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MARKOV TYPE POLYNOMIAL INEQUALITY FOR SOME GENERALIZED HERMITE WEIGHT

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ABSTRACT. In this paper we study some weighted polynomial inequalities of Markov type in L^2-norm. We use the properties of the system of generalized Hermite polynomials $\{H_n^{(\alpha)}(x)\}_{n=0}^{\infty}$. The polynomials $H_n^{(\alpha)}(x)$ are orthogonal in $\mathbb{R} = (-\infty, \infty)$ with respect to the weight function

$$W(x) = |x|^{2\alpha} e^{-x^2}, \qquad \alpha > -\frac{1}{2}.$$

The classical Hermite polynomials $H_n(x)$ present the special case for $\alpha = 0$.

1. Introduction

Let $\|f\|_\infty = \sup_{a \le x \le b} |f(x)|$ and let Π_n be a set of all algebraic polynomials of the degree at most n. For sufficiently smooth function $f(x)$ on $\mathbb{R}$, Kolmogorov in [5] established the inequality

$$\left\| f^{(k)} \right\|_\infty \le C_{n,k} \left\| f^{(n)} \right\|_\infty^{\frac{k}{n}} \|f\|_\infty^{1-\frac{k}{n}}, \qquad 0 < k < n$$

with some constant $C_{n,k}$.
The classical Markov's inequality (cf. [7])

$$\|p'\|_\infty \le n^2 \|p\|_\infty, \qquad -1 \le x \le 1,\ p(x) \in \Pi_n \tag{1.1}$$

and its extension

$$\left\|p^{(k)}\right\|_\infty \le \frac{1}{(2k-1)!!} \prod_{i=0}^{k-1} (n^2 - i^2) \|p\|_\infty, \qquad p(x) \in \Pi_n,\ 1 \le k \le n$$

are typical examples of inequalities connecting norms of a polynomial and its derivatives. In (1.1) the equality holds only at $x = \pm 1$ and only when $p(x) = cT_n(x)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind of a degree n

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and c is an arbitrary constant. These facts motivated many authors to investigate polynomial analogues of Kolmogorov's and Markov's inequalities in norms taken with some weight functions (cf. [1], [2], [4], [6], [13]).

Consider now an inner product of the functions $f(x)$, $g(x)$ with the weight function $w(x)$ denoted $(f,g)_w = \int_a^b f(x)g(x)w(x)\,dx$ and weighted L^2-norm of $f(x)$ denoted $\|f\|_w^2 = (f,f)_w$.

In the present work we need some properties of the system of generalized Hermite polynomials orthogonal in $\mathbb{R} = (-\infty,\infty)$ with respect to the weight function

$$W(x) = |x|^{2\alpha} e^{-x^2}, \qquad \alpha > -\frac{1}{2}. \tag{1.2}$$

This weight function has the singular point $x = 0$ and classical Hermite polynomials $H_n(x)$ are the special case of considered polynomials $H_n^{(\alpha)}(x)$ for $\alpha = 0$. Let $\{\tilde{H}_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ be a system of polynomials orthonormal with the weight (1.2). These polynomials satisfy the three-term recurrence relations (cf. [3])

$$x\tilde{H}_{2k}^{(\alpha)}(x) = \sqrt{\frac{2k+1+2\alpha}{2}}\,\tilde{H}_{2k+1}^{(\alpha)}(x) + \sqrt{k}\,\tilde{H}_{2k-1}^{(\alpha)}(x), \tag{1.3}$$

$$x\tilde{H}_{2k+1}^{(\alpha)}(x) = \sqrt{k+1}\,\tilde{H}_{2k+2}^{(\alpha)}(x) + \sqrt{\frac{2k+1+2\alpha}{2}}\,\tilde{H}_{2k}^{(\alpha)}(x) \tag{1.4}$$

and

$$\frac{d}{dx}\tilde{H}_{2k}^{(\alpha)}(x) = 2\sqrt{k}\,\tilde{H}_{2k-1}^{(\alpha)}(x), \tag{1.5}$$

$$\frac{d}{dx}\tilde{H}_{2k+1}^{(\alpha)}(x) = -\frac{2\alpha}{x}\tilde{H}_{2k+1}^{(\alpha)}(x) + 2\sqrt{\frac{2k+1+2\alpha}{2}}\,\tilde{H}_{2k}^{(\alpha)}(x), \qquad x \neq 0, \tag{1.6}$$

where $\tilde{H}_{-1}^{(\alpha)}(x) = 0$ and $k = 0, 1, 2, \dots$

2. Preliminary lemmas

Lemma 2.1.

$$\left\|\frac{d}{dx}\tilde{H}_{2k}^{(\alpha)}\right\|_W^2 = 4k, \tag{2.1}$$

$$\left\|\frac{d}{dx}\tilde{H}_{2k+1}^{(\alpha)}\right\|_W^2 = 4\alpha^2\left\|\frac{\tilde{H}_{2k+1}^{(\alpha)}}{x}\right\|_W^2 - 8\alpha + 2(2k+1+2\alpha). \tag{2.2}$$

P r o o f. Because the interval $\mathbb{R}$ is symmetric and the weight function $W(x)$ is even, we can write

$$\tilde{H}_n^{(\alpha)}(-x) = (-1)^n \tilde{H}_n^{(\alpha)}(x)$$

(cf. [11, p. 15]) and clearly,

$$\frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \in \Pi_{2k}.$$

Now the proof follows from (1.3), (1.5), (1.6) and from the orthogonal properties

$$(\tilde{H}_n^{(\alpha)}, Q_m)_W = 0\,, \qquad Q_m(x) \in \Pi_{n-1},$$
$$(\tilde{H}_i^{(\alpha)}, \tilde{H}_j^{(\alpha)})_W = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. □

Lemma 2.2.

$$\left\|\frac{\tilde{H}_{2k+1}^{(\alpha)}}{x}\right\|_W^2 = \frac{2}{2k+1+2\alpha} + \frac{2^2 k}{(2k+1+2\alpha)(2k-1+2\alpha)} + \frac{2^3 k(k-1)}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)} + \dots + \frac{2^{k+1} k(k-1)\dots 2.1}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}. \tag{2.3}$$

P r o o f. We use an induction on k. For $k=0$ with regard to (1.3) we have

$$\left\|\frac{\tilde{H}_1^{(\alpha)}}{x}\right\|_W^2 = \frac{2}{1+2\alpha}\left\|\tilde{H}_0^{(\alpha)}\right\|_W^2 = \frac{2}{1+2\alpha}\,.$$

Suppose that for some $k>0$ there is

$$\left\|\frac{\tilde{H}_{2k-1}^{(\alpha)}}{x}\right\|_W^2 = \frac{2}{2k-1+2\alpha} + \frac{2^2(k-1)}{(2k-1+2\alpha)(2k-3+2\alpha)} + \frac{2^3(k-1)(k-2)}{(2k-1+2\alpha)(2k-3+2\alpha)(2k-5+2\alpha)} + \dots + \frac{2^k(k-1)(k-2)\dots 2.1}{(2k-1+2\alpha)(2k-3+2\alpha)(2k-5+2\alpha)\dots(1+2\alpha)}.$$

Applying this induction hypothesis, (1.3), and the basic general properties of orthogonal polynomial, we have

$$\begin{aligned}\left\|\frac{\tilde{H}_{2k+1}^{(\alpha)}}{x}\right\|_W^2 &= \frac{2}{2k+1+2\alpha}\left\|\tilde{H}_{2k}^{(\alpha)} - \sqrt{k}\frac{\tilde{H}_{2k-1}^{(\alpha)}}{x}\right\|_W^2\\ &= \frac{2}{2k+1+2\alpha}\left\{\left\|\tilde{H}_{2k}^{(\alpha)}\right\|_W^2 + k\left\|\frac{\tilde{H}_{2k-1}^{(\alpha)}}{x}\right\|_W^2\right\}\\ &= \frac{2}{2k+1+2\alpha}\left\{1 + k\left\|\frac{\tilde{H}_{2k-1}^{(\alpha)}}{x}\right\|_W^2\right\}\end{aligned}$$

$$= \frac{2}{2k+1+2\alpha} + \frac{2^2 k}{(2k+1+2\alpha)(2k-1+2\alpha)} + \dots$$

$$\dots + \frac{2^{k+1} k(k-1)\dots 2.1}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}$$

and lemma is proved. □

Lemma 2.3. *The inequalities*

$$\left\| \frac{\tilde{H}^{(\alpha)}_{2k+1}}{x} \right\|^2_W \le \frac{2k+2}{2k+1+2\alpha}, \qquad \alpha \ge \frac{1}{2},$$

$$\left\| \frac{\tilde{H}^{(\alpha)}_{2k+1}}{x} \right\|^2_W < \frac{2^{k+1}(k+1)!}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}, \qquad \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

hold.

Proof. In (2.3) let us denote

$$\left\| \frac{\tilde{H}^{(\alpha)}_{2k+1}}{x} \right\|^2_W = B_0 + B_1 + B_2 + \dots + B_k.$$

It is obvious that we have a recurrence relation

$$B_j = B_{j-1} \, \frac{2[k-(j-1)]}{2k-(2j-1)+2\alpha}, \tag{2.4}$$

where $j = 1, 2, \dots, k$.

For $\alpha \ge \frac{1}{2}$ from (2.4) we have $B_{j-1} \ge B_j$, and clearly we get the first inequality

$$\left\| \frac{\tilde{H}^{(\alpha)}_{2k+1}}{x} \right\|^2_W \le (k+1)B_0 = \frac{2k+2}{2k+1+2\alpha}.$$

Let $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Then $B_{j-1} < B_j$, and the inequality

$$\left\| \frac{\tilde{H}^{(\alpha)}_{2k+1}}{x} \right\|^2_W < (k+1)B_k = \frac{2^{k+1}(k+1)!}{(2k+1+2\alpha)(2k-1+2\alpha)(2k-3+2\alpha)\dots(1+2\alpha)}$$

holds. The proof is complete. □

3. Main theorems

Using the obtained results, we can formulate polynomial inequalities of Markov type in weighted L^2-norm of polynomials $p_n(x) \in \Pi_n$.

Theorem 3.1. *Let $W(x)$ be given by (1.2). Then there exists a constant*

$$C = C(\alpha)$$

such that

$$\|p_n'\|_W \le Cn\|p_n\|_W \tag{3.1}$$

for all polynomials $p_n(x) \in \Pi_n$.

P r o o f. First, using Lemma 2.3 for $\alpha \ge \frac{1}{2}$, we get the relation

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 = O(1). \tag{3.2}$$

The case $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ is more complicated. Now we use the definition of Gamma function due to Gauss

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)(x+2)\dots(x+n)}.$$

Then Lemma 2.3 implies

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-\alpha}}(k+1)B_k \\
&= \lim_{k\to\infty} \frac{1}{k^{\frac{1}{2}-\alpha}} \, \frac{k+1}{k^{\alpha+\frac{1}{2}}} \, \frac{k!\, k^{\alpha+\frac{1}{2}}}{\left(k+\frac{1}{2}+\alpha\right)\left(k-\frac{1}{2}+\alpha\right)\left(k-\frac{3}{2}+\alpha\right)\dots\left(\frac{1}{2}+\alpha\right)} \\
&= \Gamma\left(\frac{1}{2}+\alpha\right),
\end{aligned}$$

from where we deduce

$$\left\| \frac{\tilde{H}_{2k+1}^{(\alpha)}}{x} \right\|_W^2 = O(k^{\frac{1}{2}-\alpha}). \tag{3.3}$$

Second, every $p_n(x) \in \Pi_n$ can be uniquely represented as a linear combination of the system $\{\tilde{H}_i^{(\alpha)}(x)\}_{i=0}^n$ (cf. [11])

$$p_n(x) = \sum_{i=0}^n c_i \tilde{H}_i^{(\alpha)}(x),$$

where $c_i = \left(\tilde{H}_i^{(\alpha)}, p_n\right)_W$. From this we have

$$p_n'(x) = \sum_{i=0}^n c_i \frac{d}{dx} \tilde{H}_i^{(\alpha)}(x)$$

and by Schwarz inequality,

$$\begin{aligned} {p'_n}^2(x) &= \left[\sum_{i=0}^{n} c_i \frac{d}{dx}\tilde{H}_i^{(\alpha)}(x)\right]^2 \\ &\le \sum_{i=0}^{n} c_i^2 \sum_{i=0}^{n}\left[\frac{d}{dx}\tilde{H}_i^{(\alpha)}(x)\right]^2 \\ &= \|p_n\|_W^2 \sum_{i=0}^{n}\left[\frac{d}{dx}\tilde{H}_i^{(\alpha)}(x)\right]^2. \end{aligned}$$

Now, we multiply the last inequality by $W(x)$ and after integrating on $\mathbb{R}$ we obtain

$$\|p'_n\|_W^2 \le \|p_n\|_W^2 \sum_{i=0}^{n} \left\|\frac{d}{dx}\tilde{H}_i^{(\alpha)}\right\|_W^2. \tag{3.4}$$

From (2.1) we have

$$\left\|\frac{d}{dx}\tilde{H}_{2k}^{(\alpha)}\right\|_W^2 = O(2k). \tag{3.5}$$

From (2.2) with regard to (3.2) and (3.3) we get

$$\left\|\frac{d}{dx}\tilde{H}_{2k+1}^{(\alpha)}\right\|_W^2 = O(2k+1). \tag{3.6}$$

Using (3.4), (3.5) and (3.6), we have

$$\|p'_n\|_W^2 \le \|p_n\|_W^2 \sum_{i=0}^{n} \left\|\frac{d}{dx}\tilde{H}_i^{(\alpha)}\right\|_W^2 \le C_1 \|p_n\|_W^2 \sum_{i=0}^{n} i \le C_2 n^2 \|p_n\|_W^2, \tag{3.7}$$

where $C_1,\ C_2$ are positive constants independent of n and of x, but dependent of α. Finally, from (3.7), we obtain

$$\|p'_n\|_W \le Cn\|p_n\|_W. \qquad \square$$

Now, we will deal with the some generalization of the weight function (1.2). Let

$$W_\delta(x) = \delta(x)|x|^{2\alpha}e^{-x^2} = \delta(x)W(x) \tag{3.8}$$

be the weight function, where the factor $\delta(x)$ satisfies the condition

$$0 < m \le \delta(x) \le M, \qquad x \in \mathbb{R}. \tag{3.9}$$

Using Theorem 3.1 and (3.9), we have basic estimates

$$\|p'_n\|_{W_\delta}^2 \le M\|p'_n\|_W^2 \le MC^2n^2\|p_n\|_W^2 \le \frac{M}{m}C^2n^2\|p_n\|_{W_\delta}^2 \le C_3 n^2 \|p_n\|_{W_\delta}^2$$

with some positive constant C_3. Based on the above considerations, we state the following theorem.

Theorem 3.2. *For all polynomials $p_n(x) \in \Pi_n$ and for some constant $C^* = C^*(\alpha)$ the inequality*

$$\|p_n'\|_{W_\delta} \le C^* n \|p_n\|_{W_\delta}$$

holds. The weight function $W_\delta(x)$ is given by (3.8).

In [9] Mirsky considered the case with an arbitrary weight function

$$w : (a,b) \to \mathbb{R}_+, \quad -\infty \le a < b \le \infty$$

for which all moments are finite, i.e., $\mu_k = \int_a^b x^k w(x)\,dx < \infty$. His general result is, however, qualitative. For the simplest and nonsingular Hermite weight function

$$w(x) = e^{-x^2}$$

($\alpha = 0$ in (1.2)) the estimate of the type (3.1) becomes (cf. [8, p. 570])

$$\frac{\|p_n'\|_w}{\|p_n\|_w} = O(n^{\frac{3}{2}}).$$

The contrast between this estimate, our results, and the classical result of Schmidt [10] and Turán [12] $\|p_n'\|_w \le \sqrt{2n}\|p_n\|_w$ is evident.

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