

# MEASURES AND IDEMPOTENTS IN THE NON-COMMUTATIVE SITUATION

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ABSTRACT. We investigate measures on sequential orthomodular posets with values in a vector space or a (not necessarily commutative) algebra with reasonable sequential topologies, using a universal property. Unfortunately, the universal measure and the universal multiplicative measure need not coincide any more as in the commutative situation. This may have applications in quantum physics.

## 1. Introduction

In [1] we investigated an adjunction between multiplicative measures and the idempotents of a commutative sequentially convex algebra, where the idempotents form a Boolean algebra. This is no longer true for non-commutative algebras; the idempotents of a not necessarily commutative algebra need not form a lattice, but only an *orthomodular poset* (cf. [2]). Nevertheless, in a sequentially convex algebra we have a sequential orthomodular poset of idempotents, which depends functorially on the algebra. We will define measures on orthomodular posets and will see that there always exists a measure with a *universal property*. Moreover, *polymeasures*, i.e., sequentially continuous maps on a product of finitely many sequential orthomodular posets which are measures in each component, correspond to linear maps on the sequentially convex tensor product of the targets of the universal measures.

If we assign to every sequentially convex algebra its *sequential orthomodular poset*, we obtain a functor, which even has a *right adjoint*. The counit of this adjunction can be interpreted as a *universal multiplicative measure* as in the commutative case. But in contrast, in general, the universal multiplicative measure need not coincide with the universal measure. Nevertheless, the left adjoint

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preserves coproducts by categorical reasons. It is not clear whether the results can be applied.

## 2. The universal measure

A subset  $Z$  of a topological space  $X$  is called *sequentially closed* if it contains all limits of sequences of elements of  $Z \subset X$ . Obviously, every closed subset is sequentially closed, but the converse is not true in general. The space  $X$  is called *sequential* if in  $X$  every sequentially closed set is closed. Every sequentially continuous map from a sequential space into an arbitrary topological space is continuous (cf. [5]). Every intersection and every finite union of sequentially closed sets is sequentially closed; hence we can define a new topology on  $X$  whose closed sets are the sequentially closed sets of  $X$ . We call  $X$  with this new topology the *sequential modification* of  $X$ ; its topology is always sequential and (in general strictly) finer than the original topology; the convergence of a sequence to a point means the same in both topologies. The sequential modification defines a *coreflection*, i.e., a right adjoint to the (full) embedding from the category of sequential topological spaces into the category of all topological spaces (cf. [8]).

The topological product  $X \times Y$  of two sequential spaces  $X, Y$  need not be sequential (cf. [5]). But its sequential modification, which we shall call  $X \amalg Y$  is a *sequential product*, i.e., a categorical product inside the category of sequential spaces. The topology of  $X \amalg Y$  is the unique sequential topology such that  $(x_n, y_n) \rightarrow (x, y)$  if and only if  $x_n \rightarrow x$  in  $X$  and  $y_n \rightarrow y$  in  $Y$ . Every open or closed subset of a sequential space  $X$  is sequential, but not every subspace of  $X$  is sequential (cf. [5]); a subset with the sequential modification of the subspace topology is often called a *sequential subspace*.

A sequential space is called *weakly Hausdorff* if every sequence converges to at most one point. This is strictly stronger than  $T_1$  and strictly weaker than  $T_2$ . All sequential products and all sequential subspaces of weakly Hausdorff spaces are weakly Hausdorff.

An *orthomodular poset* is a poset  $X$  with a top element  $1$  and a bottom element  $0$  together with an order-reversing map  $X \rightarrow X$ ,  $x \mapsto x^\perp$  subject to the following conditions:

- (i)  $x^{\perp\perp} = x$  for all  $x \in X$ .
- (ii) If  $x \leq x^\perp$ , then  $x = 0$ .
- (iii) If  $x \leq y^\perp$ , then the supremum of  $x$  and  $y$  exists.
- (iv) If  $x \leq y$ , then there exists a  $z \in X$  with  $z \leq y^\perp$  such that  $y$  is the supremum of  $x$  and  $z$ .

We call  $x^\perp$  the *complement* of  $x$ . Moreover, we write  $x \perp x$  and call  $x$  and  $y$  *orthogonal* if  $x \leq y^\perp$ , or equivalently,  $y \leq x^\perp$ ; so orthogonality is a symmetric relation. For  $x \perp y$  we denote its supremum by  $x + y$ . Then (iv) means that for  $x \leq y$  there exists a  $z$  with  $x \perp z$  and  $x + z = y$ . Note that this  $z$  is always uniquely determined, namely  $z = (x + y^\perp)^\perp$ . But without condition (iv) it is not clear that  $x + (x + y^\perp)^\perp = y$  always holds. Our results would remain true without (iv), but it should be included in an appropriate definition of orthomodularity. Two complementary elements of an orthomodular poset always have meet 0 because  $x \leq y$  and  $y \leq x^\perp$  always implies  $y \leq x^\perp \leq y^\perp$ , hence  $y = 0$  by (ii). Since  $^\perp$  is an anti-isomorphism of the order,  $x = x^{\perp\perp}$  and  $x^\perp$  always have join  $1 = 0^\perp$ .

A *morphism of orthomodular posets* is a map from orthomodular poset to another one which preserves order, top, bottom, complementation, and the join of any two orthogonal elements. A *weakly Hausdorff sequential orthomodular poset* (WHSOP) is an orthomodular poset together with a weakly Hausdorff sequential topology such that  $\{(x, y) \in X \times X \mid x \leq y\}$  is closed,  $X \rightarrow X, x \mapsto x^\perp$  is continuous and the formation of joins is a continuous map  $D \rightarrow X$  for  $D := \{(x, y) \in X \times X \mid x \perp y\}$ . The topology of  $D$  is the usual subspace topology because  $D \subset X \times X$  is closed and therefore already sequential as a topological subspace. A *WHSOP-morphism* is a morphism of orthomodular posets which is also (sequentially) continuous.

A *sequentially convex space* (see [1]) is a real vector space  $E$  together with a sequential topology such that the addition  $E \times E \rightarrow E$  and the multiplication with a scalar  $\mathbb{R} \times E \rightarrow E$  are (sequentially) continuous, subject to two more axioms. From local compactness of  $\mathbb{R}$  it follows that  $\mathbb{R} \times E$  is always sequential, hence  $\mathbb{R} \times E = \mathbb{R} \times E$ . But  $E \times E$  need not be sequential; so it may differ from  $E \times E$ . The first axiom is that  $E$  is sequentially complete, i.e., every Cauchy sequence converges. Here a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $E$  is called a Cauchy sequence if and only if for every 0-neighbourhood  $U \subset E$  there exists an  $n_0 \in \mathbb{N}$  with  $x_{n_1} - x_{n_2} \in U$  for all  $n_1, n_2 \geq n_0$ . This is equivalent to the statement that the difference of any two subsequences converges to 0. The second axiom says that the topology of  $E$  is a sequential modification of a locally convex topology. This means that a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $E$  already converges to 0 if for every *convex* 0-neighbourhood  $U \subset E$  there exists an  $n_0 \in \mathbb{N}$  with  $x_n \in U$  for  $n \geq n_0$ . So a sequentially convex space is a vector space together with a weakly Hausdorff sequential topology and sequentially continuous vector space operations satisfying these two axioms.

For a WHSOP  $X$  and a sequentially convex space  $E$  we call a map  $\mu : X \rightarrow E$  a *measure*  $f$  and only if it is continuous and also satisfies  $\mu(x + y) = \mu(x) + \mu(y)$  for all  $x, y \in X$  with  $x \perp y$ . Observe that this always implies  $\mu(0) = 0$  because  $0 + 0 = 0$  in  $X$ . First, we observe the following

**THEOREM 2.1.** *For WHSOP  $X$  there exist a sequentially convex space  $L_{(\infty)}X$  and a measure  $\chi_X : X \rightarrow L_{(\infty)}X$  with the following universal property: For every sequentially convex space  $X$  and every measure  $\mu : X \rightarrow E$  there exists a unique continuous linear map  $j : L_{(\infty)}X \rightarrow E$  with  $j \circ \chi_X = \mu$ .*

**Proof.** For fixed  $X$  the functor from the category of sequentially convex spaces to the category of sets, which assigns to every sequentially convex space  $E$  the set for all measures  $X \rightarrow E$  has a left adjoint by the Adjoint Functor Theorem (cf. [3], [6], [7]). Application to a singleton set yields that this functor is representable; this is the statement of the theorem.  $\square$

### 3. Spaces of measures

We can endow the set of all measures  $X \rightarrow E$  with a nice sequential topology, the so-called *topology of continuous convergence*. Its nice behavior is based on the fact that the category of *cartesian closedness* of the fact that sequential spaces (or of weakly Hausdorff sequential spaces) form a *cartesian closed category*. For two sequential spaces  $X, Y$ , we say that a sequence of continuous maps  $(f_n)_{n \in \mathbb{N}} : X \rightarrow Y$  *converges continuously* to  $f : X \rightarrow Y$  if for every sequence  $(x_n)_{n \in \mathbb{N}}$  which converges to  $x$  in  $X$  it follows that  $(f_n(x_n))_{n \in \mathbb{N}}$  converges to  $f(x)$  in  $Y$ . Then continuous convergence is induced by a topology, i.e., there is a topology on the set of all continuous maps  $X \rightarrow Y$  such that a sequence converges to a map  $f : X \rightarrow Y$  in this topology if and only if it converges to  $f$  continuously. Then there is a unique sequential topology with this property; we denote the resulting space of continuous maps  $X \rightarrow Y$  by  $\mathcal{C}_s(X, Y)$ . For another sequential space  $Z$  it follows that a map  $h : Z \rightarrow \mathcal{C}_s(X, Y)$  is continuous if and only if the corresponding map  $Z \times X \rightarrow Y$ ,  $(z, x) \mapsto h(z)(x)$  is continuous (cf. [3], [4], [7], [8]).

For a WHSOP  $X$  and sequentially convex space  $E$  the subset  $\mathcal{M}(X, E) \subset \mathcal{C}_s(X, E)$  of all measures  $X \rightarrow E$  is closed and therefore even sequential in the usual subspace topology. It can be regarded as a *space of measures*, and for a sequential space  $I$  we can consider a continuous map  $\mu : I \rightarrow \mathcal{M}(X, E) \subset \mathcal{C}_s(X, E)$  as a *continuous family of measures*. By cartesian closedness it corresponds to a continuous map  $\tilde{\mu} : I \times X \rightarrow E$ ; the property that  $\mu$  attains values in  $\mathcal{C}(X, E)$  is equivalent to the statement that  $\tilde{\mu}$  is a measure in the second component. By interchange of arguments, such maps correspond to continuous maps  $X \times I \rightarrow E$  which are measures in the first component. Again by cartesian closedness, such continuous maps correspond to certain continuous maps  $X \rightarrow \mathcal{C}_s(I, E)$ .

It is easy to see that  $\mathcal{C}_s(I, E)$  is a vector space under pointwise operations; moreover, these operations are even sequentially continuous. From [1] we use the following

**LEMMA 3.1.** *For a sequential space  $I$  and a sequentially convex space  $E$  the space  $\mathcal{C}_s(I, E)$  with pointwise operations is sequentially convex. For two sequentially convex spaces  $E_0, E_1$  the closed subspace  $\mathcal{L}_s(E_0, E_1) \subset \mathcal{C}_s(E_0, E_1)$  of linear continuous maps is again sequentially convex.  $\square$*

In the above situation the measure property of the continuous map  $X \text{III} \rightarrow E$  in the first component means that the corresponding map  $X \rightarrow \mathcal{C}_s(I, E)$  is a measure. Now we can use cartesian closedness and the above universal property:

**THEOREM 3.2.** *For a WHSOP  $X$  and a sequentially convex space  $E$  there is a homeomorphism  $\mathcal{M}(X, E) \cong \mathcal{L}_s(L_{(\infty)}(X, E))$ . This isomorphism is natural, i.e., compatible with morphisms in both components.*

**PROOF.** As seen above, for a sequential space  $I$  there is a bijection between continuous maps  $I \rightarrow \mathcal{C}(X, E)$  and measures  $X \rightarrow \mathcal{C}_s(I, E)$ . Now by 2.1 the set of all such maps is in bijection with the set of all continuous linear maps  $L_{(\infty)}X \rightarrow \mathcal{C}_s(I, E)$ . By cartesian closedness, such maps correspond to continuous maps  $L_{(\infty)}X \text{III} \rightarrow E$  which are linear in the first component, hence to continuous maps  $\text{II}L_{(\infty)}X \rightarrow E$  which are linear in the second component. By cartesian closedness they correspond to continuous linear maps  $I \rightarrow \mathcal{L}_s(L_{(\infty)}X, E)$ .

This means that for given  $X$  and  $E$  the contravariant hom-functors from the category of sequential spaces to the category of sets represented by the objects  $\mathcal{M}(X, E)$  and  $\mathcal{C}_s(L_{(\infty)}X, E)$  coincide; hence by Yoneda (cf. [3], [6], [7]) both objects are canonically isomorphic. In order to see this directly, apply the two functors to the objects  $\mathcal{M}(X, E)$  and  $\mathcal{C}_s(L_{(\infty)}X, E)$  itself; then application to the identity morphisms yields continuous maps in both directions. Now the uniqueness condition in 2.1 gives that these two maps are inverse to each other. Naturality is obvious.  $\square$

Observe that  $\mathcal{M}(X, E) \subset \mathcal{C}_s(X, E)$  is always a vector space (under pointwise operations). Then we can check that the above bijection is also a vector space homomorphism. Now 3.2 shows that it is even sequentially convex; this can also be seen directly.

## 4. Polymeasures

It may be interesting to consider the case that a continuous family of measures is parametrized by a measure itself. This means to consider continuous

maps  $X_1 \amalg X_2 \rightarrow E$  for WHSOPs  $X_1, X_2$  and a sequentially convex space  $E$ . We may even generalize this to more factors, i.e., for  $n \in \mathbb{N}$ , WHSOPs  $X_1, \dots, X_n$  and a sequentially convex space  $E$ , we call a continuous map  $X_1 \amalg \dots \amalg X_n$  a *polymeasure* if it is a measure in each component. We can also define this for  $n = 0$ ; the empty product is a singleton set  $\{1\}$  and every map  $\{1\} \rightarrow E$  is a polymeasure; so these polymeasures correspond to the elements of  $E$ .

We need the notion of the *tensor product* of sequentially convex spaces; it is a completion of the (algebraic) tensor product of vector spaces in a suitable topology. It can be defined uniquely by a *universal property* for *continuous bilinear maps*; its existence follows easily from the Adjoint Functor Theorem (cf. [3], [6], [7]). Moreover, there is a categorical relationship (called *adjunction*) to the above internal hom-functor. We recall from [1]:

**THEOREM 4.1.** *For two sequentially convex spaces  $E_1, E_2$  there exist a sequentially convex space  $E_1 \hat{\otimes} E_2$  and a continuous bilinear map  $b: E_1 \amalg E_2 \rightarrow E_1 \hat{\otimes} E_2$ ,  $(u, v) \mapsto u \hat{\otimes} v$  such that for each sequentially convex space  $E_3$  and every continuous bilinear map  $b: E_1 \amalg E_2 \rightarrow E_3$  there exists a unique continuous linear map  $h: E_1 \hat{\otimes} E_2 \rightarrow E_3$  with  $h(u \hat{\otimes} v) = b(u, v)$  for all  $u \in E_1, v \in E_2$ . A map  $l: E_1 \rightarrow \mathcal{L}_s(E_2, E_3)$  is continuous and linear if and only if the corresponding map  $E_1 \hat{\otimes} E_2 \rightarrow E_3$ ,  $(u, v) \mapsto l(u)(v)$  is continuous and linear.  $\square$*

This yields a so-called *symmetric monoidal closed* structure ([4], cf. [3], [7]) on the category of sequentially convex spaces. It has certain *coherence properties*, in particular  $(E_1 \hat{\otimes} E_2) \hat{\otimes} E_3 \cong E_1 \hat{\otimes} (E_2 \hat{\otimes} E_3)$  (for all sequentially convex spaces)  $E_1, E_2, E_3$ , and these isomorphisms satisfy some so-called *coherence conditions*. By Mac Lanes's *coherence theorem* (cf. [7]), all reasonable diagrams commute; so we can omit the brackets without causing misunderstandings. One can even treat the data in such a way, by natural isomorphisms, that the coherence morphisms become identities. The base field  $\mathbb{R}$  in the usual topology and vector space structure is a *unit object* in the symmetric monoidal structure, it behaves like a neutral element with respect to the tensor product; so we consider it as a tensor product of 0 elements.

**THEOREM 4.2.** *For  $n \in \mathbb{N}$ , WHSOPs  $X_1, \dots, X_n$ , and a sequentially convex space  $E$  there is a natural bijection between polymeasures  $X_1 \amalg \dots \amalg X_n \rightarrow E$  and linear maps  $L_{(\infty)} X_1 \hat{\otimes} \dots \hat{\otimes} L_{(\infty)} X_n \rightarrow E$ .*

**Proof.** We proceed by induction on  $n$ , simultaneously for all  $E$ . The case  $n = 0$  is trivial. If the statement is true for some  $n \in \mathbb{N}$ , assume you have a polymeasure  $p: X_1 \amalg \dots \amalg X_{n+1} \rightarrow E$  of  $n + 1$  arguments. Then the corresponding map  $X_1 \amalg \dots \amalg X_n \rightarrow \mathcal{M}(X_{n+1}, E)$ , which maps  $(x_1, \dots, x_n)$  to the map  $x_{n+1} \mapsto p(x_1, \dots, x_n, x_{n+1})$  is a polymeasure of  $n$  arguments; so by induction hypothesis it corresponds to a continuous linear map  $L_{(\infty)} X_1 \hat{\otimes} \dots \hat{\otimes} L_{(\infty)} X_n \rightarrow \mathcal{M}(X_{n+1}, E)$ .

By 3.2 we have a canonical isomorphism

$$\mathcal{M}(X_{n+1}, E) \cong \mathcal{L}_s(L_{(\infty)}X_{n+1}, E).$$

So our  $p$  corresponds to a continuous linear map

$$L_{(\infty)}X_1 \hat{\otimes} \cdots \hat{\otimes} L_{(\infty)}X_n \rightarrow \mathcal{L}_s(L_{(\infty)}X_{n+1}, E),$$

hence by 4.2 to a continuous linear map

$$L_{(\infty)}X_1 \hat{\otimes} \cdots \hat{\otimes} L_{(\infty)}X_n \hat{\otimes} L_{(\infty)}X_{n+1} \rightarrow E.$$

□

## 5. Idempotents and involutions

The tensor product of commutative (associative and unital) sequentially convex algebras is a coproduct; this can be used to define product measures. This uses the facts that on the one hand, the universal measure is also a universal multiplicative measure and on the other hand, the tensor product of sequentially convex spaces only uses the topology and the vector space structure, not the multiplication and the unit element. For non-commutative algebras the situation is quite different: The tensor product of algebras exists, but is no longer a coproduct of algebras. Moreover, there exists a universal multiplicative measure, but in general, it does not live on the same sequentially convex space as the universal measure. But maybe for quantum physics the coproduct is more important than the tensor product; the latter seems to model different types of events which can be observed independently, while one is probably more interested in dependence. Maybe the coproduct has a reasonable physical interpretation.

A *sequentially convex algebra* is a sequentially convex space  $E$  together with a bilinear continuous associative (but not necessarily commutative) multiplication. A *homomorphism* of sequentially convex algebras is a continuous algebra homomorphism preserving the unit element. An *idempotent* in a sequentially convex algebra  $E$  is an element  $u \in E$  with  $u^2 = u$ . We always have the idempotents 0 and 1. For an idempotent  $x$  it follows that  $x^\perp := 1 - x$  is also idempotent, and for idempotents  $x, y$  we define  $x \leq y \Leftrightarrow xy = x = yx$ . Then the idempotents form an orthomodular poset  $\Theta E$ ; we have  $x \perp y \Leftrightarrow xy = yx = 0$  for  $x, y \in \Theta E$ . Moreover,  $\Theta E \subset E$  is a closed subset and therefore sequential in the usual subspace topology; it even turns out to be a WHSOP.

Note that in general it is not a lattice; it does not admit all binary joins and meets. A counterexample is the sequentially convex algebra of real  $4 \times 4$ -matrices (cf. [2]) in the Euclidean topology. This is different from the commutative case, where the idempotents always even form a Boolean algebra. For a WHSOP  $X$ , a sequentially convex algebra  $E$  and a WHSOP-morphism  $X \rightarrow \Theta E$  the

composite  $X \rightarrow \Theta E \hookrightarrow E$  is a measure  $\mu \in \mathcal{M}(X, E)$ , which also satisfies  $\mu(1) = 1$  and  $\mu(x)\mu(y) = 0$  for  $x \perp y$ . We call a measure with these two properties *multiplicative*. Conversely, for a multiplicative measure  $\mu : X \rightarrow E$  we have  $x^\perp \perp x$  for all  $x \in \Theta E$ , hence  $\mu(x^\perp)\mu(x) = 0$ , and we obtain  $\mu(x)^2 = \mu(x)^2 + \mu(x^\perp)\mu(x) = (\mu(x) + \mu(x^\perp))\mu(x) = \mu(x + x^\perp)\mu(x) = \mu(1)\mu(x) = \mu(x)$  for all  $x \in X$ , i.e.,  $\mu$  attains only idempotent values, and a straightforward computation shows that it even induces a WHSOP-morphism  $X \rightarrow \Theta E$ .

The above assignment even defines a functor from the category of sequentially convex algebras to the category of WHSOPs. It follows by routine arguments from the Adjoint Functor Theorem (cf. [3], [6], [7]) that this functor has a left adjoint. This yields the following

**THEOREM 5.1.** *For every WHSOP  $X$  there exist a sequentially convex algebra  $L_{(\infty)}^\sharp X$  and a multiplicative measure  $\chi_X^\sharp : L_{(\infty)}^\sharp X \rightarrow L_{(\infty)}^\sharp X$  such that for every sequentially convex algebra  $E$  and every multiplicative measure  $\mu : X \rightarrow E$  there exists a unique sequential algebra homomorphism  $l : L_{(\infty)}^\sharp X \rightarrow E$  with  $l \circ \chi_X^\sharp = \mu$ .  $\square$*

By general reasons the left adjoint functor  $L_{(\infty)}^\sharp$  preserves colimits, in particular coproducts (cf. [3], [6], [7]). In contrast to the commutative case,  $L_{(\infty)}^\sharp X$  may differ from  $L_{(\infty)} X$ . Moreover, coproducts of Boolean algebras in the category of orthomodular posets need not be Boolean algebras. Start with two four-element discrete Boolean algebras, i.e., Boolean algebras with two atoms. Then their (sequentially continuous) Boolean coproduct is a discrete Boolean algebra  $B$  with four atoms, i.e., with 16 elements, and  $L_{(\infty)} B \cong L_{(\infty)}^\sharp B$  is a four-dimensional vector space. On the other hand, the coproduct of WHSOPs is the discrete six-element orthomodular lattice  $X$  with a top element 1, a bottom element 0, and four incomparable elements  $a, b, a^\perp, b^\perp$ . Then a measure  $\mu : X \rightarrow E$  into a sequentially convex space  $E$  is given by the three vectors  $\mu(a), \mu(b), \mu(1)$ , which can be chosen arbitrarily; on the other elements,  $\mu$  is given by  $\mu(0) = 0$ ,  $\mu(a^\perp) = \mu(1) - \mu(a)$ ,  $\mu(b) = \mu(1) - \mu(b^\perp)$ . Therefore  $L_{(\infty)} X$  is a three-dimensional vector space. But  $L_{(\infty)}^\sharp X$  is the (non-commutative) algebra on two generators  $t, t'$  with the relations  $t^2 = t$ ,  $t'^2 = t'$ . Therefore it has a basis of all finite alternating (possibly empty) strings of the elements  $t$  and  $t'$ . Therefore it is (countably) infinite-dimensional, its topology is the finest vector space topology; it coincides with the finest locally convex topology.

Finally we claim that the universal multiplicative measure also has the property in the category of sequentially convex algebras with *involution*. An involution on a sequentially convex algebra  $E$  is a continuous linear map  $E \rightarrow E$ ,  $u \mapsto u^*$  with  $u^{**} = u$  for all  $x \in E$  and  $(uv)^* = v^*u^*$  for all  $x, y$ . This always implies  $1^* = 1$ . A  $u \in E$  is called *symmetric* if  $u^* = u$  holds. In the classical Hilbert space model for quantum dynamics,  $E$  is the vector space of all bounded linear

operators into itself (in a reasonable topology) and the involution is the formation of the adjoint operator. Then the symmetric elements are the self-adjoint (sometimes called hermitean) operators. If  $E_1, E_2$  are sequentially convex algebras with involution, we say that a homomorphism  $f : E_1 \rightarrow E_2$  preserves the involution if and only if  $f(u^*) = f(u)^*$  holds for all  $x \in E_1$ .

**THEOREM 5.2.** *For a WHSOP  $X$  there is a unique involution on  $L_{(\infty)}^{\sharp}X$  such that  $\chi_X^{\sharp}$  attains only symmetric values. If  $\mu : X \rightarrow E$  is an arbitrary multiplicative measure with values in a sequentially convex algebra with an involution, then the unique sequential algebra homomorphism  $l : L_{(\infty)}^{\sharp}X \rightarrow E$  with  $l \circ \chi_X^{\sharp}$  preserves this involution if and only if all values of  $\mu$  are symmetric.*

*Proof.* For a sequentially convex algebra we can also consider the *opposite algebra*  $E^{\text{op}}$ , which has the same underlying vector space with the same topology and the same unit element, but the opposite multiplication “ $*$ ” with  $u * v := vu$  for  $u, v \in E$ . Then  $E^{\text{op}}$  is also a sequentially convex algebra, and a multiplicative measure  $\mu : X \rightarrow E$  is also a multiplicative measure as a map  $X \rightarrow E^{\text{op}}$ . In particular,  $\chi_X^{\sharp} : X \rightarrow (L_{(\infty)}^{\sharp}X)^{\text{op}}$  is also a multiplicative measure. Now 5.1 yields a unique homomorphism  $s : L_{(\infty)}^{\sharp}X \rightarrow (L_{(\infty)}^{\sharp}X)^{\text{op}}$  with  $s \circ \chi_X^{\sharp} = \chi_X^{\sharp}$  as a (continuous linear) map. But then the map  $s$  is also a homomorphism from  $(L_{(\infty)}^{\sharp}X)^{\text{op}} \rightarrow L_{(\infty)}^{\sharp}X$ ; hence  $s \circ s : L_{(\infty)}^{\sharp}X \rightarrow L_{(\infty)}^{\sharp}X$  is also a homomorphism of sequentially convex algebras, and it satisfies  $s \circ s \circ \chi_X^{\sharp} = s \circ \chi_X^{\sharp} = \chi_X^{\sharp}$ . But the identity map has the same properties; hence the uniqueness statement of 5.1 gives  $s \circ s = \text{id}_{L_{(\infty)}^{\sharp}X}$ . Now  $x \mapsto x^* := s(x)$  is the desired involution, and it is unique by uniqueness of  $s$ .

Now let  $E$  be a sequentially convex algebra with involution and let  $\mu : X \rightarrow E$  be a multiplicative measure. Now we have two homomorphisms of sequentially convex algebras  $L_{(\infty)}^{\sharp}X \rightarrow E^{\text{op}}$  defined by  $u \mapsto l(u^*)$  and by  $u \mapsto l(u)^*$ . By assumption, we have  $l(\chi_x^{\sharp}(x))^* = l(\chi_x^{\sharp})(x) = \mu(x)$  for all  $x \in X$ . If  $\mu$  assumes only symmetric values, we also have  $l(\chi_x^{\sharp}(x))^* = \mu(x)^* = \mu(x)$  for all  $x \in X$ . Thus the above two homomorphism  $L_{(\infty)}^{\sharp}X \rightarrow E^{\text{op}}$  coincide by the uniqueness condition in 5.1; hence  $l$  preserves the involution. Conversely, if  $l$  preserves the involution, we have  $\mu(x)^* = l(\chi_X(x)^*) = l(\chi_X(x)) = \mu(x)$  for all  $x \in X$ , i.e., all values of  $\mu$  are symmetric.  $\square$

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