VERSITA

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/v10127-011-0025-1
Tatra Mt. Math. Publ. **49** (2011), 67–70

A DECOMPOSITION OF BOUNDED, WEAKLY MEASURABLE FUNCTIONS

Surjit Singh Khurana

ABSTRACT. Let $(X, \mathcal{A}, \mu)$ be a complete probability space, ρ a lifting, $\mathcal{T}_\rho$ the associated Hausdorff lifting topology on X and E a Banach space. Suppose $F\colon (X, \mathcal{T}_\rho) \to E''_\sigma$ be a bounded continuous mapping. It is proved that there is an $A \in \mathcal{A}$ such that $F\chi_A$ has range in a closed separable subspace of E (so $F\chi_A\colon X \to E$ is strongly measurable) and for any $B \in \mathcal{A}$ with $\mu(B) > 0$ and $B \cap A = \emptyset$, $F\chi_B$ cannot be weakly equivalent to a E-valued strongly measurable function. Some known results are obtained as corollaries.

1. Introduction and notation

In this paper R stands for the set of real numbers, K will denote the field of real or complex numbers (we will call them scalars). For locally convex spaces, results and notations of [5] will be used; for a locally convex space F with F' its dual, if $x \in F$ and $f \in F'$, $f(x)$ will also be denoted by $< f, x >$ and $< x, f >$. $(X, \mathcal{A}, \mu)$ is a complete probability space. We fix a lifting ρ for this measure space and taking the lifting topology $\mathcal{T}_\rho$ on X ([8, p. 58]; [3, p. 88]) (the open sets in this topology are $\{\rho(A) \setminus Q : A \in \mathcal{A},\ \mu(Q) = 0\}$); this space is a Baire space and the measure μ on this space is τ-smooth ([9], [10]). For a topological space Y with $A \subset Y$, $Y \setminus A$ will also be denoted by A', E is a Banach space with E', E'' its dual and bidual. We will denote by E''_σ the space E'' with $\sigma(E'', E')$ topology. A function $f\colon X \to E$ is said to be weakly measurable if $g \circ f$ is measurable for every $g \in E'$. For measure theory we will use the results and notations of [1]. For a bounded weakly measurable function $f\colon X \to E$, with $\|f\| \le C$ for some $C > 0$, we get, for every $g \in E'$, a bounded continuous $\rho(g \circ f)\colon (X, \mathcal{T}_\rho) \to K$ such that $g \circ f = \rho(g \circ f)$ *a.e.* (μ). So $|\rho(g \circ f)| \le C$ for all $g \in E'$ with $\|g\| \le 1$. Define $F\colon X \to E''\colon < g, F(x) >= \rho(g \circ f)(x)$. This function is easily checked to

2010 Mathematics Subject Classification: Primary 46G15, 28A51, 28B05; Secondary 28C15.
Keywords: liftings, weakly measurable functions, weakly equivalent functions, vector measures with finite variations.

be well-defined and is continuous as a mapping $F\colon (X, \mathcal{T}_\rho) \to E''_\sigma$. In the measure space $(X, \mathcal{A}, \mu)$, two weakly measurable functions f_1, f_2 are said to be weakly equivalent, written as $f_1 \equiv f_2$, if for every $g \in E'$, $g \circ f_1 = g \circ f_2$, *a.e.*$(\mu)$, thus the functions f, F are weakly equivalent.

In [7], using Stonian transforms, an interesting result about the decomposition of weakly measurable functions $f\colon X \to E$ is proved. Using liftings, we put the result in a different setting. This result is also used to get a simpler method for the decomposition of E-valued measures of finite variations as done in [4].

2. Main theorem

Now we come to the main theorem.

Theorem 1. *Suppose $F\colon (X, \mathcal{T}_\rho) \to E''_\sigma$ is a bounded continuous mapping. Then there is a unique $A \in \mathcal{A}$ (in the sense that $A_1 = A_2$ if $\mu(A_1 \Delta)A_2 = 0$) such that $F\chi_A$ has range in a closed separable subspace of E (so $F\chi_A\colon X \to E$ is strongly measurable) and for any $B \in \mathcal{A}$ with $\mu(B) > 0$ and $B \cap A = \emptyset$, $F\chi_B$ cannot be weakly equivalent to a strongly measurable function.*

P r o o f. The closed unit ball S of E'' is the intersection of closed cylindrical sets of E''_σ and so $F^{-1}(S)$ is $\mathcal{T}_\rho$-closed. From this it follows that for any closed separable subspace $Q \subset E$, $F^{-1}(Q) \in \mathcal{A}$. Let $\mathcal{S}$ be the collection of all closed separable subspaces of E. Define an order in $\mathcal{S}$: $E_1 \geq E_2$ if $\rho\big(F^{-1}(E_1)\big) \supset \rho\big(F^{-1}(E_2)\big)$. Let $\mathcal{P}$ is a totally ordered subset of $\mathcal{S}$ and let $c = \sup \mu\big(\{F^{-1}(P) : P \in \mathcal{P}\}\big)$. Take an increasing sequence $\{P_n : n \in N\} \subset \mathcal{P}$ such that $\sup \mu\big(F^{-1}(P_n)\big) = c$. Thus $P = \overline{\cup P_n}$ (closure in E) is a maximal element of $\mathcal{P}$. By Zorn lemma, $\mathcal{S}$ has a maximal element, say E_0, a closed separable subspace of E. Let $A = F^{-1}(E_0)$. Then $F\chi_A\colon X \to E$ is strongly measurable function.

Take a $B \in \mathcal{A}$, $\mu(B) > 0$ such that $A \cap B = \emptyset$. Assume $F\chi_B$ is weakly equivalent to a strongly measurable function f_0. Since f_0 is strongly measurable, by Egoroff's theorem, there is a $C \subset B$, $C \in \mathcal{A}$ with $\mu(C) > 0$, and a sequence of measurable simple functions $\big\{f_n\colon X \to (E, \|.\|)\big\}$ such that $f_n \to f_0$ uniformly on C with norm topology on E. From this it follows that there is an absolutely convex compact subset $Q \subset E$ such that $f_0(C) \subset Q$. Put $f = f_0\chi_C$. Define $\bar{f}\colon X \to E''$, $< g, \bar{f}(x) >= \rho(g \circ f)(x)$. This function $\bar{f}\colon (X, \mathcal{T}_\rho) \to E''_\sigma$ is easily seen to be well-defined and continuous. We claim $\bar{f}$ is Q-valued. If this is not true, there is an $x \in X$, a $g \in E'$ such that $Rl\big(< g, \bar{f}(x) >\big) > \sup Rl\ g(Q) \geq \sup \big|g\big(f(X)\big)\big|$. Since $\sup\big|\big(\rho(g \circ f)\big)(X)\big| = \sup\big|\big((g \circ f)\big)(X)\big|$, this is a contradiction.

Thus $F\chi_C \equiv \bar{f}$ and so $F\chi_{\rho(C)} \equiv \bar{f}$. Because of the continuity of $F\chi_{\rho(C)}$ and $\bar{f}$, we get $F\chi_{\rho(C)} = \bar{f}$. So the range of $F\chi_C$ is contained in a closed separable subspace G_0 of E. Let $E_1 = \overline{span(E_0 \cup G_0)}$ (closure in E). Now $F^{-1}(E_1) \supset (A \cup C)$ and $\mu(A \cup C) = \mu(A) + \mu(C) > \mu(A)$, a contradiction.

To prove the uniqueness of A, suppose $A_1 \in \mathcal{A}$ also satisfies the conditions of the theorem and $\mu(A_1 \Delta A) > 0$. Suppose $\mu(A_1 \setminus A) > 0$. Put $B = A_1 \setminus A$. Then $\mu(B) > 0$ and the range of $F\chi_B$ is contained in a closed separable subspace of E. Thus $F\chi_B$ is weakly equivalent to a strongly measurable. This contradicts the definition of A. In a similar way, we can prove that $\mu(A \setminus A_1) > 0$ leads to a contradiction. □

Remark 2. A can be obtained in another way which gives a more insight about its connection to E:

Since μ is τ-smooth, the measure μF^{-1} on E''_σ will also be τ-smooth. Because of boundedness of F, this will be a Radon measure. Since $\big(E, \sigma\big(E, E'\big)\big)$ in universally measurable ([6, Theorem 3.4, p. 8]), E is measurable in E''_σ relative to the measure μF^{-1}. Thus $F^{-1}(E) = A_0 \in \mathcal{A}$. Put $A = A_0 \cap \rho(A_0)$. Thus A is an open set in X and $\mu(A_0 \Delta A) = 0$. If for some $g \in E'$, $g \circ F\chi_A = 0$, *a.e.* (μ), then, using the openness of A and continuity of F, one could easily verify that $g \circ F\chi_A = 0$ on X. From [2, Theorem 5, p. 390] it follows that range of $F\chi_A$ is contained in a separable subspace of E. This gives additional insight about A relative to E. Also $(F\chi_{A'})^{-1}(E) = A' \cap A_0 = A' \cap (A \cup Q) = A' \cap Q$ (Q a null set), is a closed null set and so is of first category (cf. [7, Definition 203, p. 430]).

The main result of [7] can be put in the following form.

Corollary 3. *Suppose $f\colon X \to E$ is a bounded weakly measurable function. Then there is an $A \in \mathcal{A}$ such that $f\chi_A$ is weakly equivalent to a strongly measurable function and for any $B \in \mathcal{A}$ with $B \cap A = \emptyset$ and $\mu(B) > 0$, $f\chi_B$ is not weakly equivalent to a strongly measurable function. This A is unique in the sense that if there is another A_1 with this property, then $\mu(A \Delta A_1) = 0$.*

Proof. Using liftings, wee get a bounded, continuous $F\colon (X, \mathcal{T}_\rho) \to E''_\sigma$. Using Theorem 1, it is a routine verification. □

Remark 4. This result is proved in [7, Theorem 2.5, p. 430] without the introduction of the set A. If we decompose $f = \chi_A f + \chi_{A'} f$, then trivially $\big(g \circ (\chi_A f)\big)\big(g \circ (\chi_{A'} f)\big) = 0$ on X.

In [4] a result about the decomposition of vector measures with finite variations is proved. Using Theorem 1, this result comes easily.

Corollary 5. *Let $\nu\colon \mathcal{A} \to E$ be a measure of bounded variation. Putting $\mu = |\nu|$, we assume that $(X, \mathcal{A}, \mu)$ is a complete probability space. Then there is an $A \in \mathcal{A}$ such that $\chi_A\, \nu$ has RN derivative and for any $B \in \mathcal{A}$ with $B \cap A = \emptyset$ and $\mu(B) > 0$, $\chi_B\, \mu$ does not have RN derivative. This A is unique in the sense that if there is another A_1 with this property, then $\mu(A \Delta A_1) = 0$.*

P r o o f. Using liftings and the RN property of scalar-valued measures, we get a bounded, continuous $F\colon (X, \mathcal{T}_\rho) \to E''_\sigma$. Using Theorem 1, the proof is a routine procedure. $\square$

REFERENCES

[1] DIESTEL, J.—UHL, J. J.: *Vector Measures*. in: Mathematical Surveys, Vol. 15, AMS, Providence, R.I., 1977.

[2] KHURANA, S. S.: *Pointwise compactness and measurability*, Pacific. J. Math. **83** (1979), 387–391.

[3] OXTOBY, J.: *Measure and Category*. Springer-Verlag, New York, 1971.

[4] RYBAKOV, V. I.: *The separation from a vector measure of the part representable by Bochner integral*, Math. Notes **17** (1975), 476–482.

[5] SCHAEFER, H. H.: *Topological Vector Spaces*. Springer-Verlag, New York, 1986.

[6] SCHWARTZ, L.: *Certaine proprietes des mesures sur les espsces de Banach*, Seminaire d'analyse fonctionnelle **23** (1975–1976), 1–12.

[7] SENTILLES, F.D.: *Decomposition of weakly measurable functions*, Indiana Univ. Math. J. **32** (1983), 425–437.

[8] IONESCU TULCEA, A.—IONESCU TULCEA, C.: *Topics in the Theory of Lifting*. Springer-Verlag, New York, 1969.

[9] WHEELER, R. F.: *A survey of Baire measures and strict topologies*, Expo. Math. **2** (1983), 97–190.

[10] VARADARAJAN, V. S.: *Measures on topological spaces*, Amer. Math. Soc. Transl. Ser. 2 **48** (1965), 161–228.

Received December 29, 2010

Department of Mathematics
University of Iowa
Iowa City, Iowa 52242
U. S. A.
E-mail: khurana@math.uiowa.edu