

# BROOKS-JEWETT-TYPE THEOREMS FOR THE POINTWISE IDEAL CONVERGENCE OF MEASURES WITH VALUES IN $(I)$ -GROUPS

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ABSTRACT. Some Brooks-Jewett, Vitali-Hahn-Saks and Nikodým convergence-type theorems in the context of  $(I)$ -groups with respect to ideal convergence are proved. Moreover, an example is given.

## 1. Introduction

The theory of convergence with respect to ideals was introduced in [15] and is widely studied in the literature, in particular in problems concerning limit and integrals, even in abstract structures (see [8], where also a version of the basic matrix theorem is given in the context of  $(I)$ -groups). Observe that, in general, ideal convergence is strictly weaker than ordinary convergence (see [15]).

Here we present some versions of limit theorems of the type Brooks-Jewett, Vitali-Hahn-Saks and Nikodým in the context of  $(I)$ -groups, in which the existence of the “pointwise” limit measure is required only with respect to the convergence generated by a fixed  $P$ -ideal  $\mathcal{I}$ . For classical results on such theorems in the context of  $(I)$ -groups see, for instance, [5], [7]. Note that the ideal  $\mathcal{I}_d$  of the subsets of the natural numbers having zero asymptotic density is a  $P$ -ideal (see [12], [15]), and that  $\mathcal{I}_d$ -convergence coincides with the statistical convergence (see [13]). Observe that in general order and  $(D)$ -convergence are not topological: for instance, in the space  $L^0(X, \mathcal{B}, \mu)$ , where  $\mu$  is positive,  $\sigma$ -additive and  $\sigma$ -finite, they coincide with almost everywhere convergence. Some conditions under which order convergence coincides with topological convergence can be found in [10], and some comparison results between these two kinds of convergence are given in [16].

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We prove a sort of “weak” uniform  $(s)$ -boundedness ( $\sigma$ -additivity, absolute continuity) of the involved measures with respect to the given ideal, and we show by a counterexample that the existence of the “ $(\mathcal{I})$ -limit” measure, even when it is equal to zero and when our involved  $(l)$ -group is the real line, is not enough to get the classical uniform  $(s)$ -boundedness. Our technique is similar to the one used in [3].

## 2. Limit theorems

The following concepts and results were given in [8].

A Dedekind complete  $(l)$ -group  $R$  is said to be *super Dedekind complete* if every subset  $R_1 \subset R$ ,  $R_1 \neq \emptyset$  bounded from above contains a countable subset having the same supremum as  $R_1$ .

A sequence  $(p_n)_n$  of positive elements of  $R$  is an  $(O)$ -sequence if it is decreasing and  $\inf_n p_n = 0$ . A sequence  $(x_n)_n$  in  $R$  is said to be *order-convergent* (or  $(O)$ -convergent) to  $x$  if there exists an  $(O)$ -sequence  $(p_n)_n$  in  $R$  with  $|x_n - x| \leq p_n$ , for all  $n \in \mathbb{N}$ , and in this case we will write  $(O) \lim_n x_n = x$ .

A bounded double sequence  $(a_{t,l})_{t,l}$  in  $R$  is called  $(D)$ -sequence or *regulator* if for all  $t \in \mathbb{N}$  we have  $a_{t,l} \downarrow 0$  as  $l \rightarrow +\infty$ . A sequence  $(x_n)_n$  in  $R$  is said to be  $(D)$ -convergent to  $x \in R$  (and we write  $(D) \lim_n x_n = x$ ) if there exists a  $(D)$ -sequence  $(a_{t,l})_{t,l}$  in  $R$ , such that to every  $\varphi \in \mathbb{N}^{\mathbb{N}}$  there corresponds  $n_0 \in \mathbb{N}$  such that

$$|x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for all } n \in \mathbb{N}, n \geq n_0.$$

An  $(l)$ -group  $R$  is said to be *weakly  $\sigma$ -distributive* if for every  $(D)$ -sequence  $(a_{t,l})_{t,l}$  we have

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left( \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \right) = 0.$$

We recall the Fremlin lemma (see [17, Theorem 3.2.3]), whose role, under the hypothesis of equiboundedness, is the one of “replacing” countably many  $(D)$ -sequences with one regulator.

**LEMMA 2.1.** *Let  $R$  be a Dedekind complete  $(l)$ -group,  $(a_{i,l}^{(n)})_{i,l}$ ,  $n \in \mathbb{N}$ , be a sequence of regulators in  $R$ . Then for every  $u \in R$ ,  $u \geq 0$  there exists a  $(D)$ -sequence  $(a_{i,l})_{i,l}$  in  $R$  such that*

$$u \wedge \left[ \bigvee_{q=1}^{\infty} \left( \sum_{n=1}^q \left( \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}^{(n)} \right) \right) \right] \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{for all } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Let  $X$  be any nonempty set. A family of sets  $\mathcal{I} \subset \mathcal{P}(X)$  is called an *ideal* of  $X$  if  $A \cup B \in \mathcal{I}$  whenever  $A, B \in \mathcal{I}$  and for each  $A \in \mathcal{I}$  and  $B \subset A$  we get  $B \in \mathcal{I}$ . An ideal  $\mathcal{I}$  is said to be *non-trivial* if  $\mathcal{I} \neq \emptyset$  and  $X \notin \mathcal{I}$ . A non-trivial ideal  $\mathcal{I}$  is said to be *admissible* if it contains all singletons. An admissible ideal  $\mathcal{I}$  is said to be a *P-ideal* if for any sequence  $(A_j)_j$  in  $\mathcal{I}$  there are sets  $B_j \subset X$ ,  $j \in \mathbb{N}$ , such that the symmetric difference  $A_j \Delta B_j$  is finite for all  $j \in \mathbb{N}$  and  $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$  (see also [1]).

It is known that the ideal  $\mathcal{I}_d := \{A \subset \mathbb{N} : d(A) = 0\}$ , where  $d$  denotes the asymptotic density, is a *P-ideal*, as well as the ideal  $\mathcal{I}_{\text{fin}}$  of all finite subsets of  $\mathbb{N}$ , while there are also other examples of *P-ideals*, known in the literature (see for example [15]).

Now, given a *fixed* admissible ideal  $\mathcal{I}$ , together with its dual filter  $\mathcal{F} = \mathcal{F}(\mathcal{I}) := \{X \setminus I : I \in \mathcal{I}\}$ , we introduce the  $(D)$ -convergence related with it.

When we deal with an ideal  $\mathcal{I}$ , we always suppose that  $\mathcal{I}$  is admissible, without saying it explicitly.

A sequence  $(x_n)_n$  in  $R$   $(D\mathcal{I})$ -converges to  $x \in R$  if there exists a  $(D)$ -sequence  $(a_{t,l})_{t,l}$  such that for all  $\varphi \in \mathbb{N}^{\mathbb{N}}$  the following holds

$$\left\{ n \in \mathbb{N} : |x_n - x| \leq \bigvee_{t=1}^{\infty} a_{t, \varphi(t)} \right\} \in \mathcal{F}.$$

From now on, we always suppose that  $R$  is a super Dedekind complete weakly  $\sigma$ -distributive  $(l)$ -group. Examples of such spaces are  $\mathbb{R}^{\mathbb{N}}$  and  $L^0(X, \mathcal{S}, \mu)$  with  $\mu$  positive,  $\sigma$ -additive and  $\sigma$ -finite (see also [8]).

The following results were proved in [8].

**PROPOSITION 2.2.** *Let  $\mathcal{I}$  be any fixed admissible ideal of  $\mathbb{N}$ . If  $(D) \lim_n x_n = x$ , then  $(D\mathcal{I}) \lim_n x_n = x$ .*

*Moreover, if  $(x_n)_n$  is a monotone sequence in  $R$  and  $x \in R$ , then*

$$(D\mathcal{I}) \lim_n x_n = x \quad \text{if and only if} \quad (D) \lim_n x_n = x.$$

**PROPOSITION 2.3.** *Let  $(x_{i,j})_{i,j}$  be a bounded double sequence in  $R$ ,  $\mathcal{I}$  be any *P-ideal*,  $\mathcal{F} = \mathcal{F}(\mathcal{I})$  be its dual filter, and let us suppose that  $(D\mathcal{I}) \lim_i x_{i,j} = x_j$  for every  $j \in \mathbb{N}$ .*

*Then there exists  $B_0 \in \mathcal{F}$  such that  $(D) \lim_{h \rightarrow +\infty, h \in B_0} x_{h,j} = x_j$  for all  $j \in \mathbb{N}$  and with respect to a same  $(D)$ -sequence  $(\hat{a}_{t,l})_{t,l}$ .*

Let  $G$  be any infinite set,  $\mathcal{A} \subset \mathcal{P}(G)$  be an algebra, closed with respect to countable disjoint unions,  $\nu: \mathcal{A} \rightarrow [0, +\infty]$  be a finitely additive measure,  $m: \mathcal{A} \rightarrow R$  be a positive finitely additive measure.

We say that  $m$  is  $\nu$ -*absolutely continuous* [resp.  $\sigma$ -*additive*] if

$$(D\mathcal{I}) \lim_j m(H_j) = 0$$

whenever  $(H_j)_j$  is a decreasing sequence in  $\mathcal{A}$ , such that  $\lim_j \nu(H_j) = 0$  [resp.  $H_j \downarrow \emptyset$ ]. Note that, in this case, monotonicity of the sequence  $(m(H_j))_j$  and Proposition 2.2 guarantee us that

$$\inf_j m(H_j) = (D) \lim_j m(H_j) = (DI) \lim_j m(H_j) = 0.$$

We now recall the Maeda-Ogasawara-Vulikh representation theorem, which links “pointwise” and “lattice” suprema and infima (see [2]).

**THEOREM 2.4.** *Given a Dedekind complete  $(l)$ -group  $R$ , there exists a compact extremely disconnected topological space  $\Omega$ , unique up to homeomorphisms, such that  $R$  can be embedded as a solid subgroup of  $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^\Omega : f \text{ is continuous, and } \{\omega : |f(\omega)| = +\infty\} \text{ is nowhere dense in } \Omega\}$ . Moreover, if  $(a_\lambda)_{\lambda \in \Lambda}$  is any family such that  $a_\lambda \in R$  for all  $\lambda$ , and  $a = \sup_\lambda a_\lambda \in R$  (where the supremum is taken with respect to  $R$ ), then  $a = \sup_\lambda a_\lambda$  with respect to  $\mathcal{C}_\infty(\Omega)$ , and the set  $\{\omega \in \Omega : (\sup_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega)\}$  is meager in  $\Omega$ .*

We now prove the following result.

**THEOREM 2.5.** *Let  $G$  be any infinite set;  $\mathcal{A} \subset \mathcal{P}(G)$  be an algebra, closed with respect to countable disjoint unions,  $(m_i : \mathcal{A} \rightarrow R)_i$  be an equibounded sequence of positive finitely additive measures,  $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$  be a  $P$ -ideal. Suppose that  $m_0(E) := (DI) \lim_i m_i(E)$  exists in  $R$  for every  $E \in \mathcal{A}$ , and that  $m_0$  is  $\sigma$ -additive on  $\mathcal{A}$ .*

*Then for every disjoint sequence  $(C_j)_j$  in  $\mathcal{A}$  there exists  $A \in \mathcal{F}(\mathcal{I})$  such that*

$$(O) \lim_j \left[ \sup_{i \in A} m_i(C_j) \right] = (D) \lim_j \left[ \sup_{i \in A} m_i(C_j) \right] = 0.$$

**Proof.** Let  $(C_j)_j$  be any fixed disjoint sequence in  $\mathcal{A}$ , and  $\mathcal{L}$  be the  $\sigma$ -algebra generated by the  $C_j$ 's. For any  $B \in \mathcal{L}$  there is  $P \subset \mathbb{N}$  with  $B = \bigcup_{j \in P} C_j$ . By hypothesis, since we are considering substantially only countably many “objects” and  $\mathcal{I}$  is a  $P$ -ideal, from Lemma 2.1 and Proposition 2.3 there are a set  $A \in \mathcal{F}(\mathcal{I})$  and a regulator  $(\eta_{t,l})_{t,l}$  with  $(D) \lim_{i \in A} m_i(C_j) = m_0(C_j)$ ,  $j \in \mathbb{N}$ ;

$$(D) \lim_{i \in A} \left[ m_i \left( \bigcup_{j=1}^{\infty} C_j \right) \right] = m_0 \left( \bigcup_{j=1}^{\infty} C_j \right);$$

$$(D) \lim_{i \in A} \left[ m_i \left( \bigcup_{j \leq k} C_j \right) \right] = m_0 \left( \bigcup_{j \leq k} C_j \right), \quad k \in \mathbb{N}$$

with respect to  $(\eta_{t,l})_{t,l}$ .

Let now  $\mathcal{B}$  be the countable class whose elements are all finite unions of the  $C_j$ 's and the set  $\bigcup_{j=1}^{\infty} C_j$ . Proceeding analogously as above and thanks to

[4, Theorem 3.4], an  $(O)$ -sequence  $(p_l)_l$  (depending of  $(C_j)_j$ ) can be found, with the property that to every  $l \in \mathbb{N}$  and  $H \in \mathcal{B}$  there corresponds  $i_0 \in A$  with  $|m_i(H) - m_0(H)| \leq p_l$  whenever  $i \in A, i \geq i_0$ .

By Theorem 2.4 we get the existence of a meager set  $N' \subset \Omega$  such that the sequence  $(p_l(\omega))_l$  is an  $(O)$ -sequence for any  $\omega \notin N'$ , and thus

$$\sup_{i \geq 0} \left[ \sup_{E \in \mathcal{A}} m_i(E)(\omega) \right] \in \mathbb{R}$$

for each  $\omega \notin N'$ , and for all  $\varepsilon > 0$ ,  $\omega \notin N'$  and  $H \in \mathcal{B}$  there is  $i_0 \in A$  such that  $|m_i(H)(\omega) - m_0(H)(\omega)| \leq p_l(\omega)$  whenever  $i \in A, i \geq i_0$ . Furthermore, by  $\sigma$ -additivity of  $m_0$  and Theorem 2.4 again, there exists a meager set  $N \subset \Omega$ , without loss of generality  $N \supset N'$ , such that

$$\lim_k \left[ m_0 \left( \bigcup_{j>k} C_j \right) (\omega) \right] = 0 \quad \text{for all } \omega \notin N.$$

For each  $\varepsilon > 0$  and  $\omega \in \Omega \setminus N$  there exists  $k_0(\varepsilon, \omega)$  such that  $m_0(\bigcup_{j \geq k_0} C_j)(\omega) \leq \varepsilon$ , and hence, by positivity of  $m_0$ ,  $m_0(\bigcup_{j \geq k_0, j \in P} C_j)(\omega) \leq \varepsilon$ . Moreover there is  $i_0 \in A, i_0 = i_0(\varepsilon, \omega, k_0)$  with the property that, for all  $i \in A, i \geq i_0$ ,

$$\begin{aligned} \left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| &\leq \varepsilon, \\ \left| m_i \left( \bigcup_{j \leq k_0} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0} C_j \right) (\omega) \right| &\leq \varepsilon, \\ \left| m_i \left( \bigcup_{j=1}^{\infty} C_j \right) (\omega) - m_0 \left( \bigcup_{j=1}^{\infty} C_j \right) (\omega) \right| &\leq \varepsilon, \end{aligned}$$

and therefore

$$\left| m_i \left( \bigcup_{j>k_0} C_j \right) (\omega) - m_0 \left( \bigcup_{j>k_0} C_j \right) (\omega) \right| \leq 2\varepsilon.$$

Let now  $B \in \mathcal{L}$ . For all  $i \in A, i \geq i_0$  we have

$$\begin{aligned} 0 &\leq |m_i(B)(\omega) - m_0(B)(\omega)| \\ &= \left| m_i \left( \bigcup_{j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \in P} C_j \right) (\omega) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| \\
 &\quad + m_0 \left( \bigcup_{j > k_0, j \in P} C_j \right) (\omega) + m_i \left( \bigcup_{j > k_0, j \in P} C_j \right) (\omega) \\
 &\leq \left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| \\
 &\quad + m_0 \left( \bigcup_{j > k_0} C_j \right) (\omega) + m_i \left( \bigcup_{j > k_0} C_j \right) (\omega) \\
 &\leq \left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| \\
 &\quad + \left| m_i \left( \bigcup_{j > k_0} C_j \right) (\omega) - m_0 \left( \bigcup_{j > k_0} C_j \right) (\omega) \right| \\
 &\quad + 2 m_0 \left( \bigcup_{j > k_0} C_j \right) (\omega) \leq 5\varepsilon.
 \end{aligned}$$

Thus,  $\lim_{i \in A} m_i(B)(\omega) = m_0(B)(\omega)$  for all  $\omega \in \Omega \setminus N$  and  $B \in \mathcal{L}$  (Note that  $A$  and  $N$ , in general, depend (only) on the given sequence  $(C_j)_j$ ). So, the finitely additive real-valued measures  $m_i(\cdot)(\omega)$ ,  $i \in A$ ,  $\omega \notin N$ , satisfy the hypotheses of the classical version of the Brooks-Jewett theorem on  $\mathcal{L}$ . Thus for all disjoint sequences  $(C_j)_j$  in  $\mathcal{L}$  we get:  $\lim_j [\sup_{i \in A} m_i(C_j)(\omega)] = 0$  for all  $\omega \notin N$ . Since  $N$  is meager, we get  $(O) \lim_j [\sup_{i \in A} m_i(C_j)] = (D) \lim_j [\sup_{i \in A} m_i(C_j)] = 0$ .  $\square$

We now give some versions of the Vitali-Hahn-Saks and Nikodým convergence theorems.

**THEOREM 2.6.** *Let  $G$  be any infinite set and  $\mathcal{A} \subset \mathcal{P}(G)$  be any algebra;  $\nu: \mathcal{A} \rightarrow [0, +\infty]$  be a finitely additive measure;  $(m_i: \mathcal{A} \rightarrow \mathbb{R})_i$  be an equibounded sequence of positive  $\nu$ -absolutely continuous [resp.  $\sigma$ -additive] measures,  $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$  be a  $P$ -ideal. Suppose that  $m_0(E) := (DI) \lim_i m_i(E)$  exists in  $\mathbb{R}$  for every  $E \in \mathcal{A}$ , and that  $m_0$  is  $\nu$ -absolutely continuous [ $\sigma$ -additive] on  $\mathcal{A}$ .*

*Then for every decreasing sequence  $(H_j)_j$  in  $\mathcal{A}$  with  $\nu(H_j) \downarrow 0$  [ $H_j \downarrow \emptyset$ ] there exists  $A \in \mathcal{F}(\mathcal{I})$  such that*

$$(O) \lim_j \left[ \sup_{i \in A} m_i(H_j) \right] = (D) \lim_j \left[ \sup_{i \in A} m_i(H_j) \right] = 0.$$

*P r o o f.* We prove the result concerning absolute continuity; analogously one can check the result involving  $\sigma$ -additivity. Let  $(H_j)_j$  be a fixed decreasing sequence in  $\mathcal{A}$ , with  $\nu(H_j) \downarrow 0$ . For each  $j \geq 1$ , put  $C_j := H_j \setminus H_{j+1}$ ,  $C_0 := \bigcap_{j=1}^{\infty} H_j$ ; let  $\mathcal{L}$  be the  $\sigma$ -field generated by  $C_j$ ,  $j \geq 0$ . Since  $\mathcal{I}$  is a  $P$ -ideal and we deal with only countably many “objects”, proceeding analogously as in Theorem 2.5 we get the existence of a meager set  $N \subset \Omega$  such that, for all  $\omega \notin N$ ,  $\sup_{i \geq 0} [\sup_{E \in \mathcal{A}} m_i(E)(\omega)] \in \mathbb{R}$ , and of a set  $A \in \mathcal{F}(\mathcal{I})$  with  $\lim_{i \in A} m_i(E)(\omega) = m_0(E)(\omega)$  for every set  $E$  belonging to the algebra  $\mathcal{L}_0$  generated by  $H_j$ ,  $j \in \mathbb{N}$ , in  $H_1$ ;  $\lim_k m_i(H_k)(\omega) = 0$  for all  $i \in \mathbb{N} \cup \{0\}$ . Analogously as in [3, Theorem 2.6], it is possible to show that each  $m_i|_{\mathcal{L}_0}$  admits a countably additive extension  $\widetilde{m}_i$  to  $\mathcal{L}$ , and that such extension is the unique finitely additive map, defined on  $\mathcal{L}$  and agreeing with  $m_i$  on  $\mathcal{L}_0$ .

Fix  $B \in \mathcal{L}$ . Then there is  $P \subset \mathbb{N}$  with  $B = \bigcup_{j \in P} C_j$ . For every  $i \geq 0$ , set

$$\widetilde{m}_i(B) := \sup_n m_i(\bigcup_{j \in P, j \leq n} C_j) = (O) \lim_n m_i(\bigcup_{j \in P, j \leq n} C_j).$$

It is easy to check that  $\widetilde{m}_i$  is an extension of  $m_i$ . Moreover, if  $\widehat{m}_i$  is another extension of  $m_i$ , we have:

$$\widehat{m}_i(B) \geq \sup_n m_i(\bigcup_{j \in P, j \leq n} C_j),$$

by virtue of monotonicity of  $\widehat{m}_i$ ;

$$\widehat{m}_i(B) = m_i(\bigcup_{j \in P, j \leq n} C_j) + \widehat{m}_i(\bigcup_{j \in P, j > n} C_j) \leq m_i(\bigcup_{j \in P, j \leq n} C_j) + m_i(H_n)$$

for all  $n \in \mathbb{N}$ . Thus we get

$$0 \leq \widehat{m}_i(B) - m_i(\bigcup_{j \in P, j \leq n} C_j) \leq m_i(H_n)$$

for all  $n \in \mathbb{N}$ . As  $m_i(H_n) \downarrow 0$ , we obtain

$$\widehat{m}_i(B) = (O) \lim_n m_i(\bigcup_{j \in P, j \leq n} C_j) = \widetilde{m}_i(B)$$

which proves the claim about  $\widetilde{m}_i$ .

We now prove that  $\widetilde{m}_0(B)(\omega) = \lim_{i \in A} \widetilde{m}_i(B)(\omega)$  for all  $\omega \in \Omega \setminus N$  and  $B \in \mathcal{L}$ . Fix  $\omega \notin N$  and  $B \in \mathcal{L}$ . Let  $P \subset \mathbb{N}$  be related to  $B$  as before. For each  $\varepsilon > 0$  there exists  $k_0(\varepsilon, \omega)$  such that  $m_0(H_{k_0})(\omega) < \varepsilon$ . Moreover, there exists  $i_0 \in A$ ,  $i_0 = i_0(\varepsilon, \omega, k_0)$  such that for all  $i \in A$ ,  $i \geq i_0$ ,

$$\left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| \leq \varepsilon,$$

$$|m_i(H_{k_0})(\omega) - m_0(H_{k_0})(\omega)| \leq \varepsilon.$$

Thus for all  $i \in A$ ,  $i \geq i_0$ , we have

$$0 \leq |\widetilde{m}_i(B)(\omega) - \widetilde{m}_0(B)(\omega)|$$

$$\begin{aligned}
 &\leq \left| m_i \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) - m_0 \left( \bigcup_{j \leq k_0, j \in P} C_j \right) (\omega) \right| \\
 &\quad + \widetilde{m}_i \left( \bigcup_{j > k_0, j \in P} C_j \right) (\omega) + \widetilde{m}_0 \left( \bigcup_{j > k_0, j \in P} C_j \right) (\omega) \\
 &\leq \varepsilon + |m_i(H_{k_0})(\omega) - m_0(H_{k_0})(\omega)| + 2m_0(H_{k_0})(\omega) \leq 4\varepsilon.
 \end{aligned}$$

So we get:  $\lim_{i \in A} \widetilde{m}_i(B)(\omega) = \widetilde{m}_0(B)(\omega)$  for all  $\omega \in \Omega \setminus N$  and  $B \in \mathcal{L}$ , that is the claim.

Fix now  $\omega \notin N$  and let  $\mathcal{L} \ni A_s \downarrow$ , with  $\nu(A_s) \downarrow 0$ . Note that every  $A_s$  is a union of some  $C_j$ .

It is not difficult to show that there is a subsequence  $(A_{s_k})_{k \geq 2}$  with  $\widetilde{m}_i(A_{s_k})(\omega) \leq \widetilde{m}_i(H_k)(\omega)$  for any  $i$  and  $k \in \mathbb{N}$  (see also [7]).

Indeed, if there is  $q_2 \in \mathbb{N}$  with  $A_s \supset C_1$  for  $s > q_2$ , then  $\nu(C_1) = 0$  and hence  $\widetilde{m}_i(C_1)(\omega) = 0$  for all  $i \in \mathbb{N} \cup \{0\}$ . So, in this case, we get

$$\widetilde{m}_i(A_s)(\omega) = \widetilde{m}_i(A_s \setminus C_1)(\omega) \leq \widetilde{m}_i(H_2)(\omega)$$

for all  $s > q_2$  and  $i \in \mathbb{N} \cup \{0\}$ . Otherwise there is  $l_2 > 1$  such that  $A_{l_2} \subset H_2$ , and hence  $\widetilde{m}_i(A_{l_2})(\omega) \leq \widetilde{m}_i(H_2)(\omega)$  for all  $i$ . In any case, for at least an index  $s_2$  and for any  $i$  we have:  $\widetilde{m}_i(A_{s_2})(\omega) \leq \widetilde{m}_i(H_2)(\omega)$ .

In the following step, we get still two cases. If there is a positive integer  $q_3$  with  $A_s \supset C_1 \cup C_2$  for all  $s > q_3$ , then  $\nu(C_1 \cup C_2) = 0$  and thus  $\widetilde{m}_i(C_1 \cup C_2)(\omega) = 0$  for all  $i$ . Therefore in this case, for each  $s > q_3$  and  $j \in \mathbb{N}$ ,

$$\widetilde{m}_i(A_s)(\omega) = \widetilde{m}_i(A_s \setminus (C_1 \cup C_2))(\omega) \leq \widetilde{m}_i(H_3)(\omega).$$

If not, then there exists  $l_3 > s_2$  with  $A_{l_3} \subset H_3$ , and so  $\widetilde{m}_i(A_{l_3})(\omega) \leq \widetilde{m}_i(H_3)(\omega)$  for all  $i$ . In any case,  $\widetilde{m}_i(A_{s_3})(\omega) \leq \widetilde{m}_i(H_3)(\omega)$  for at least an integer  $s_3 > s_2$  and for all  $i$ . Proceeding by induction, to every  $k \geq 2$  there corresponds an element  $A_{s_k}$  with  $\widetilde{m}_i(A_{s_k})(\omega) \leq \widetilde{m}_i(H_k)(\omega)$  for any  $i$ , and  $s_k < s_{k+1}$  for all  $k \geq 2$ . This proves the claim.

Thanks to  $\nu$ -absolute continuity of every  $\widetilde{m}_i$ , we get

$$0 \leq \inf_k \widetilde{m}_i(A_k)(\omega) \leq \inf_k \widetilde{m}_i(A_{s_k})(\omega) \leq \inf_k \widetilde{m}_i(H_k)(\omega) = 0$$

for each  $i$ . Hence the maps  $\widetilde{m}_i(\cdot)(\omega)$ ,  $\omega \notin N$ ,  $i \geq 0$ , are  $\nu$ -absolutely continuous on  $\mathcal{L}$ . By the classical Vitali-Hahn-Saks theorem, the mappings  $\widetilde{m}_i(\cdot)(\omega)|_{\mathcal{L}}$ ,  $i \in A$ ,  $\omega \in \Omega \setminus N$ , are  $\nu$ -uniformly absolutely continuous for all  $\omega \notin N$ . Hence,  $\inf_j \left[ \sup_{i \in A} m_i(H_j)(\omega) \right] = 0$  for all  $\omega \notin N$ . As  $N$  is meager, we have

$$\inf_j \left[ \sup_{i \in A} m_i(H_j) \right] = (O) \lim_j \left[ \sup_{i \in A} m_i(H_j) \right] = (D) \lim_j \left[ \sup_{i \in A} m_i(H_j) \right] = 0,$$

that is the assertion.  $\square$



**Remark 2.7.** We ask whether if  $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$  is any fixed admissible ideal,  $m_i, i \in \mathbb{N}$ , are  $\sigma$ -additive positive measures and  $(DI) \lim_i m_i(E)$  exists for every  $E \in \mathcal{A}$ , then for every disjoint sequence  $(C_j)_j$  in  $\mathcal{A}$  one has

$$(DI) \lim_j \left[ \sup_{i \in \mathbb{N}} m_i(C_j) \right] = 0.$$

The answer is in general negative, even for real-valued measures, as the following example shows.

Let  $G = \mathbb{N}$ ,  $\mathcal{A} = \mathcal{P}(\mathbb{N})$ ,  $R = \mathbb{R}$ ,  $H := \{h_1 < \dots < h_s < h_{s+1} < \dots\}$  be an infinite set belonging to  $\mathcal{I}$  and such that  $\mathbb{N} \setminus H$  is infinite. Since  $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ , then  $H$  does exist. For every  $i \notin H$  and  $E \subset \mathbb{N}$ , set  $m_i(E) = 0$ . For any  $s \in \mathbb{N}$  and  $E \subset \mathbb{N}$ , set  $m_{h_s}(E) = 1$  if  $s \in E$  and 0 otherwise. Note that  $m_0(E) := (DI) \lim_i m_i(E) = 0$  for each  $E \subset \mathbb{N}$ . Moreover, it is easy to check that the  $m_i$ 's are  $\sigma$ -additive positive equibounded measures. Indeed, given  $i \in \mathbb{N}$  and any disjoint sequence  $(C_j)_j$  of subsets of  $\mathbb{N}$ , the entity  $m_i(C_j)$  can be different from zero (and in this case is equal to 1) at most for one index  $j$ , because for all  $s \in \mathbb{N}$  we get that  $m_i(\{s\}) \neq 0$  if and only if  $i = h_s$ .

For every  $j \in \mathbb{N}$  set  $C_j := \{j\}$ . Of course, the  $C_j$ 's are pairwise disjoint. For each  $j \in \mathbb{N}$  we get:  $1 \geq \sup_{i \in \mathbb{N}} m_i(C_j) \geq m_{h_j}(C_j) = 1$ . This proves the claim.  $\square$

Observe that, as a consequence of [11, Theorem 3.2 and Remarks 3.3, 3.4], and since every real-valued bounded map has  $\mathcal{I}$ -limit for each maximal ideal  $\mathcal{I}$  ([6, Proposition 2.2]), then, at least when  $R = \mathbb{R}$  and the involved  $P$ -ideal  $\mathcal{I}$  is maximal, the existence of the  $\mathcal{I}$ -limit measure can be included in the thesis of our results, because it is an immediate consequence of equiboundedness. If we assume the continuum hypothesis, then there are ideals which are at the same time maximal and  $P$ -ideals (see [14]).

**Open problem.** Under which conditions can one obtain  $\sigma$ -additivity of the  $\mathcal{I}$ -limit measure in the conclusions (without assuming necessarily that the involved  $P$ -ideal  $\mathcal{I}$  is maximal)?

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