** (2011), 1–7

ANOTHER PROOF OF HUREWICZ THEOREM

Miroslav Repický

ABSTRACT. A Hurewicz theorem says that every coanalytic non-G_δ set C in a Polish space contains a countable set Q without isolated points such that $\overline{Q} \cap C = Q$. We present another elementary proof of this theorem and generalize it for κ-Suslin sets. As a consequence, under Martin's Axiom, we obtain a characterization of $\mathbf{\Sigma}^1_2$ sets that are the unions of less than the continuum closed sets.

1. Hurewicz schemes

Several proofs of the Hurewicz theorem are known. The original proof by W. Hurewicz [2] is based on the notion of "Häufungsystem" (which we call Hurewicz scheme below). The proof presented by A. Kechris [3, Theorem 21.18 and Theorem 21.22] is based on the so called separation game. Recently Michal Staš [5] gave another simple and elementary proof of this theorem. Our proof is relative to the proof by M. Staš but, in addition, we introduce a notion of a D-proper mapping between metric spaces and obtain another variant of his characterization. We present a natural generalization to κ-Suslin sets.

Definition 1.1. Let X be a metric space.

(1) A mapping $\varphi\colon {}^{<\omega}\omega \to X$ is an *H-scheme on X* (i.e., *Hurewicz scheme*) if the following two conditions are satisfied:
   - (a) $\varphi(s^\frown n) \neq \varphi(s^\frown m)$ for all $s \in {}^{<\omega}\omega$ and $n \neq m$,
   - (b) $\varphi(s) = \lim_{n\to\infty} \varphi(s^\frown n)$ for all $s \in {}^{<\omega}\omega$.

(2) An H-scheme φ is *separated* if there is a system of open sets $\{V_s : s \in {}^{<\omega}\omega\}$ such that for all $s \in {}^{<\omega}\omega$,
   - (c) $\varphi(s) \in V_s \setminus \bigcup_{n\in\omega} \overline{V_{s^\frown n}}$, and
   - (d) $\{\overline{V_{s^\frown n}} : n \in \omega\}$ is a disjoint system of subsets of V_s.

2010 Mathematics Subject Classification: Primary 03E15; Secondary 03E17, 03E50.
Keywords: Hurewicz scheme, D-proper mapping, analytic set.
The work has been supported by grant of Slovak Grant Agency VEGA 1/0032/09.

We say that the system $\{V_s : s \in {}^{<\omega}\omega\}$ is *normal* if

(e) $\lim_{s\in{}^{<\omega}\omega} \operatorname{diam}(V_s) = 0$, i.e., $(\forall\varepsilon > 0)(\forall^\infty s \in {}^{<\omega}\omega)\ \operatorname{diam}(V_s) < \varepsilon$.

(3) An H-scheme $\varphi : {}^{<\omega}\omega \to X$ is *complete* (in X), if for every $g \in {}^{\omega}\omega$, the sequence $\{\varphi(g\restriction n)\}_{n\in\omega}$ converges in X.

Definition 1.2. For an H-scheme $\varphi : {}^{<\omega}\omega \to X$ we denote

$$H(\varphi) = \overline{\operatorname{rng}(\varphi)} \setminus \operatorname{rng}(\varphi),$$
$$G(\varphi) = \{x \in X : (\exists f \in {}^{\omega}\omega)\ x = \lim_{n\to\infty} \varphi(f\restriction n)\}.$$

It is easy to see that $H(\varphi)$ is a G_δ set and, if X is a Polish space, then $G(\varphi)$ is analytic. For every H-scheme φ we can define a (partial) function $f_\varphi : {}^{\omega}\omega \to X$ by $f_\varphi(x) = \lim_{n\to\infty} \varphi(x\restriction n)$, if the limit exists. The set $G(\varphi)$ is the range of f_φ. The function f_φ is total if and only if φ is a complete H-scheme. If φ is an H-scheme on a complete metric space separated by a normal system of open sets, then f_φ is total, injective, and continuous.

By Hurewicz [2], an H-scheme φ is *normal*, if it satisfies condition $\overline{\operatorname{rng}(\varphi)} = G(\varphi) \cup \operatorname{rng}(\varphi)$, i.e., $H(\varphi) \subseteq G(\varphi)$. We need a stronger property because of the next lemma.

Lemma 1.3. *Let $\varphi : {}^{<\omega}\omega \to X$ be an H-scheme on a metric space X separated by a normal system of open sets $\{V_s : s \in {}^{<\omega}\omega\}$. If $\langle s_n : n \in \omega\rangle$ is a sequence in ${}^{<\omega}\omega$, $T = \{s \in {}^{<\omega}\omega : \exists^\infty n\ s \subseteq s_n\}$, and $x \in X$, then $x = \lim_{n\to\infty} \varphi(s_n)$ if and only if one of the following conditions holds:*

(i) *there is $s \in {}^{<\omega}\omega$ such that $T = \{s\restriction k : k \le |s|\}$, $\forall^\infty n\ s \subseteq s_n$, and $x = \varphi(s)$, or*

(ii) *there is $g \in {}^{\omega}\omega$ such that $T = \{g\restriction k : k \in \omega\}$, $\forall k\ \forall^\infty n\ g\restriction k \subseteq s_n$, and $x = \lim_{n\to\infty} \varphi(g\restriction n) \notin \operatorname{rng}(\varphi)$.*

In particular, $G(\varphi) = H(\varphi)$.

Proof. Assume that $x = \lim_{n\to\infty} \varphi(s_n)$. For every $s \in T$ we have $x \in \overline{V_s}$. By condition (d) it follows that T does not contain any two incomparable elements. If T is finite, let s be its maximal element. There are increasing sequences $\{n_k\}_{k\in\omega}$ and $\{m_k\}_{k\in\omega}$ such that $s^\frown m_k \subseteq s_{n_k}$ for all k. Then

$$d\big(\varphi(s), \varphi(s_{n_k})\big) \le d\big(\varphi(s), \varphi(s^\frown m_k)\big) + \operatorname{diam}(V_{s^\frown m_k})$$

and hence

$$\varphi(s) = \lim_{k\to\infty} \varphi(s_{n_k}) = x.$$

If t is a proper subsequence of s, then $\varphi(t) \in V_t \setminus \overline{V_s}$ and so $\varphi(t) \neq x$. Hence we cannot apply the same argument for any t below s which means that $\forall^\infty n$ $s \subseteq s_n$.

If T is infinite, then there must be $g \in {}^{\omega}\omega$ such that $T = \{g{\restriction}k : k \in \omega\}$. Then for every k, $\varphi(g{\restriction}k) \in V_{g\restriction k} \setminus \overline{V_{g\restriction(k+1)}}$, hence $\varphi(g{\restriction}k) \neq x$ and, like in the previous case, $\forall k\ \forall^{\infty} n\ g{\restriction}k \subseteq s_n$. Then $x = \lim_{n\to\infty} \varphi(g{\restriction}n)$ because $x \in \bigcap_{n\in\omega} \overline{V_{g\restriction n}}$. □

Example 1.4. The Cantor space ${}^{\omega}2$ can be represented as a topological closure of an H-scheme separated by a normal system of open sets. By induction let us define $t_s \in {}^{<\omega}2$ for $s \in {}^{<\omega}\omega$: $t_{\emptyset} = \emptyset$ and $t_{s^\frown n} = t_s{}^\frown 0^n 1$. Then $\varphi_C : {}^{<\omega}\omega \to {}^{\omega}2$ defined by $\varphi_C(s) = t_s{}^\frown\overline{0}$ ($\overline{0}$ is the infinite sequence of 0's) is an H-scheme separated by the normal system of clopen sets $\{[t_s] : s \in {}^{<\omega}\omega\}$. Clearly, $\overline{\operatorname{rng}(\varphi_C)} = {}^{\omega}2$ and $G(\varphi_C) = H(\varphi_C) = {}^{\omega}2 \setminus \operatorname{rng}(\varphi_C) \simeq {}^{\omega}\omega$.

Theorem 1.5. *If X is a metric space and $\varphi : {}^{<\omega}\omega \to X$ is a complete H-scheme separated by a normal system of open sets, then $H(\varphi) = G(\varphi) \simeq {}^{\omega}\omega$ and $\overline{\operatorname{rng}(\varphi)}$ is a compact perfect set homeomorphic to the Cantor space.*

Proof. Let $\psi : {}^{<\omega}\omega \to Y$ be any other complete H-scheme on a metric space Y separated by a normal system of open sets. There is a homeomorphism $f : \overline{\operatorname{rng}(\varphi)} \to \overline{\operatorname{rng}(\psi)}$ such that $f\big(\varphi(s)\big) = \psi(s)$ and $f\big(f_{\varphi}(g)\big) = f_{\psi}(g)$ for all $s \in {}^{<\omega}\omega$ and $g \in {}^{\omega}\omega$. This is possible because the convergence of $\{\varphi(s_n)\}_{n\in\omega}$ and $\{\psi(s_n)\}_{n\in\omega}$ depends on the sequence $\{s_n\}_{n\in\omega}$ in the way described by Lemma 1.3. Apply this fact to the H-scheme on the Cantor space from Example 1.4. □

2. Hurewicz theorem

Definition 2.1. Let X be a metric space. An H-scheme $\varphi' : {}^{<\omega}\omega \to X$ is a *subscheme* of an H-scheme $\varphi : {}^{<\omega}\omega \to X$, if there is an injective mapping $h : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $|h(s)| = |s|$, $t \subseteq s$ implies $h(t) \subseteq h(s)$, and $\varphi'(s) = \varphi\big(h(s)\big)$ for all $s \in {}^{<\omega}\omega$.

Subschemes of H-schemes preserve normality, separateness, and completeness. If φ' is a subscheme of an H-scheme φ, then $\operatorname{rng}(\varphi') \subseteq \operatorname{rng}(\varphi)$ and $G(\varphi') \subseteq G(\varphi)$. The notion of a subscheme is similar to the notion of "Restsystem" in [2].

Lemma 2.2. *Every H-scheme has a subscheme separated by a normal system of open sets.*

Proof. We define open balls V_s and a function $h\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ by induction on $|s|$ for $s \in {}^{<\omega}\omega$ so that

$$\varphi\big(h(s)\big) \in V_s \quad \text{and} \quad \operatorname{diam}(V_s) < 2^{-(n+s(n))} \qquad \text{for} \quad s \in {}^{n+1}\omega.$$

Set $h(\emptyset)=\emptyset$ and let $V_\emptyset$ be an open ball with center $\varphi\big(h(\emptyset)\big)$. Let $s\in{}^{<\omega}\omega$ be arbitrary and let us assume that $h(s)$ and V_s are defined and $\varphi\big(h(s)\big)\in V_s$. Let $\{k_n^s\}_{n\in\omega}$ be an increasing sequence such that

$$\varphi\big(h(s)^\frown k_n^s\big)\in V_s \quad\text{and}\quad d\Big(\varphi\big(h(s)^\frown k_{n+1}^s\big),\varphi\big(h(s)\big)\Big)<d\Big(\varphi\big(h(s)^\frown k_n^s\big),\varphi\big(h(s)\big)\Big).$$

Set $h(s^\frown n)=h(s)^\frown k_n^s$ and choose open balls $V_{s^\frown n}\subseteq V_s\setminus\big\{\varphi(s)\big\}$ with centers $\varphi\big(h(s^\frown n)\big)$ for $n\in\omega$ so that their closures are pairwise disjoint, do not contain $\varphi\big(h(s)\big)$, and their diameters are sufficiently small. □

Definition 2.3. Let $f:Y\to X$ and $f(Y)\subseteq D\subseteq X$. The function f is said to be *D-proper*, if for every nonempty open set $U\subseteq Y$ the set $\overline{f(U)}\setminus D$ is nonempty and has no isolated points.

Lemma 2.4. *Let X and Y be metric spaces, let $f:Y\to X$ be continuous, and let $f(Y)\subseteq D\subseteq X$. If Y is complete and f is D-proper, then there exists a complete H-scheme $\varphi:{}^{<\omega}\omega\to X\setminus D$ such that $G(\varphi)\subseteq f(Y)$.*

Proof. By induction on $|s|$ for $s\in{}^{<\omega}\omega$ we define a sequence $\big\langle V_s : s\in{}^{<\omega}\omega\big\rangle$ of nonempty open balls in Y and an H-scheme $\varphi:{}^{<\omega}\omega\to X\setminus D$ such that

(1) $\operatorname{diam}(V_s)<2^{-|s|}$ and $\operatorname{diam}\big(\overline{f(V_s)}\big)<2^{-|s|}$,

(2) $\{\overline{V_{s^\frown n}} : n\in\omega\}$ is a disjoint system of subsets of V_s,

(3) $\varphi(s)\in\overline{f(V_s)}\setminus D$.

Let $V_\emptyset\subseteq Y$ be any open ball such that $\operatorname{diam}(V_\emptyset)<1$ and $\operatorname{diam}\big(\overline{f(V_\emptyset)}\big)<1$, and let $\varphi(\emptyset)\in\overline{f(V_\emptyset)}\setminus D$. This is possible because f is continuous and D-proper.

Let us assume that V_s and $\varphi(s)\in\overline{f(V_s)}\setminus D$ have been constructed for a given $s\in{}^m\omega$. Since $\varphi(s)$ is not an isolated point of $\overline{f(V_s)}\setminus D$, we can choose a disjoint sequence $\langle U_n : n\in\omega\rangle$ of open balls in X such that $U_n\cap(\overline{f(V_s)}\setminus D)\neq\emptyset$, $U_n\subseteq B(\varphi(s),2^{-m-1})$, and $\varphi(s)\notin\overline{U_n}$; hence $\operatorname{diam}(U_n)<2^{-m-1}$. For each $n\in\omega$ let us choose an open ball $V_{s^\frown n}\subseteq f^{-1}(U_n)\cap V_s$ in Y with diameter $<2^{-m-1}$ such that $\overline{f(V_{s^\frown n})}\subseteq U_n$ and let $\varphi(s^\frown n)\in\overline{f(V_{s^\frown n})}\setminus D$.

Clearly, φ is an H-scheme and conditions (1)–(3) are satisfied. We prove that $G(\varphi)\subseteq f(Y)$. Let $g\in{}^\omega\omega$. Since Y is complete there is a unique $y\in\bigcap_{m\in\omega}\overline{V_{g\restriction m}}$. Then $f(y)=\lim_{m\to\infty}\varphi(g\restriction m)$ because $f(y)\in f(\overline{V_{g\restriction m}})\subseteq\overline{f(V_{g\restriction m})}$ and hence $d\big(f(y),\varphi(g\restriction m)\big)<2^{-m}$. □

Let κ be an infinite cardinal number. For a set $D\subseteq X$ let $I_{<\kappa}(D)$ denote the ideal over D generated by unions of $<\kappa$ closed sets in X which are subsets of D and let $I_\kappa(D)=I_{<\kappa^+}(D)$. Let us recall that $\operatorname{cov}(\mathcal{M})$ denotes the least cardinality of a family of meager sets covering the real line.

Theorem 2.5. *Let $\omega \le \kappa < \operatorname{cov}(\mathcal{M})$. Let X be a metric space, let $A \subseteq D \subseteq X$, and let A be a κ-Suslin set. The following conditions are equivalent:*

1. $A \notin I_{<\operatorname{cov}(\mathcal{M})}(D)$.
2. $A \notin I_\kappa(D)$.
3. *There is a continuous D-proper mapping $f\colon Y \to X$ for some complete metric space Y such that $f(Y) \subseteq A$ and the weight of Y is less or equal to κ.*
4. *There is a continuous D-proper mapping $f : Y \to X$ for some complete metric space Y such that $f(Y) \subseteq A$.*
5. *There is a complete H-scheme $\varphi : {}^{<\omega}\omega \to X \setminus D$ such that $G(\varphi) \subseteq A$.*
6. *There is a complete H-scheme $\varphi : {}^{<\omega}\omega \to X \setminus D$ separated by a normal system of open sets such that $G(\varphi) \subseteq A$.*
7. *There is a compact perfect set $P \subseteq X$ such that $P \cap D \subseteq A$, $P \setminus D$ is countable, and $\overline{P \setminus D} = P$.*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (4) are trivial, the implication (4) ⇒ (5) is Lemma 2.4 and (5) ⇒ (6) follows by Lemma 2.2.

(2) ⇒ (3) Let $g : {}^{\omega}\kappa \to X$ be a continuous function such that $g({}^{\omega}\kappa) = A$. As ${}^{\omega}\kappa$ has a base of open sets of size κ, there is a maximal open set U_0 in ${}^{\omega}\kappa$ such that $g(U_0) \in I_\kappa(D)$. Denote $Y = {}^{\omega}\kappa \setminus U_0$ and $f = g\restriction Y$. The function $f\colon Y \to X$ is continuous, Y is a complete metric space, and $f(Y) \subseteq A$. We verify that f is D-proper.

Let V be an open set in Y, i.e., $V = U \setminus U_0$ for some open set $U \subseteq {}^{\omega}\kappa$. If $\overline{f(V)} \subseteq D$, then $g(U) = \big(g(U) \setminus g(U_0)\big) \cup \big(g(U) \cap g(U_0)\big) \subseteq g(V) \cup g(U_0) \subseteq \overline{f(V)} \cup g(U_0) \in I_\kappa(D)$ and hence $V \subseteq U \subseteq U_0$. It follows that if V is a nonempty open set in Y, then $\overline{f(V)} \setminus D \neq \emptyset$. We prove that $\overline{f(V)} \setminus D$ has no isolated points. On the contrary, assume that $x \in \overline{f(V)} \setminus D$ is an isolated point and B is an open ball in X such that $B \cap \overline{f(V)} \setminus D = \{x\}$. The set $F = \overline{f(V)} \cap B \setminus \{x\}$ is an F_σ set in X contained in D, the set $V' = V \cap f^{-1}(B \setminus \{x\})$ is open in Y, and $g(V') \subseteq f(V) \cap B \setminus \{x\} \subseteq F \in I_\kappa(D)$. Let $U' \subseteq {}^{\omega}\kappa$ be such an open set that $V' = U' \setminus U_0$. Then $g(U') = \big(g(U') \setminus g(U_0)\big) \cup \big(g(U') \cap g(U_0)\big) \subseteq g(V') \cup g(U_0) \in I_\kappa(D)$. It follows that $V' = \emptyset$ because $V' \subseteq U' \subseteq U_0$. Then $f(V) \cap B \setminus \{x\} = \emptyset$ and $x \in f(V) \subseteq D$ because $x \in \overline{f(V)}$. This is a contradiction.

(6) ⇒ (7) If $\varphi : {}^{<\omega}\omega \to X \setminus D$ is a complete H-scheme separated by a normal system of open sets, then by Theorem 1.5, $P = \overline{\operatorname{rng}(\varphi)}$ is a compact perfect set, and since $G(\varphi) = H(\varphi) \subseteq A \subseteq D$, then $P \cap D = G(\varphi)$ and $P \setminus D = \operatorname{rng}(\varphi)$.

(7) ⇒ (1) Assume that P is a compact perfect set satisfying (7). Then $P \setminus A = P \setminus D$ is a countable dense subset of P. So, if $A \in I_{<\operatorname{cov}(\mathcal{M})}(D)$, then $P \cap A$ and

hence also P is the union of $< \operatorname{cov}(\mathcal{M})$ nowhere dense subsets of P. This is impossible and therefore $A \notin I_{<\operatorname{cov}(\mathcal{M})}(D)$. □

Let us note that the proof of the theorem requires neither that the metric space X is complete nor that X has a countable base. If we use another definition of a κ-Suslin set, then there may be some restrictions (for example, a set in a metric space of weight $\leq \kappa$ is a result of the Suslin operation $\mathcal{A}^\kappa$ applied to closed sets if and only if it is a continuous image of ${}^\omega\kappa$, see [4, Proof of Theorem 2B.1]).

Condition (3) as a variant of (4) was suggested by the referee.

Michal Staš [5] has proved the equivalence of conditions closely related to (2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) for analytic sets (i.e., ω_1-Suslin sets). His proof of the implication (2) $\Rightarrow$ (6) contains a direct construction of an H-scheme by induction using the following lemma:

Lemma 2.6. *Let X be a space with a countable base of open sets. Let $A \subseteq D \subseteq X$. Then for every open set $U \subseteq X$ such that $A \cap U \notin I_\omega(D)$ there are infinitely many points $p \in U \setminus D$ such that $A \cap V \notin I_\omega(D)$ for all neighbourhoods V of p.* □

Let us look at the meaning of the clause "infinitely many points $p \in U$" in the lemma. Let us note that the set A in the lemma need not be analytic. But if A is analytic and $A \cap U \notin I_\omega(D)$, then by Theorem 2.5 there is a continuous D-proper mapping $f\colon Y \to X$ defined on a Polish space Y such that $f(Y) \subseteq A \cap U$. For every open set $V \subseteq X$ such that $f(Y) \cap V \neq \emptyset$ the function $f\restriction\big(f^{-1}(V)\big)$ is again D-proper. Hence for every $p \in \overline{f(Y)} \setminus D$ and for every neighbourhood V of p we have $A \cap V \notin I_\omega(D)$.

It is well-known that every $\mathbf{\Sigma}^1_2$ set is ω_1-Suslin. By Theorem 2.5, if $\operatorname{cov}(\mathcal{M}) > \omega_1$, then a $\mathbf{\Sigma}^1_2$ set A in a (possibly non-complete) metric space is the union of $\leq \omega_1$ closed sets if and only if there is no compact perfect set P such that $P \setminus A$ is countable dense in P. (Take $D = A$ in the theorem.) In particular, we have:

Corollary 2.7. *If Martin's Axiom holds, then a $\mathbf{\Sigma}^1_2$ set A is the union of $< 2^\omega$ closed sets if and only if there is no compact perfect set P such that $P \setminus A$ is countable dense in P.* □

REFERENCES

[1] BARTOSZYŃSKI, T.—JUDAH, H.: *Set Theory. On the Structure of the Real Line.* A. K. Peters, Ltd., Wellesley, MA, 1995.

[2] HUREWICZ, W.: *Relativ perfekte Teile von Punktmengen und Mengen (A)*, Fund. Math. **12** (1928), 78–109.

[3] KECHRIS, A. S.: *Classical Descriptive Set Theory*. Springer-Verlag, New York, 1994.

[4] MOSCHOVAKIS, Y. N.: *Descriptive Set Theory,* in: Stud. Logic Found. Math., Vol. 100, North-Holland Publ. Comp., Amsterdam, 1980.

[5] STAŠ, M.: *Hurewicz scheme*, Acta Univ. Carolin. Math. Phys. **49** (2008), 75–78.

Received February 25, 2010

Mathematical Institute
Slovak Academy of Sciences
Jesenná 5
SK–041–54 Košice
SLOVAKIA

Department of Computer Science
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK–041-54 Košice
SLOVAKIA
E-mail: repicky@kosice.upjs.sk