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GENERALIZED OSCILLATIONS FOR GENERALIZED CONTINUITIES

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ABSTRACT. Let $(X, \mathfrak{g})$ be a generalized topological space, (Y, d) a metric one and $f\colon X \to Y$ a function. We can define a generalized oscillation of f at $x \in X$ as $k_f^{\mathfrak{g}}(x) = \inf\{\operatorname{diam} f(A) : A \in \mathfrak{g},\, x \in A\}$. We discuss some properties of the generalized oscillation.

In the literature, the concept of a topological space is generalized by replacing open sets by other kinds of subsets. In many cases, generalized open sets have the property that the arbitrary unions of them belong to the same class of sets. This property is postulated in the concept of generalized topology by Á. Császár in [5]. Further, he introduces the notion of generalized continuous function between generalized topological spaces [7]. In [4], the sets of points of generalized continuities are investigated using generalized oscillations. In this paper, we will investigate some properties of generalized oscillations.

We recall some notions. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. We call a class $\mathfrak{g} \subset \mathcal{P}(X)$ a generalized topology [5] (briefly GT), if $\emptyset \in \mathfrak{g}$ and the arbitrary union of elements of $\mathfrak{g}$ belongs to $\mathfrak{g}$. A GT $\mathfrak{g}$ is strong if $X \in \mathfrak{g}$. A set X with GT $\mathfrak{g}$ is called a generalized topological space (briefly, GTS) and is denoted by $(X, \mathfrak{g})$. For $x \in X$ we denote $\mathfrak{g}(x) = \{A \in \mathfrak{g} : x \in A\}$.

For a GTS $(X, \mathfrak{g})$, the elements of $\mathfrak{g}$ are called $\mathfrak{g}$-open and their complements are $\mathfrak{g}$-closed. For $A \subset X$, we denote by $i_{\mathfrak{g}}(A)$ the union of all $\mathfrak{g}$-open sets contained in A and by $c_{\mathfrak{g}}(A)$ the intersection of all $\mathfrak{g}$-closed sets containing A. A set A is said to be $\mathfrak{g}$-semi-open ($\mathfrak{g}$-pre-open, $\mathfrak{g}$-α-open, $\mathfrak{g}$-β-open), if $A \subset c_{\mathfrak{g}}\big(i_{\mathfrak{g}}(A)\big)$ ($A \subset i_{\mathfrak{g}}(c_{\mathfrak{g}}(A))$, $A \subset i_{\mathfrak{g}}(c_{\mathfrak{g}}(i_{\mathfrak{g}}(A)))$, $A \subset c_{\mathfrak{g}}(i_{\mathfrak{g}}(c_{\mathfrak{g}}(A))))$, respectively [6]. We denote the class of all $\mathfrak{g}$-semi-open ($\mathfrak{g}$-pre-open, $\mathfrak{g}$-α-open, $\mathfrak{g}$-β-open) sets by $\sigma(\mathfrak{g})$ ($\pi(\mathfrak{g}), \alpha(\mathfrak{g}), \beta(\mathfrak{g})$). If $(X, \mathfrak{g})$ is a topological space, we obtain the families of semi-open sets $SO(X)$, pre-open sets $PO(X)$, α-open sets $\alpha(X)$ and β-open sets $\beta(X)$. $SO(X)$, $PO(X)$, $\alpha(X)$ and $\beta(X)$ are GT (in fact, $\alpha(X)$ is a topology).

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By [7], if $(X, \mathfrak{g})$ and $(Y, \mathfrak{h})$ are GTS's, then a mapping $f\colon X \to Y$ is called $(\mathfrak{g}, \mathfrak{h})$-continuous, if $f^{-1}(V) \in \mathfrak{g}$ for each $V \in \mathfrak{h}$. If $(Y, \mathfrak{h})$ is a topological space, for $\mathfrak{g} = SO(X)$ we have the family of quasicontinuous functions, for $\mathfrak{g} = PO(X)$ we have pre-continuous functions, for $\mathfrak{g} = \alpha(X)$ α-continuous functions and for $\mathfrak{g} = \beta(X)$ we obtain β-continuous functions. A function $f\colon X \to Y$ is $(\mathfrak{g}, \mathfrak{h})$-continuous at $x \in X$ if for each $V \in \mathfrak{h}\big(f(x)\big)$ there is $U \in \mathfrak{g}(x)$ such that $f(U) \subset V$. By [7], a function f is $(\mathfrak{g}, \mathfrak{h})$-continuous if it is such at each point. Denote by $(\mathfrak{g}, \mathfrak{h})(f)$ the family of all $(\mathfrak{g}, \mathfrak{h})$-continuity points of f.

If (Y, d) is a metric space we can characterize the set $(\mathfrak{g}, \mathfrak{h})(f)$. From now, we will assume that (Y, d) is a metric space. We will use the notion $\mathfrak{g}$-continuity for $(\mathfrak{g}, d)$-continuity and $\mathfrak{g}(f)$ for continuity points $(\mathfrak{g}, d)(f)$.

Definition 1 ([4]). Let $(X, \mathfrak{g})$ be a GTS, let (Y, d) be a metric space and let $f\colon X \to Y$ be a function. The function $k_f^{\mathfrak{g}}\colon X \to [0, \infty]$ defined by

$$k_f^{\mathfrak{g}}(x) = \inf\big\{\operatorname{diam} f(A) : A \in \mathfrak{g}(x)\big\}$$

is called the $\mathfrak{g}$-oscillation of f (remark that $\inf \emptyset = \infty$), where $\operatorname{diam}(Z) = \sup\big\{d(u, v) : u,\, v \in Z\big\}$ is the diameter of Z.

Definition 2 ([4]). A function $f\colon X \to [-\infty, \infty]$ is said to be upper $\mathfrak{g}$-continuous [lower $\mathfrak{g}$-continuous] at x if for each $a > f(x)$ $[a < f(x)]$ there is a set $A \in \mathfrak{g}(x)$ such that $f(y) < a$ $[f(y) > a]$ for each $y \in A$. A function is upper [lower] $\mathfrak{g}$-continuous if it is such at each point. Denote by $u\mathfrak{g}(f)$ $[l\mathfrak{g}(f)]$ the set of all upper [lower] $\mathfrak{g}$-continuity points of f.

Definition 3 ([4]). Let $f\colon X \to [-\infty, \infty]$ be a function. Define $M_f^{\mathfrak{g}}, m_f^{\mathfrak{g}}, \tilde{k}_f^{\mathfrak{g}} : X \to [-\infty, \infty]$ as

$M_f^{\mathfrak{g}}(x) = \inf\big\{\sup\{f(y) : y \in A\} : A \in \mathfrak{g}(x)\big\}$ for $x \in \bigcup \mathfrak{g}$ and $M_f^{\mathfrak{g}}(x) = \infty$ for $x \in X \setminus \bigcup \mathfrak{g}$,

$m_f^{\mathfrak{g}}(x) = \sup\big\{\inf\{f(y) : y \in A\} : A \in \mathfrak{g}(x)\big\}$ for $x \in \bigcup \mathfrak{g}$ and $m_f^{\mathfrak{g}}(x) = -\infty$ for $x \in X \setminus \bigcup \mathfrak{g}$,

$\tilde{k}_f^{\mathfrak{g}}(x) = M_f^{\mathfrak{g}}(x) - m_f^{\mathfrak{g}}(x)$ (we put $\infty - \infty = 0$).

In [4] it is shown, that for $f\colon X \to [-\infty, \infty]$, $M_f^{\mathfrak{g}}$ and $\tilde{k}_f^{\mathfrak{g}}$ are upper $\mathfrak{g}$-continuous, $m_f^{\mathfrak{g}}$ is lower $\mathfrak{g}$-continuous, $x \in u\mathfrak{g}(f)$ if and only if $f(x) = M_f^{\mathfrak{g}}(x)$, $x \in l\mathfrak{g}(f)$ if and only if $f(x) = m_f^{\mathfrak{g}}(x)$ and $x \in u\mathfrak{g}(f) \cap l\mathfrak{g}(f)$ if and only if $\tilde{k}_f^{\mathfrak{g}}(x) = 0$. Moreover, $\tilde{k}_f^{\mathfrak{g}} \le k_f^{\mathfrak{g}}$. For a function $f\colon X \to Y$, $k_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous and f is $\mathfrak{g}$-continuous at x if and only if $k_f^{\mathfrak{g}}(x) = 0$; so, the set $\mathfrak{g}(f)$ of $\mathfrak{g}$-continuity points of f is the countable intersection of sets belonging to $\mathfrak{g}$. We will give some further properties of $k_f^{\mathfrak{g}}$.

Proposition 1. *If $f\colon X \to Y$ is $\mathfrak{g}$-continuous at x then $k_f^{\mathfrak{g}}$ and is $\mathfrak{g}$-continuous at x, too.*

P r o o f. Let $c > 0$. Since f is $\mathfrak{g}$-continuous at x we have $k_f^{\mathfrak{g}}(x) = 0$. Since $k_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous there is $A \in \mathfrak{g}(x)$ such that $k_f^{\mathfrak{g}}(y) < c$ for each $y \in A$. So, $0 \le k_f^{\mathfrak{g}}(y) < c$ for all $y \in A$ and $k_f^{\mathfrak{g}}$ is $\mathfrak{g}$-continuous. □

Definition 4. Let X be a topological space with the topology $\mathcal{T}$ and let $\mathfrak{g}$ be a strong GT on X. We will say that $\mathfrak{g}$ satisfies P1 if each nonempty member in $\mathfrak{g}$ has the nonempty interior and $\mathfrak{g}$ satisfies P2 if $U \cap A \in \mathfrak{g}$ for all $U \in \mathcal{T}$ and $A \in \mathfrak{g}$.

Proposition 2. *Let $\mathfrak{g}$ be a strong GT on X.*

(i) *If $\mathfrak{g}$ satisfies P2 then $\mathcal{T} \subset \mathfrak{g}$.*

(ii) *If $\mathfrak{g} \subset SO(X)$ then $\mathfrak{g}$ satisfies P1.*

(iii) *If $\mathfrak{g}$ satisfies P1 and P2 then $\mathcal{T} \subset \mathfrak{g} \subset SO(X)$.*

(iv) *There is $\mathfrak{g}$ such that $\mathcal{T} \subset \mathfrak{g} \subset SO(X)$ and $\mathfrak{g}$ does not satisfy P2.*

P r o o f. (i) and (ii) are obvious.
(iii): We have $\mathcal{T} \subset \mathfrak{g}$ by (i). Assume that there is $A \in \mathfrak{g}$ such that $A \notin SO(X)$. Then $A \setminus \operatorname{Cl}\operatorname{Int} A \neq \emptyset$. Since $X \setminus \operatorname{Cl}\operatorname{Int} A \in \mathcal{T}$, by P2 we have $A \cap (X \setminus \operatorname{Cl}\operatorname{Int} A) = A \setminus \operatorname{Cl}\operatorname{Int} A \in \mathfrak{g}$. By P1 we have $\operatorname{Int}(A \setminus \operatorname{Cl}\operatorname{Int} A) \neq \emptyset$. On the other hand, $\operatorname{Int}(A \setminus \operatorname{Cl}\operatorname{Int} A) = \operatorname{Int} A \setminus \operatorname{Cl}\operatorname{Int} A = \emptyset$.
(iv): Let $X = [0,1]$ with the usual toplogy $\mathcal{T}$, C be the Cantor set and $A \in \mathfrak{g}$ if A is open or $X \setminus C \subset A$. Then $\mathfrak{g}$ is a strong GT and $\mathcal{T} \subset \mathfrak{g} \subset SO(X)$. If $X \setminus C = \bigcup_{i=1}^{\infty}(a_i, b_i)$ then $A = \bigcup_{i=1}^{\infty}[a_i, b_i] \in \mathfrak{g}$, however $A \cap [0, 1/2) \notin \mathfrak{g}$. □

Definition 5. Let $f, F\colon X \to \mathbb{R}$ be functions. A function F is $\mathfrak{g}$-primitive for f if the $\mathfrak{g}$-oscillation of F is equal to f.

The question of the existence of $\mathfrak{g}$-primitive for the usual oscillation was investigated by J. E w e r t and S. P. P o n o m a r e v. They have shown that in each metric space, each upper semicontinuous real function vanishing at isolated points has $\mathfrak{g}$-primitive [9]. We give a partial answer for the existence of $\mathfrak{g}$-primitive; especially, this holds for $\mathfrak{g} = SO(X)$.

Theorem 1. *Let $(X, \mathcal{T})$ be a Baire first countable T_1 space without isolated points. Let a strong GT $\mathfrak{g}$ on X satisfy P1 and P2. Let $g\colon X \to \mathbb{R}$ be a function. Then $g = k_f^{\mathfrak{g}}$ for some function $f\colon X \to \mathbb{R}$ if and only if g is nonnegative and upper $\mathfrak{g}$-continuous.*

P r o o f. *Necessity.* By [4] g is nonnegative and upper $\mathfrak{g}$-continuous.
Sufficiency. By Proposition 2 we have $\mathfrak{g} \subset SO(X)$, so g is upper quasicontinuous and by [3] it is cliquish and hence $X \setminus \mathcal{T}(g)$ is of first category. Since X is Baire, $\mathcal{T}(g)$ is dense in X, and since X is T_1, $\mathcal{T}(g)$ has no isolated point. Hence, according to [1], $\mathcal{T}(g) = A \cup B$, where A and B are disjoint and dense in $\mathcal{T}(g)$.

Evidently, they are dense in X, too. Define a function $f\colon X \to \mathbb{R}$ as $f(x) = 0$ for $x \in A$ and $f(x) = g(x)$ otherwise.

We will show that $k_g^{\mathfrak{g}} = f$. Let $x \in X$ and $\varepsilon > 0$. Put $D = g^{-1}((-\infty, g(x)+\varepsilon))$. Since g is upper $\mathfrak{g}$-continuous and $x \in D$ we have $D \in \mathfrak{g}(x)$. For each $y \in D$ we have $0 \le f(y) \le g(y) < g(x) + \varepsilon$ and hence $\operatorname{diam} f(D) < g(x) + \varepsilon$ and $k_f^{\mathfrak{g}}(x) \le g(x) + \varepsilon$.

Now, let $x \in A$. Then $x \in \mathcal{T}(g)$ and there is an open neighbourhood U of x such that $g(y) > g(x) - \varepsilon$ for each $y \in U$. Let $E \in \mathfrak{g}(x)$. Then by P2 we have $U \cap E \in \mathfrak{g}$ and since $x \in E \cap U$, by P1 we have $\operatorname{Int}(U \cap E) \neq \emptyset$. There are $z_1 \in \operatorname{Int}(U \cap E) \cap A$ and $z_2 \in \operatorname{Int}(U \cap E) \cap B$. We have $f(z_1) = 0$ and $f(z_2) = g(z_2)$, therefore $\operatorname{diam} f(E) \ge \operatorname{diam} f\big(\operatorname{Int}(U \cap E)\big) \ge f(z_2) - f(z_1) = g(z_2) > g(x) - \varepsilon$. This yields $k_f^{\mathfrak{g}}(x) \ge g(x) - \varepsilon$.

If $x \in X \setminus A$ then $f(x) = g(x)$. For arbitrary $E \in \mathfrak{g}(x)$ we have $\operatorname{Int}(E) \neq \emptyset$ and for $z \in \operatorname{Int}(E) \cap A$ we have $\operatorname{diam} f(E) \ge f(x) - f(z) = g(x)$ and $k_f^{\mathfrak{g}}(x) \ge g(x)$. Therefore for each $x \in X$ we have $g(x) - \varepsilon \le k_f^{\mathfrak{g}}(x) \le g(x) + \varepsilon$ and hence $k_f^{\mathfrak{g}}(x) = g(x)$. □

Theorem is not true for an arbitrary strong GT $\mathfrak{g}$ (e.g., if $\mathfrak{g}$ is the family of all pre-open or β-open sets, see [2, Theorem 7]).

Proposition 3. *Let $\mathfrak{g}$ be a strong GT on a topological space X satisfying P2. Let $f\colon X \to \mathbb{R}$ be an upper $\mathfrak{g}$-continuous function bounded from below. Then $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = 0$ for each accumulation point of X.*

Proof. Let x be an accumulation point of X, let U be an open neighbourhood of x and $\varepsilon > 0$. Then $c = \inf f(U) \in \mathbb{R}$. Let $y \in U$ be such that $f(y) < c+\varepsilon$. Then there is $A \in \mathfrak{g}(y)$ such that $f(z) < c + \varepsilon$ for each $z \in A$. Further, $y \in A \cap U \in \mathfrak{g}$ and hence $A \cap U \in \mathfrak{g}(y)$. For each $z \in A \cap U$ we have $c \le f(u) < c + \varepsilon$. Hence $\operatorname{diam} f(A \cap U) \le \varepsilon$ and $k_f^{\mathfrak{g}}(y) \le \varepsilon$. So, $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = 0$. □

The conditions cannot be omitted. Let $X = \mathbb{Q} = \{q_1, q_2, \dots\}$ with the usual topology $\mathcal{T}$ and let $\mathfrak{g} = \mathcal{T}$. Let $f\colon X \to \mathbb{R}$, $f(q_n) = -n$. Then f is upper $\mathfrak{g}$-continuous, however $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = -\infty$ for each $x \in X$. Further, let $X = [0, 1]$ with the usual topology $\mathcal{T}$ and $\mathfrak{g} = \{A \subset X : A = \emptyset \text{ or } 0 \in A\}$ and $f(x) = x$. Then f is bounded upper $\mathfrak{g}$-continuous, however $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = x$ for each $x \in X$.

By [4], for $f\colon X \to \mathbb{R}$, $f = M_f^{\mathfrak{g}}$ if and only if f is upper $\mathfrak{g}$-continuous and $f = m_f^{\mathfrak{g}}$ if and only if f is lower $\mathfrak{g}$-continuous. Similar results we can obtain for $k_f^{\mathfrak{g}}$ and $\tilde{k}_f^{\mathfrak{g}}$.

Proposition 4. *Let X be a GTS and $f\colon X \to \mathbb{R}$. Then $\tilde{k}_f^{\mathfrak{g}} = f$ if and only if f is upper $\mathfrak{g}$-continuous and $m_f^{\mathfrak{g}}(x) = 0$ for each $x \in X$.*

P r o o f. Let $\tilde{k}_f^{\mathfrak{g}} = f$. Then f is upper $\mathfrak{g}$-continuous. Hence $f = M_f^{\mathfrak{g}}$ and for each $x \in X$ we have $m_f^{\mathfrak{g}}(x) = M_f^{\mathfrak{g}}(x) - \tilde{k}_f^{\mathfrak{g}}(x) = f(x) - f(x) = 0$. Conversely, let f be upper $\mathfrak{g}$-continuous and $m_f^{\mathfrak{g}}(x) = 0$ for each $x \in X$. Then $f(x) = M_f^{\mathfrak{g}}(x) = M_f^{\mathfrak{g}}(x) - m_f^{\mathfrak{g}}(x) = \tilde{k}_f^{\mathfrak{g}}(x)$. □

Proposition 5. *Let X be a topological space, let $\mathfrak{g}$ be a strong GT on X satisfying P1 and P2 and let $f\colon X \to \mathbb{R}$ be a function. Then $k_f^{\mathfrak{g}} = f$ if and only if f is nonnegative upper $\mathfrak{g}$-continuous, $\liminf_{u\to x} f(u) = 0$ for each accumulation point x of X and $f(x) = 0$ for each isolated point x of X.*

P r o o f. Let $k_f^{\mathfrak{g}} = f$. Then f is nonnegative and by [4], it is upper $\mathfrak{g}$-continuous. If x is isolated point of X then by Proposition 2 $\{x\} \in \mathfrak{g}$ and so f is $\mathfrak{g}$-continuous at x and $k_f^{\mathfrak{g}}(x) = f(x) = 0$. If x is an accumulation point of X then by Proposition 3 we have $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = 0 = \liminf_{u\to x} f(u)$.

Conversely, let f be nonnegative upper $\mathfrak{g}$-continuous, $f(x) = 0$ for isolated points of X and $\liminf_{u\to x} f(u) = 0$ for accumulation points of X.

If x is an isolated point of X then $\{x\} \in \mathfrak{g}$ and hence f is $\mathfrak{g}$ continuous at x, so $f(x) = 0 = k_f^{\mathfrak{g}}(x)$. Now, let x be an accumulation point of X. Let $\varepsilon > 0$. Then there is $A \in \mathfrak{g}(x)$ such that $0 \le f(u) < f(x) + \varepsilon$ for each $u \in A$. This yields $\operatorname{diam} f(A) \le f(x) + \varepsilon$ and $k_f^{\mathfrak{g}}(x) \le f(x) + \varepsilon$.

If $f(x) = 0$ then $k_f^{\mathfrak{g}}(x) = f(x) = 0$. Now, let $f(x) > 0$. Let $B \in \mathfrak{g}(x)$. Then by P1 we have $\operatorname{Int}(B) \neq \emptyset$ and there is $z \in \operatorname{Int}(B)$ with $f(z) < \varepsilon$ (otherwise, z is an accumulation point and $\liminf_{u\to z} f(u) \ge \varepsilon$, a contradiction). We have $|f(x) - f(z)| \ge f(x) - f(z) > f(x) - \varepsilon$ and hence $\operatorname{diam} f(B) \ge f(x) - \varepsilon$ for each $B \in \mathfrak{g}(x)$, so $k_f^{\mathfrak{g}}(x) \ge f(x) - \varepsilon$. Therefore $k_f^{\mathfrak{g}}(x) = f(x)$. □

Corollary 1. *Let $\mathfrak{g}$ satisfy P1 and P2. If $f\colon X \to \mathbb{R}$ is locally bounded then $k_{k_f^{\mathfrak{g}}}^{\mathfrak{g}} = k_f^{\mathfrak{g}}$.*

P r o o f. Let $x \in X$. Then there is an open neighbourhood U of x such that $\operatorname{diam} f(U) < \infty$. If $A \in \mathfrak{g}(x)$ then $A \cap U \in \mathfrak{g}(x)$ and $\operatorname{diam}(f(A \cap U) < \infty$, so $k_f^{\mathfrak{g}} : X \to \mathbb{R}$. The function $k_f^{\mathfrak{g}}$ is nonnegative upper $\mathfrak{g}$-continuous, by Proposition 3 $\liminf_{u\to x} k_f^{\mathfrak{g}}(u) = 0$ for each accumulation point of X and $k_f^{\mathfrak{g}}(x) = 0$ for isolated points, so by Proposition 5 we obtain $k_{k_f^{\mathfrak{g}}}^{\mathfrak{g}} = k_f^{\mathfrak{g}}$. □

This is not true for an arbitrary $\mathfrak{g}$. If $\mathfrak{g}$ be the family of all pre-open or β-open sets, then by [2, Theorem 9] there is a function $f\colon X \to \mathbb{R}$ such that $k_{k_f^{\mathfrak{g}}}^{\mathfrak{g}} \neq k_f^{\mathfrak{g}}$.

In [10], the set of all continuity points is characterized (using generalized oscillation) for functions with values in weakly developable spaces. I do not know that its generalization for GT characterized $\mathfrak{g}$-continuity points. Nevertheless, for

developable spaces we can define a "good" $\mathfrak{g}$-oscillation (i.e., the set of $\mathfrak{g}$-continuity points is the countable intersection of sets from $\mathfrak{g}$); of course, if the range space is metrizable, this $\mathfrak{g}$-oscillation does not reduce to our $\mathfrak{g}$-oscillation $k_f^{\mathfrak{g}}$.

Let Y be developable space with a development $(\mathcal{G}_n)_n$. Without loss of generality we can suppose that $\mathcal{G}_1 = \{Y\}$ and $\mathcal{G}_{n+1}$ is a refinement of $\mathcal{G}_n$. For a subset A of X we put $\omega_f(A) = \inf\bigl\{1/n : \text{ there is } V \in \mathcal{G}_n \text{ such that } f(A) \subset V\bigr\}$ and define a function $\omega_f^{\mathfrak{g}} \colon X \to [0,1]$ as $\omega_f^{\mathfrak{g}}(x) = \inf\bigl\{\omega_f(A) : A \in \mathfrak{g}(x)\bigr\}$.

Proposition 6. *A function $f \colon X \to Y$ is $\mathfrak{g}$-continuous at x if and only if $\omega_f^{\mathfrak{g}}(x) = 0$.*

Proof. Let f be $\mathfrak{g}$-continuous at x and $\varepsilon > 0$. Let $1/n < \varepsilon$ and let $V \in \mathcal{G}_n$ be such that $f(x) \in V$. Then there is $A \in \mathfrak{g}(x)$ such that $f(A) \subset V$. This yields $\omega_f(A) \le 1/n < \varepsilon$ and hence $\omega_f^{\mathfrak{g}}(x) \le \omega_f(A) < \varepsilon$. So, $\omega_f^{\mathfrak{g}}(x) = 0$.

Conversely, let $\omega_f^{\mathfrak{g}}(x) = 0$. Let H be a neighbourhood of $f(x)$. Then there is $n \in \mathbb{N}$ such that $st\bigl(f(x), \mathcal{G}_n\bigr) = \bigcup\bigl\{G \in \mathcal{G}_n : f(x) \in G\bigr\} \subset H$. Since $\omega_f^{\mathfrak{g}}(x) < 1/n$ there is $A \in \mathfrak{g}(x)$ such that $\omega_f(A) < 1/n$. Hence there is $k \in \mathbb{N}$ such that $f(A) \subset V$ for some $V \in \mathcal{G}_k$. From the definition of ω_f we have $k \ge n$ and hence there is $W \in \mathcal{G}_n$ with $V \subset W$. Therefore $f(x) \in V \subset W \subset st\bigl(f(x), \mathcal{G}_n\bigr) \subset H$ and $f(A) \subset H$. Therefore f is $\mathfrak{g}$-continuous at x. $\square$

Proposition 7. *The function $\omega_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous.*

Proof. Let $\omega_f^{\mathfrak{g}}(x) < a$. Then there is $A \in \mathfrak{g}(x)$ with $\omega_f(A) < a$. For each $y \in A$ we have $\omega_f^{\mathfrak{g}}(y) \le \omega_f(A) < a$, thus $\omega_f^{\mathfrak{g}}$ is upper $\mathfrak{g}$-continuous at x. $\square$

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