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ONE MORE DIFFERENCE BETWEEN MEASURE AND CATEGORY

Jacek Hejduk

ABSTRACT. The first part of the paper contains some ideas of the density topologies in the measurable spaces. The second part is devoted to the difference between measure and category for the abstract density space related to the separation axioms.

Let X be a nonempty set, $\mathcal{A} \subset 2^X$ be an arbitrary family of sets, and Φ be an operator such that

$$\Phi\colon \mathcal{A} \to 2^X.$$

If the family $\mathcal{T}_\Phi = \big\{A \in \mathcal{A} : A \subset \Phi(A)\big\}$ is a topology, then we say that the topology $\mathcal{T}_\Phi$ is generated by the operator Φ.

Let $(X, \mathcal{S}, \mathcal{J})$ be a measurable space, where S is the σ-algebra and $\mathcal{J} \subset \mathcal{S}$ is the proper σ-ideal. The fact that for any sets $A, B \in \mathcal{S}$ we have $A \triangle B \in \mathcal{J}$ will be denoted by $A \sim B$.

Definition 1. We shall say that a topology τ on X is the abstract density topology on $(X, \mathcal{S}, \mathcal{J})$, if there exists a lower density operator $\Phi\colon \mathcal{S} \to \mathcal{S}$ such that $\mathcal{T}_\Phi = \tau$.

Definition 2. An operator $\Phi\colon \mathcal{S} \to \mathcal{S}$ is called the lower density operator on $(X, \mathcal{S}, \mathcal{J})$ if:

i) $\Phi(\emptyset) = \emptyset$, $\Phi(X) = X$,

ii) $\forall A, B \in \mathcal{S} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

iii) $\forall A, B \in \mathcal{S} \quad A \sim B \implies \Phi(A) = \Phi(B)$,

iv) $\forall A \in \mathcal{S} \quad A \sim \Phi(A)$.

The following states when a topology τ is the abstract density topology.

Theorem 3 (cf. [7]). *A topology τ on X is the abstract density topology on $(X, \mathcal{S}, \mathcal{J})$ if and only if the following conditions are satisfied:*

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a) $A \in \mathcal{J} \iff A$ *is* τ*-nowhere dense and* τ*-closed,*

b) $Ba(\tau) = \mathcal{S}$,

where $Ba(\tau)$ *is a family of Baire sets and* $K(\tau)$ *is a family of meager sets with respect to the topology* τ.

The abstract density topology can be derived directly from the lower density operator.

Theorem 4 (cf. [7]). *If* $\Phi \colon \mathcal{S} \to \mathcal{S}$ *is the lower density operator on* $(X, \mathcal{S}, \mathcal{J})$ *and the pair* $(\mathcal{S}, \mathcal{J})$ *has the hull property then the family* $\mathcal{T}_\Phi = \{A \subset X \colon A \subset \Phi(A)\}$ *is a topology on* X.

It is clear that $\mathcal{T}_\Phi$ is the abstract density topology on $(X, \mathcal{S}, \mathcal{J})$.

Examples. A density topology (see [11]), $\mathcal{I}$-density topology (see [11]), $\langle s \rangle$--density topology with respect to measure and category (see [2], [5]), density topology with respect to a sequence tending to 0 (see [10]) are generated by the method described above.

A result similar to Theorem 3 can be obtained in the case, when $\mathcal{S}$ is an algebra in 2^X and $\mathcal{I} \subset \mathcal{S}$ is a proper ideal.

Theorem 5. *A topology* τ *on* X *is the abstract density topology on* $(X, \mathcal{S}, \mathcal{I})$ *if and only if the following conditions are satisfied:*

a) $A \in \mathcal{I} \iff A$ *is* τ*-nowhere dense and* τ*-closed,*

b) $\mathcal{S} = \{A \subset X : A = V \triangle B,\ V \in \tau,\ B \in \mathcal{N}(\tau)\}$, *where* $\mathcal{N}(\tau)$ *is the family of* τ*-nowhere dense sets.*

Proof. *Necessity.* Let τ be the abstract density topology on $(X, \mathcal{S}, \mathcal{I})$. It means that there exists an operator $\Phi \colon \mathcal{S} \to \mathcal{S}$ satisfying conditions (i–iv) such that $\tau = \mathcal{T}_\Phi = \{A \subset X \colon A \subset \Phi(A)\}$. If $A \in \mathcal{I}$ then it is clear that $X - A \in \mathcal{T}_\Phi$. It implies that A is τ-closed and also $\mathrm{int}_\tau A = \emptyset$. Hence the set A is τ-nowhere dense. Let us suppose that the set A is τ-nowhere dense and τ-closed and $A \notin \mathcal{I}$. It is clear that $A \in \mathcal{S} \setminus \mathcal{I}$. Moreover, the set $A \cap \Phi(A)$ is $\mathcal{T}_\Phi$ open and non-empty. It contradicts that A is τ-nowhere dense. The proof of condition a) is completed.

Now we shall prove condition b). It is obvious that $\mathcal{N}(\tau) = \mathcal{I}$. By this we have that the family $\tau \triangle \mathcal{N}(\tau) = \{A \subset X : A = V \triangle B,\ V \in \tau,\ B \in \mathcal{N}(\tau)\} \subset \mathcal{S}$. Let $A \in \mathcal{S}$. Then $A = A \cap \Phi(A) \cup A - \Phi(A)$. Since $A \cap \Phi(A) \in \mathcal{T}_\Phi$, $A - \Phi(A) \in \mathcal{I}$ and the family $\tau \triangle \mathcal{N}(\tau)$ is an algebra of sets, we get that $A \in \mathcal{S}$. The proof of condition b) is completed.

Sufficiency. We prove that there exists an operator $\Phi \colon \mathcal{S} \to \mathcal{S}$ which is the lower density operator on $(X, \mathcal{S}, \mathcal{I})$. Let $A \in \mathcal{S}$. Since $A = V \triangle B$, where $V \in \tau$ and $B \in \mathcal{N}(\tau)$ we have that the set A has a unique representation in form $G \triangle P$, where P is a regular τ-open set and P is τ-nowhere dense (cf. [8, Th. 4.5, 4.6]).

Let $\Phi(A) = G$. Obviously, $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$. Let $A, B \in \mathcal{S}$ and $A = G_1 \triangle P_1$, $B = G_2 \triangle P_2$, when G_1, G_2 are τ-regular open and P_1, P_2 are τ-nowhere dense set. There exists τ-nowhere dense set P_3 such that

$$(V_1 \triangle P_1) \cap (V_2 \triangle P_2) = (V_1 \cap V_2) \triangle P_3.$$

It implies that

$$\Phi(A \cap B) = \Phi(A) \cap \Phi(B).$$

If $A, B \in \mathcal{S}$ and $A \sim B$ then we conclude that $\Phi(A) = \Phi(B)$. The condition that $\Phi(A) \sim A$ is also fulfilled. In that way operator Φ is the lower density operator on $(X, \mathcal{S}, \mathcal{I})$. Now, we shall prove that $\tau = \mathcal{T}_\Phi$. Let $A \in \tau$. By Theorem 4.5 in [8] we have that $A = G - P$, where G is τ-regular open, P is τ-nowhere dense and $P \subset G$. Hence $A \in \mathcal{S}$ and $\Phi(A) = G \supset A$. It implies that $A \in \mathcal{T}_\Phi$. Let us suppose that $A \in \mathcal{T}_\Phi$. Then $A \in \mathcal{S}$ and $A \subset \Phi(A)$. Since $A = G \triangle P$, where G is τ-regular open and P is τ-nowhere dense then $G \triangle P \subset \Phi(A) = G$ and we get that $P \setminus G = \emptyset$ and $A = G \setminus P$. The equality $\mathcal{N}(\tau) = \mathcal{I}$ and condition a) imply that P is τ-closed, so that $A \in \tau$. □

Remark. The lower density operator Φ on $(X, \mathcal{S}, \mathcal{I})$ such that $\mathcal{T}_\Phi = \tau$ is unique.

Proof. Let Φ_1, Φ_2 be the abstract density operators on $(X, \mathcal{S}, \mathcal{I})$ such that $\mathcal{T}_{\Phi_1} = \tau = \mathcal{T}_{\Phi_2}$. We show that

$$\forall A \in \mathcal{S} \quad \Phi_1(A) = \Phi_2(A).$$

Let $A \in \mathcal{S}$. Then by condition iii), $\Phi_1(A) \in \tau$. Hence $\Phi_1(A) \in \mathcal{T}_\Phi$. It means that $\Phi_1(A) \subset \Phi_2\big(\Phi_1(A)\big) = \Phi_2(A)$. Analogously we get that $\Phi_2(A) \subset \Phi_1(A)$. □

Theorem 6. *If an ideal $\mathcal{I}$ contains all singletons then the operator Φ described in theorem above has the following form*

$$\forall A \in \mathcal{S} \quad \Phi(A) = \operatorname{int}\Big\{x \in X : x \in \operatorname{int}\big(A \cup \{x\}\big)\Big\}.$$

Proof. Let $A \in \mathcal{S}$. Let us denote $\Phi_1(A) = \operatorname{int}\big\{x \in X : x \in \operatorname{int}(A \cup \{x\})\big\}$. First, we show that if $A, B \in \mathcal{S}$, $A \sim B$ then $\Phi_1(A) = \Phi_1(B)$. It is clear that $A = B \triangle C$, where $C \in \mathcal{N}(\tau)$. We prove that $\Phi_1(A) \subset \Phi_1(B)$. Let $x \in \Phi_1(A)$. Hence $x \in \Phi_1(B \triangle C)$ and $x \in \operatorname{int}\big(B \triangle C \cup \{x\}\big) \subset \operatorname{int}\big(B \cup C \cup \{x\}\big)$. We show that $x \in \operatorname{int}\big(B \cup \{x\}\big)$. There exists τ-open set $W_x \ni x$ such that $W_x \subset B \cup C \cup \{x\}$. Hence $W_x - \big(C - \{x\}\big) \subset B \cup \{x\}$, $x \in W_x - \big(C - \{x\}\big)$, and $W_x - \big(C - \{x\}\big)$ is τ-open. Then $x \in \operatorname{int}\big(B \cup \{x\}\big)$ and $\Phi_1(A) \subset \Phi_1(B)$. Similarly, we have $\Phi_1(B) \subset \Phi_1(A)$. Let $A = V \triangle B$, where V is τ-regular open set and $B \in \mathcal{N}(\tau)$. Then $\Phi(A) = V$ and $\Phi_1(A) = \Phi_1(V)$. It remains to prove that $V = \Phi_1(V)$. Since $V \subset \Phi_1(V)$, it is sufficient to show that $\Phi_1(V) \subset V$. Let $x \in \Phi_1(V)$. Then $x \in \bar{V}$. Certainly, let $W \in \tau$, and $x \in W$. Since $x \in \operatorname{int}\big(V \cup \{x\}\big)$, then there exists a set $W_x \in \tau$, $W_x \subset V \cup \{x\}$ and $x \in W_x$. It is clear that $x \in W \cap W_x$ then $W \cap W_x \neq \emptyset$ and $W \cap W_x - \{x\} \neq \emptyset$, because $\mathcal{I}$ contains all singletons and $W \cap W_x \subset \Phi(W \cap W_x)$. We get that $W \cap W_x - \{x\} \subset V$ and $W \cap V \neq \emptyset$. Hence $x \in \bar{V}$. Simultaneously, the set $\{x \in X : x \in \operatorname{int}(V \cup \{x\})\}$ is open. Then $\Phi_1(V) = \operatorname{int} \Phi_1(V) \subset \operatorname{int} \bar{V} = V$. □

Corollary. *Let (X, τ) be a topological space. Then the following conditions are equivalent:*

a) *there exists exactly one pair $(\mathcal{S}, \mathcal{I})$, $\mathcal{S}$-algebra, $\mathcal{I}$ – a proper ideal, $\mathcal{I} \subset \mathcal{S}$ such that τ is the abstract density topology on $(X, \mathcal{S}, \mathcal{I})$;*

b) *every τ-nowhere dense set is τ-closed.*

Proof. By Theorem 5 we get that implication $a \implies b$ is true. Let $\mathcal{I} = \mathcal{N}(\tau)$ and $\mathcal{S} = \tau \triangle \mathcal{N}(\tau)$. Then again by Theorem 5 we have that τ is the abstract density topology on $(X, \mathcal{S}, \mathcal{I})$.

The pair $(\mathcal{S}, \mathcal{I})$ is unique. If there exists a pair $(\mathcal{S}_1, \mathcal{I}_1)$ such that τ is the abstract density topology on $(X, \mathcal{S}_1, \mathcal{I}_1)$, then by Theorem 5 we have that $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{I}_1 = \mathcal{I}$.

Let $X \neq \emptyset$, $\mathcal{S}$ – algebra in 2^X and let $\mathcal{I} \subset \mathcal{S}$ be a proper ideal.

Let $\Phi \colon \mathcal{S} \to 2^X$ satisfy the conditions:

1^o $\Phi(\emptyset) = \emptyset$, $\Phi(X) = X$,

2^o $\forall A, B \in \mathcal{S} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

3^o $\forall A, B \in \mathcal{S} \quad A \sim B \implies \Phi(A) = \Phi(B)$,

4^o $\forall A \in \mathcal{S} \quad \Phi(A) - A \in \mathcal{I}$. □

It is clear that a lower density operator Φ on $(X, \mathcal{S}, \mathcal{I})$ satisfies conditions 1^o–4^o. The conditions 1^o–4^o are sufficient to get a topology on X similarly, as in the case of the abstract density operator. We have the following theorem.

Theorem 7 (cf. [4]). *Let $\Phi \colon \mathcal{S} \to 2^X$ satisfy conditions 1^o–4^o. If the pair $(\mathcal{S}, \mathcal{I})$ has the hull property then the family*

$$\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}$$

is a topology on X.

Examples. By the method suggested in the above theorem, ψ-density topology (see [9]), f-density topology (see [1]), f-symmetrical density topology (see [3]) were introduced.

In the following two theorems we obtain an answer to the question: When is $\mathcal{T}_\Phi$ topology the abstract density topology?

Theorem 8 (cf. [4]). *Let $\Phi \colon \mathcal{S} \to 2^X$ satisfy conditions 1^o–3^o generating $\mathcal{T}_\Phi$ topology, and let $\mathcal{N}(\mathcal{T}_\Phi)$ be a family of nowhere dense set with respect to $\mathcal{T}_\Phi$ topology. Then $\mathcal{N}(\mathcal{T}_\Phi) = \mathcal{I}$ if and only if there exists an algebra $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{I} \subset \mathcal{S}'$ and $\mathcal{T}_\Phi$ is an abstract density topology on $(X, \mathcal{S}', \mathcal{I})$.*

Theorem 9 (cf. [4]). *If $\Phi\colon \mathcal{S} \to 2^X$ is an operator fulfilling 1^o–3^o generating $\mathcal{T}_\Phi$ topology then $\mathcal{N}(\mathcal{T}_\Phi) = \mathcal{I}$ and $\mathcal{S} = \mathcal{T}_\Phi \triangle \mathcal{I} = \{A \subset X : A = V \triangle B, V \in \mathcal{T}_\Phi, B \in \mathcal{I}\}$ if and only if $\mathcal{T}_\Phi$ is an abstract density topology on $(X, \mathcal{S}, \mathcal{I})$.*

In this part of the paper we shall present the next difference between measure and category on the real line.

Let Ba be a σ-algebra of sets having the Baire property on the real line $\mathbb{R}$ with respect to the natural topology and let $\mathbb{K}$ be a σ-ideal of sets of the first category, respectively.

Let $\Phi\colon Ba \to 2^X$ satisfy the conditions:

1^o $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,

2^o $\forall A, B \in Ba \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

3^o $\forall A, B \in Ba \quad A \triangle B \in \mathbb{K} \implies \Phi(A) = \Phi(B)$.

Theorem 10. *If $\Phi\colon Ba \to 2^{\mathbb{R}}$ satisfies conditions 1^o–3^o and generates $\mathcal{T}_\Phi$ topology stronger than natural topology then the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is Hausdorff but not regular.*

First, we prove the following

Lemma 11. *If a set $W \subset R$ is $\mathcal{T}_\Phi$-open and dense then it is residual.*

P r o o f. Let $W \in \mathcal{T}_\Phi$. Then $W = V \triangle A$, where V is open and A is a set of the first category. We will prove that V is a dense set. Let us suppose the contrary: there exists a non-open set C such that $C \cap V = \emptyset$. Hence $\Phi(C \cap V) = \Phi(C) \cap \Phi(V) = \emptyset$. By condition 3^o we have that $\Phi(V) = \Phi(W)$. Therefore $\Phi(C) \cap \Phi(W) = \emptyset$. At the same time $C \subset \Phi(C)$ because the topology $\mathcal{T}_\Phi$ is stronger then the natural topology. Also $W \subset \Phi(W)$ because $W \in \mathcal{T}_\Phi$. In that way, $C \cap W \subset \Phi(C) \cap \Phi(W) = \emptyset$. It contradicts the fact that $C \cap W \neq \emptyset$. Finally the set W as dense and open in residual. □

P r o o f o f T h e o r e m 10. Obviously the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is Hausdorff. Let $A \subset \mathbb{R}$ be a dense set and of the first category. Let $x \notin A$. By condition 3^o we have that $\mathbb{R} - A \in \mathcal{T}_\Phi$. Let us suppose that the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is regular. Hence there exist the sets $W, V \in \mathcal{T}_\Phi$ such that $A \subset W$, $x \in V$ and $W \cap V = \emptyset$. By Lemma 13 the set W is residual and the set V is not the second category so that $W \cap V \neq \emptyset$. It proves that the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is not regular and Hausdorff because the topology $\mathcal{T}_\Phi$ is stronger then the natural topology.

Now, let $\mathcal{L}$ be a σ-algebra of the Lebesgue measurable sets in $\mathbb{R}$ and $\mathbb{L}$ be a σ-ideal of the Lebesgue null sets.

Let $\Phi\colon \mathcal{L} \to 2^X$ satisfies the conditions:

1^o $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,

2^o $\forall A, B \in \mathcal{L} \quad \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

3^o $\forall A, B \in \mathcal{L} \quad A \triangle B \in \mathbb{L} \implies \Phi(A) = \Phi(B)$. □

PROPERTY 12. There are operators $\Phi_1, \Phi_2 \colon \mathcal{L} \to 2^{\mathbb{R}}$ generating the topologies $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$ stronger than the natural topology and such that $(\mathbb{R}, \mathcal{T}_{\Phi_1})$ is completely regular and $(\mathbb{R}, \mathcal{T}_{\Phi_2})$ is Hausdorff but not regular.

The f-density operators introduced by M. Filipczak, T. Filipczak (see [1]) are examples of such operators.

Namely, let $f \colon (0,\infty) \to (0,\infty)$ be a function such that $\liminf_{x\to 0^+} \frac{f(x)}{x} < \infty$. We say that x is an f-density point of a set $E \in \mathcal{L}$ if

$$\lim_{\substack{h\to 0, k\to 0\\ h\geq 0, k\geq 0\\ h+k>0}} \frac{|E' \cap [x-h, x+k]|}{f(h+k)} = 0.$$

Let $\Phi_{\langle f\rangle}(E) = \{x \in \mathbb{R} : x \text{ is an } f\text{-density point of } E\}$. The operator satisfies conditions 1^o–4^o and $\mathcal{T}_{\Phi_{\langle f\rangle}} = \big\{E \in \mathcal{L} : E \subset \Phi(E)\big\}$ is a topology.

The proof of Property 12 is a consequence of the following theorem.

THEOREM 13 (cf. [1])**.** *Let $f \colon (0,\infty) \to (0,\infty)$ be a function such that*

$$\lim_{x\to 0^+} f(x) = 0 \quad \textit{and} \quad f \textit{ is nondecreasing.}$$

If $0 < \liminf_{x\to 0^+} \frac{f(x)}{x} < \infty$, then the space $(\mathbb{R}, \mathcal{T}_{\Phi_{\langle f\rangle}})$ is completely regular.

If $\liminf_{x\to 0^+} \frac{f(x)}{x} = 0$, then the space $(\mathbb{R}, \mathcal{T}_{\Phi_{\langle f\rangle}})$ is Hausdorff but not regular.

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Faculty of Mathematics and
Computer Science
Łódź University
Banacha 22
PL–90-238 Łódź
POLAND
E-mail: hejduk@math.uni.lodz.pl