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UNDERPARAMETRIZED REGRESSION MODELS

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ABSTRACT. A description of many events and processes needs a large number of parameters. However, models describing these events and processes are difficult to deal with. Therefore, for practical purposes, it is sometimes necessary to neglect some of the parameters and to use underparametrized models. Some problems arising by this are studied.

Introduction

Mathematical description of many events and processes needs a large number of parameters. However, models with large number of parameters are difficult to deal with. Therefore, for practical purposes, a part of parameters is neglected in the model. What can be expected in connection with statistical inference in such underparametrized model?

Similar problems are studied also with a rather different approach in [2], [4]–[8] etc.

Since a class of problems connected with models with a large number of parameters is very rich, the problem of estimation is studied in the following text only.

1. Notation and auxiliary statements

$\mathbf{Y}\dots n$-dimensional random vector (observation vector),

$\mathcal{F} = \{F(\cdot,\boldsymbol{\beta},\boldsymbol{\gamma}) : \boldsymbol{\beta} \in \underline{\beta}, \boldsymbol{\gamma} \in \underline{\gamma}\}\dots$ class of distribution functions affiliated to $\mathbf{Y}$,

$\boldsymbol{\beta}\dots k$-dimensional unknown vector parameter which cannot be neglected,

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$\boldsymbol{\gamma}\dots s$-dimensional unknown vector parameter which is to be neglected,

$E(\mathbf{Y}) = \int \mathbf{u} dF(\mathbf{u}; \boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathbf{X}\boldsymbol{\beta} + \mathbf{S}\boldsymbol{\gamma}$,

$\mathbf{X}\dots n \times k$ given matrix,

$\mathbf{S}\dots n \times s$ given matrix,

$\boldsymbol{\Sigma} = \text{Var}(\mathbf{Y})\dots$ covariance matrix of the observation vector $\mathbf{Y}$,

$\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$,

$\underline{\beta}, \underline{\gamma}\dots$ linear manifolds which can be either the whole space R^k and R^s, respectively, or $\underline{\beta} = \{\boldsymbol{\beta} : \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}\}$, or

$$\{\underline{\beta, \gamma}\} = \left\{ \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix} : \mathbf{b} + \mathbf{B}\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} = \mathbf{0} \right\}, \quad \text{etc.}$$

$\mathbf{B}\dots q \times k$ given matrix,

$\mathbf{G}\dots q \times s$ given matrix,

$\mathbf{M}_X\dots$ projection matrix on the orthogonal complement (in the Euclidean norm) $\mathcal{M}^{\perp}(\mathbf{X})$ of the column space $\mathcal{M}(\mathbf{X}) = \{\mathbf{X}\mathbf{u} : \mathbf{u} \in R^k\}$,

$(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^+\dots$ the Moore-Penrose generalized inverse of the matrix $\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X$ (in more detail cf. [9]),

$\mathbf{K} = \mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{G}$,

$\mathbf{L} = \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{G}$.

The covariance matrix $\boldsymbol{\Sigma}$ of the observation vector $\mathbf{Y}$ is assumed to be fixed for the whole class $\mathcal{F}$. This situation is denoted as

$$(b) \qquad \mathbf{Y} \sim_n \left[(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma} \right], \boldsymbol{\beta} \in R^k, \boldsymbol{\gamma} \in R^s. \qquad (1)$$

The underparametrized model is denoted as

$$(a) \qquad \mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \boldsymbol{\beta} \in R^k. \qquad (2)$$

The best linear unbiased estimators of the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in the model (1) are denoted as $\widehat{\boldsymbol{\beta}^{(b)}}, \widehat{\boldsymbol{\gamma}^{(b)}}$ and the best linear unbiased estimator of the parameter $\boldsymbol{\beta}$ in the model (2) is denoted as $\widehat{\boldsymbol{\beta}^{(a)}}$.

Assumption. In the following sections it is assumed that the regularity of the models (1) and (2), i.e., $r(\mathbf{X}, \mathbf{S}) = k+s < n, r(\mathbf{X}) = k < n, \boldsymbol{\Sigma}$ is positive definite.

2. Models without constraints

In this section the models (1) and (2) are considered.

Lemma 2.1. *If* $\boldsymbol{\gamma} \in \mathcal{A}$*, where*

$$\mathcal{A} = \Big\{\boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\Big\{\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\Big\}^{-} \times \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} \le 1\Big\},$$

then

$$\forall\{\mathbf{h} \in R^k\}\ \Big\{\mathbf{h}'\Big[E_{(b)}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big) - \boldsymbol{\beta}\Big]\Big\}^2 + \mathrm{Var}\big(\mathbf{h}'\widehat{\boldsymbol{\beta}^{(a)}}\big) \le \mathrm{Var}\big(\mathbf{h}'\widehat{\boldsymbol{\beta}^{(b)}}\big).$$

P r o o f. It is valid

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}^{(b)}} \\ \widehat{\boldsymbol{\gamma}^{(b)}} \end{pmatrix} = \begin{pmatrix} \mathbf{C}, & \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}, & \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \end{pmatrix}.$$

Thus

$$\widehat{\boldsymbol{\beta}^{(b)}} = \widehat{\boldsymbol{\beta}^{(a)}} - \mathbf{C}^{-1}\mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\big(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}^{(a)}}\big),$$

where $\widehat{\boldsymbol{\beta}^{(a)}} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$. (With respect to the assumption on regularity, the matrix $\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}$ is regular.) Since $\widehat{\boldsymbol{\beta}^{(a)}}$ and $\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}^{(a)}}$ are uncorrelated,

$$\begin{aligned} \mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(b)}}\big) &= \mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big) + \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1} \\ &\quad \times(\boldsymbol{\Sigma} - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}')\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} \\ &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}. \end{aligned}$$

Let $\mathbf{b} = E_{(b)}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big) - \boldsymbol{\beta}$ and let $\boldsymbol{\gamma}$ satisfy the inequality (in the Loevner sense)

$$\mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big) + \mathbf{b}\mathbf{b}' \le_L \mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(b)}}\big)$$

which is equivalent to

$$\forall\{\mathbf{h} \in R^k\}\mathrm{Var}\big(\mathbf{h}'\widehat{\boldsymbol{\beta}^{(a)}}\big) + (\mathbf{h}'\mathbf{b})^2 \le \mathrm{Var}\big(\mathbf{h}'\widehat{\boldsymbol{\beta}^{(b)}}\big).$$

The last inequality is equivalent (with respect to the Scheffé theorem [10]) to

$$\mathbf{b}'\Big[\mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(b)}}\big) - \mathrm{Var}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big)\Big]^{-}\mathbf{b} \le 1.$$

Since $\mathbf{b} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}$,

$$\operatorname{Var}\big(\widehat{\boldsymbol{\beta}^{(b)}}\big) - \operatorname{Var}\big(\widehat{\boldsymbol{\beta}^{(a)}}\big) = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}$$

and $\mathbf{b} \in \mathcal{M}\Big(\operatorname{Var}(\widehat{\boldsymbol{\beta}^{(b)}}) - \operatorname{Var}(\widehat{\boldsymbol{\beta}^{(a)}})\Big)$, the last inequality is invariant to the choice of the generalized inverse and the Scheffé theorem can be utilized. □

3. Models with constraints I

A model with constraints I will be considered in two forms

$$(c_I) \qquad \mathbf{Y} \sim_n \left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \boldsymbol{\Sigma}\right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} = \mathbf{0}, \tag{3}$$

or

$$(b_I) \qquad \mathbf{Y} \sim_n \left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\boldsymbol{\beta}\\ \boldsymbol{\gamma}\end{pmatrix}, \boldsymbol{\Sigma}\right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}. \tag{4}$$

Here it is assumed $r(\mathbf{B},\mathbf{G}) = q < k+s, r(\mathbf{B}) = q < k$.

In both cases the underparametrized model is

$$(a_I) \qquad \mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}. \tag{5}$$

In the following text estimators in models with constraints will be denoted by $\widehat{\widehat{\ }}$.

3.1. The constraints $\mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}$ in the true model

Lemma 3.1. *In the model* (5) *the BLUE of the parameter* $\boldsymbol{\beta}$ *is*

$$\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}} = \widehat{\boldsymbol{\beta}} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\big(\mathbf{B}\widehat{\boldsymbol{\beta}} + \mathbf{b}\big),$$

where $\widehat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$. *Further*

$$\operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\Big) = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}.$$

P r o o f. Cf., e.g., in [3]. □

In the model (4) the mean value of the parameter $\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}$ is

$$E_{(b_I)}\Big(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\Big) = \boldsymbol{\beta} + (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}. \tag{6}$$

Lemma 3.2. *The covariance matrix of the BLUE of the parameter* $\boldsymbol{\beta}$ *in the model* (4) *is*

$$\begin{aligned}\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(b_I)}}}\right) &= (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+} \\ &\quad + (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{XM_{B'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{B'}})^{+}\mathbf{S}\big]^{-1} \\ &\quad \times \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}.\end{aligned}$$

P r o o f. The model (4) can be rewritten as follows.

$$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{K}_B\boldsymbol{\kappa}, \quad \mathcal{M}(\mathbf{K}_B) = \mathcal{K}er(\mathbf{B}) = \{\mathbf{u} : \mathbf{B}\mathbf{u} = \mathbf{0}\},$$

where $\mathbf{K}_B$ is $k \times (k-q)$ matrix with the full rank in columns and $\mathbf{b} + \mathbf{B}\boldsymbol{\beta}_0 = \mathbf{0}$.

Since $\mathbf{K}_B(\mathbf{K}_B'\mathbf{C}\mathbf{K}_B)^{-1}\mathbf{K}_B' = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}$ and $\mathbf{M}_{XK_B} = \mathbf{M}_{XM_{B'}}$, the expression for $\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(b_I)}}}\right)$ can be easily obtained. □

The relationship (6), Lemma 3.1 and Lemma 3.2 imply

$$E_{(b_I)}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) - \boldsymbol{\beta} \in \mathcal{M}\left[\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(b_I)}}}\right) - \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right)\right],$$

and thus analogously as in Lemma 2.1 the following theorem can be stated.

Theorem 3.3. *If* $\boldsymbol{\gamma} \in \mathcal{A}_{I,b_I}$, *where*

$$\begin{aligned}\mathcal{A}_{I,b_I} = \Big\{\boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+} \\ \times \Big\{(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{XM_{B'}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{B'}})^{+}\mathbf{S}\big]^{-1} \\ \times\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\Big\}^{-} \\ \times (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} \le 1\Big\},\end{aligned}$$

then in the model (4)

$$\forall\left\{\mathbf{h} \in R^k\right\}\left\{\mathbf{h}'\left[E_{(b_I)}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) - \boldsymbol{\beta}\right]\right\}^2 + \operatorname{Var}\left(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) \le \operatorname{Var}\left(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}^{(b_I)}}}\right).$$

3.2. The constraints $\mathbf{b} + \mathbf{B}\boldsymbol{\beta} + \mathbf{G}\boldsymbol{\gamma} = \mathbf{0}$ in the true model

In the model (3) the situation is a little more complicated. The mean value $E\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right)$ in the model (3) is

$$E_{(c_I)}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) = \boldsymbol{\beta} + \left[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{G})\right]\boldsymbol{\gamma} \quad (7)$$

Lemma 3.4. *Let* $\mathbf{K} = \mathbf{B}\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{G}$. *Then*

$$\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}} - \widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}} = \left[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]\boldsymbol{\xi}$$

and

$$\begin{aligned} \boldsymbol{\xi} = &-\left[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]^{-1} \\ &\times \left[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{v} + \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}(\mathbf{B}\widehat{\boldsymbol{\beta}} + \mathbf{b})\right], \\ \mathbf{v} = \mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}, &\quad \widehat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \quad \mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}. \end{aligned}$$

P r o o f. In the model (3) the BLUE of the parameter $\boldsymbol{\beta}$ is

$$\begin{aligned} \widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}} = (\mathbf{I}, \mathbf{0})\Bigg\{&\mathbf{D}^{-1}\begin{pmatrix} \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \end{pmatrix} - \mathbf{D}^{-1}\begin{pmatrix} \mathbf{B}' \\ \mathbf{G}' \end{pmatrix} \\ &\times \left[(\mathbf{B}, \mathbf{G})\mathbf{D}^{-1}\begin{pmatrix} \mathbf{B}' \\ \mathbf{G}' \end{pmatrix}\right]^{-1}\left[(\mathbf{B}, \mathbf{G})\mathbf{D}^{-1}\begin{pmatrix} \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \end{pmatrix} + \mathbf{b}\right]\Bigg\}, \\ \mathbf{D} = \begin{pmatrix} \mathbf{X}' \\ \mathbf{S}' \end{pmatrix}&\boldsymbol{\Sigma}^{-1}(\mathbf{X}, \mathbf{S}). \end{aligned}$$

After some simple, however tedious calculations, the proof can be straightforwardly finished. □

Lemma 3.5. *The covariance matrix of the vector* $\boldsymbol{\xi}$ *from Lemma 3.4 is*

$$\mathrm{Var}(\boldsymbol{\xi}) = \left[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]^{-1}.$$

P r o o f. It is a direct consequence of the definition of the vector $\boldsymbol{\xi}$. □

Lemma 3.6.

$$\begin{aligned}\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}}\right) = \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) &+ \left[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right] \\ &\times \left[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]^{-1} \\ &\times \left[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} - \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\right].\end{aligned}$$

P r o o f. The covariance matrix of the estimator $\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}}$ is

$$\begin{aligned}\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}}\right) = (\mathbf{I},\mathbf{0})\Bigg\{\mathbf{D}^{-1} - \mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}'\\ \mathbf{G}'\end{pmatrix}\left[(\mathbf{B},\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}'\\ \mathbf{G}'\end{pmatrix}\right]^{-1} \\ \times (\mathbf{B},\mathbf{G})\mathbf{D}^{-1}\Bigg\}\begin{pmatrix}\mathbf{I}\\ \mathbf{0}\end{pmatrix}.\end{aligned}$$

This expression can be rearranged into expression given in the statement. □

Since

$$\mathbf{b}_{I,(c_I)} = E_{(c_I)}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) - \boldsymbol{\beta} \in \mathcal{M}\left[\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}}\right) - \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right)\right]$$

(cf. (7)), the following theorem can be stated.

Theorem 3.7. *If in the model* (3) $\boldsymbol{\gamma} \in \mathcal{A}_{I,(c_I)}$, *where*

$$\begin{aligned}\mathcal{A}_{I,(c_I)} = \Bigg\{\boldsymbol{\gamma} : \boldsymbol{\gamma}'&\left[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} - \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\right] \\ &\times \Bigg\{\left[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right] \\ &\times \left[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]^{-1} \\ &\times \left[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} - \mathbf{K}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}^{-1}\right]\Bigg\}^{-} \\ &\times \left[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{K}\right]\boldsymbol{\gamma} \le 1\Bigg\},\end{aligned}$$

then

$$\forall\left\{\mathbf{h} \in R^k\right\}\left\{\mathbf{h}'\left[E_{(c_I)}\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right) - \boldsymbol{\beta}\right]\right\}^2 + \operatorname{Var}\left[\mathbf{h}'\left(\widehat{\widehat{\boldsymbol{\beta}^{(c_I)}}}\right)\right] \le \operatorname{Var}\left[\left(\widehat{\widehat{\boldsymbol{\beta}^{(a_I)}}}\right)\right].$$

P r o o f. Proof is analogous as in Lemma 2.1. □

4. Models with constraints II

A model with constraints II will be considered in two forms

$$(c_{II}) \qquad \mathbf{Y} \sim_n \left[(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma} \right], \quad \mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{G}\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}, \qquad (8)$$

$$(b_{II}) \qquad \mathbf{Y} \sim_n \left[(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma} \right], \quad \mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}. \qquad (9)$$

The regularity conditions are $r(\mathbf{B}_1, \mathbf{G}, \mathbf{B}_2) = q < k_1 + s + k_2$, $r(\mathbf{B}_2) = k_2 < q$, $r(\mathbf{B}_1, \mathbf{B}_2) = q < k_1 + k_2$.
The parameters $\boldsymbol{\beta}_1$ and $\boldsymbol{\gamma}$ occurring in the mean value of the observation vector $\mathbf{Y}$ can be directly measured. However, the parameter $\boldsymbol{\beta}_2$ occurs in the constraints only.

In both cases the underparametrized model is

$$(a_{II}) \qquad \mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \qquad \mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}. \qquad (10)$$

4.1. The constraints $\mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}$ in the true model

Lemma 4.1. *The BLUE of the parameter $\boldsymbol{\beta}_1$ in the model* (10) *is*

$$\begin{aligned} \widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}} &= \widehat{\boldsymbol{\beta}}_1 - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\right)^+\left(\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1 + \mathbf{b}\right), \\ \widehat{\boldsymbol{\beta}_1} &= \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \end{aligned}$$

and its mean value in the model (9) *is*

$$\begin{aligned} E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\right) &= \boldsymbol{\beta}_1 + \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\right)^+ \\ &\quad \times \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} \\ &= \boldsymbol{\beta}_1 + \left(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\right)^+\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}. \end{aligned} \qquad (11)$$

P r o o f. The expression for $\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}$ is known, e.g., in [3]. The expression for $E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\right)$ can be obtained directly. □

Lemma 4.2. *The covariance matrix of the BLUE $\widehat{\widehat{\boldsymbol{\beta}_1^{(b_{II})}}}$ of the parameter $\boldsymbol{\beta}_1$ in the model* (9) *is*

$$\begin{aligned}\operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(b_{II})}}}\Big) &= \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\Big) + \Big(\mathbf{M}_{B_1' M_{B_2}} \mathbf{C} \mathbf{M}_{B_1' M_{B_2}}\Big)^+ \mathbf{X}' \boldsymbol{\Sigma}^{-1} \mathbf{S} \\ &\quad \times \Big[\mathbf{S}' \Big(\mathbf{M}_{X M_{B_1' M_{B_2}}} \boldsymbol{\Sigma} \mathbf{M}_{X M_{B_1' M_{B_2}}}\Big)^+ \mathbf{S}\Big]^{-1} \\ &\quad \times \mathbf{S}' \boldsymbol{\Sigma}^{-1} \mathbf{X} \Big(\mathbf{M}_{B_1' M_{B_2}} \mathbf{C} \mathbf{M}_{B_1' M_{B_2}}\Big)^+ .\end{aligned}$$

P r o o f. Let $\mathcal{K}er(\mathbf{B}_1, \mathbf{B}_2) = \mathcal{M}\left(\begin{smallmatrix}\mathbf{K}_1\\ \mathbf{K}_2\end{smallmatrix}\right)$. With respect to $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_{1,0} + \mathbf{K}_1 \boldsymbol{\kappa}$, the model (9) can be rewritten as

$$\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{1,0} \sim_n \left[(\mathbf{X}\mathbf{K}_1, \mathbf{S}) \begin{pmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma}\right]$$

and thus

$$\begin{aligned}\operatorname{Var}\big(\widehat{\boldsymbol{\kappa}^{(b_{II})}}\big) &= (\mathbf{I}, \mathbf{0}) \begin{pmatrix} \mathbf{K}_1' \mathbf{C} \mathbf{K}_1, & \mathbf{K}_1' \mathbf{X}' \boldsymbol{\Sigma}^{-1} \mathbf{S} \\ \mathbf{S}' \boldsymbol{\Sigma}^{-1} \mathbf{X} \mathbf{K}_1, & \mathbf{S}' \boldsymbol{\Sigma}^{-1} \mathbf{S} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \\ &= \big(\mathbf{K}_1' \mathbf{C} \mathbf{K}_1\big)^{-1} + \big(\mathbf{K}_1' \mathbf{C} \mathbf{K}_1\big)^{-1} \mathbf{K}' \mathbf{X}' \boldsymbol{\Sigma}^{-1} \mathbf{S} \\ &\quad \times \Big[\mathbf{S}' \big(\mathbf{M}_{X K_1} \boldsymbol{\Sigma} \mathbf{M}_{X K_1}\big)^+ \mathbf{S}\Big]^{-1} \\ &\quad \times \mathbf{S}' \boldsymbol{\Sigma}^{-1} \mathbf{X} \mathbf{K}_1 \big(\mathbf{K}_1' \mathbf{C} \mathbf{K}_1\big)^{-1}.\end{aligned}$$

Since $\mathcal{M}(\mathbf{K}_1) = \mathcal{M}\big(\mathbf{M}_{B_1' M_{B_2}}\big)$, the proof can be straightforwardly finished. $\square$

Since (cf. (11))

$$E_{(b_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\Big) - \boldsymbol{\beta}_1 \in \mathcal{M}\left[\operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(b_{II})}}}\Big) - \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\Big)\right],$$

the following theorem can be stated.

Theorem 4.3. *Let*

$$\begin{aligned}\mathcal{A}_{II,\beta_1,(b_{II})} = \Big\{ \boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\right)^{+} \\ \times \Big\{ \left(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\right)^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} \\ \times \left[\mathbf{S}'\left(\mathbf{M}_{XM_{B_1'M_{B_2}}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{B_1'M_{B_2}}}\right)^{+}\mathbf{S}\right]^{-1} \\ \times \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\right)^{+}\Big\}^{+} \\ \times \left(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\right)^{+}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} \le 1 \Big\}.\end{aligned}$$

Then

$$\begin{aligned}\boldsymbol{\gamma} \in \mathcal{A}_{II,\beta_1,(b_{II})} \Rightarrow \forall\{\mathbf{h}\in R^{k_1}\}\left\{\mathbf{h}'\left[E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\right) - \boldsymbol{\beta}_1\right]\right\}^2 \\ + \mathrm{Var}\left(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\right) \le \mathrm{Var}\left(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}_1^{(b_{II})}}}\right).\end{aligned}$$

Lemma 4.4. *In the model* (10) *the BLUE of the parameter* $\boldsymbol{\beta}_2$ *is*

$$\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}} = -\left[(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}\right]'\left(\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1 + \mathbf{b}\right),$$

where $\widehat{\boldsymbol{\beta}}_1 = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$, $\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$. *The mean value of* $\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}$ *in the model* (9) *is*

$$E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right) = \boldsymbol{\beta}_2 - \left[(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}\right]'\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}.$$

Proof. The expression for $\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}$ is known, cf., e.g., [3]. The expression for $E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right)$ is thus obvious. □

Lemma 4.5. *In the model (9) the covariance matrix of the BLUE of $\boldsymbol{\beta}_2$ is*

$$\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}}\right) = \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right) + \left[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\right]' \mathbf{B}_1 \left(\mathbf{M}_{B_1'M_{B_2}} \mathbf{C} \mathbf{M}_{B_1'M_{B_2}}\right)^+$$
$$\times\ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\left[\mathbf{S}'\left(\mathbf{M}_{XM_{B_1'M_{B_2}}} \boldsymbol{\Sigma} \mathbf{M}_{XM_{B_1'M_{B_2}}}\right)^+ \mathbf{S}\right]^{-1}$$
$$\times\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\left(\mathbf{M}_{B_1'M_{B_2}} \mathbf{C} \mathbf{M}_{B_1'M_{B_2}}\right)^+ \mathbf{B}_1'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}.$$

P r o o f. The model (9) can be rewritten as

$$\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{1,0} \sim_n \left[(\mathbf{X}\mathbf{K}_1, \mathbf{S})\begin{pmatrix}\boldsymbol{\kappa}\\ \gamma\end{pmatrix}, \boldsymbol{\Sigma}\right],$$
$$\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{\beta}_{1,0}\\ \boldsymbol{\beta}_{2,0}\end{pmatrix} + \begin{pmatrix}\mathbf{K}_1\\ \mathbf{K}_2\end{pmatrix}\boldsymbol{\kappa},$$
$$\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}} = \boldsymbol{\beta}_{2,0} + \mathbf{K}_2\widehat{\boldsymbol{\kappa}^{(b_{II})}}.$$

Thus $\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}}\right) = \mathbf{K}_2\operatorname{Var}(\widehat{\boldsymbol{\kappa}^{(b_{II})}})\mathbf{K}_2'$ implies the statement. □

Lemma 4.6.

$$E_{(b_{II})}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right) - \boldsymbol{\beta}_2 \in \mathcal{M}\left[\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}}\right) - \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right)\right].$$

P r o o f. It is valid

$$\left[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\right]' \mathbf{B}_1 \left(\mathbf{M}_{B_1'M_{B_2}} \mathbf{C} \mathbf{M}_{B_1'M_{B_2}}\right)^+ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}$$
$$= \left[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\right]^{-1}\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_1$$
$$\times \left[\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}_1\left(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\right)^+\mathbf{B}_1\mathbf{C}^{-1}\right]\mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{S}$$
$$= \left[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\right]'\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} \subset \mathcal{M}\left[\operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}}\right) - \operatorname{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\right)\right].$$

□

Thus the following theorem can be stated.

Theorem 4.7. *Let*

$$\begin{aligned}\mathcal{A}_{II,\beta_2,(b_{II})} = \Bigg\{ &\boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')} \\ &\times \Bigg\{ \Big[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big]' \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} \\ &\times \Big[\mathbf{S}'\Big(\mathbf{M}_{XM_{B_1'M_{B_2}}}\boldsymbol{\Sigma}\mathbf{M}_{XM_{B_1'M_{B_2}}}\Big)^+\mathbf{S}\Big]^{-1} \\ &\times \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Bigg\}^- \\ &\times \Big[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big]' \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} \le 1 \Bigg\}.\end{aligned}$$

Then

$$\begin{aligned}\boldsymbol{\gamma} \in \mathcal{A}_{II,\beta_2,(b_{II})} \;\Rightarrow\; \forall\{\mathbf{h}\in R^{k_2}\} &\Bigg\{\mathbf{h}'\Big[E_{(b_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big) - \boldsymbol{\beta}_2\Big]\Bigg\}^2 \\ &+ \mathrm{Var}\Big(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big) \le \mathrm{Var}\Big(\mathbf{h}'\widehat{\widehat{\boldsymbol{\beta}_2^{(b_{II})}}}\Big).\end{aligned}$$

4.2. The constraints $\mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{G}\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}$ in the true model

Let the adequate model be (8) and let the underparametrized model be (10).

Corollary 4.8. *The mean value of the BLUE of $\boldsymbol{\beta}_1$ in the model* (10) *is (cf. Lemma 4.1)*

$$E_{(c_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\Big) = \boldsymbol{\beta}_1 + \Big(\mathbf{M}_{B_1'M_{B_2}}\mathbf{C}\mathbf{M}_{B_1'M_{B_2}}\Big)^+ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}$$

in the model (8)*.*

Lemma 4.9. *The BLUE of $\boldsymbol{\beta}_1$ in* (8) *is*

$$\widehat{\widehat{\boldsymbol{\beta}_1^{(c_{II})}}} = \widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}} + \Big[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\right)^+\mathbf{L}\Big]\boldsymbol{\eta},$$

where

$$\mathbf{L} = \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{G},$$

$$\boldsymbol{\eta} = -\Big\{\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{L}'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2})^{+}\mathbf{L}\Big\}^{-1}$$
$$\times \Big[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{v} + \mathbf{L}'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2})^{+}(\mathbf{B}_1\widehat{\boldsymbol{\beta}_1} + \mathbf{b})\Big],$$

$$\widehat{\boldsymbol{\beta}}_1 = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \mathbf{v} = \mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}_1, \mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}.$$

Proof. Let $\mathbf{L}'_{B_2}$ be $(q-k_2)\times q$ matrix with the full rank in rows, $\mathbf{L}'_{B_2}\mathbf{B}_2 = \mathbf{0}, \mathbf{L}'_{B_2}\mathbf{L}_{B_2} = \mathbf{I}_{(q-k_2),(q-k_2)}$, i.e., $\mathbf{M}_{B_2} = \mathbf{L}_{B_2}\mathbf{L}'_{B_2}$. The constraints $\mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{G}\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}$ in (8) as far as the parameter $\boldsymbol{\beta}_1$ is concerned are equivalent to constraints

$$\mathbf{L}'_{B_2}\mathbf{b} + \mathbf{L}'_{B_2}\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{L}'_{B_2}\mathbf{G}\boldsymbol{\gamma} = \mathbf{0}.$$

Thus the model (8) can be rewritten as

$$\mathbf{Y} \sim {}_n\left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\gamma}\end{pmatrix}, \boldsymbol{\Sigma}\right], \mathbf{L}'_{B_2}\mathbf{b} + \mathbf{L}'_{B_2}\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{L}'_{B_2}\mathbf{G}\boldsymbol{\gamma} = \mathbf{0}. \tag{12}$$

The model (12) can be compared with the model (3) and thus regarding Lemma 3.4 and the identity

$$\mathbf{L}_{B_2}\big(\mathbf{L}'_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{L}_{B_2}\big)^{-1}\mathbf{L}'_{B_2} = \big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+},$$

the proof can be finished. □

Now the following lemma is a consequence of Lemma 3.5.

Lemma 4.10. *The covariance matrix of the vector* $\boldsymbol{\eta}$ *from Lemma 4.9 is*

$$\mathrm{Var}(\boldsymbol{\eta}) = \Big[\mathbf{S}'\big(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X\big)^{+}\mathbf{S} + \mathbf{L}'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{L}\Big]^{-1}.$$

Proof. Proof can be performed analogously as in Lemma 4.9. □

Lemma 4.11.

$$\mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(c_{II})}}}\Big) - \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{(a_{II})}}}\Big)$$
$$= \Big[\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{L}\Big]$$
$$\times \Big[\mathbf{S}'\big(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X\big)^{+}\mathbf{S} + \mathbf{L}'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{-1}\mathbf{L}\Big]^{-1}$$
$$\times \Big[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\mathbf{C}^{-1} - \mathbf{L}'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{B}_1\mathbf{C}^{-1}\Big].$$

P r o o f. The statement is a direct consequence of Lemma 4.9. However, to find the resulting formula is rather lengthy and therefore it is omitted. □

Since

$$E_{(c_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(a_{II})}\Big) - \boldsymbol{\beta}_1 \in \mathcal{M}\Big[\operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(c_{II})}\Big) - \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(a_{II})}\Big)\Big],$$

the following theorem is valid.

Theorem 4.12. *Let*

$$\begin{aligned}\mathbf{U} &= \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{L},\\ \mathbf{V} &= \mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{L}'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{L}\\ &= \big[\operatorname{Var}(\boldsymbol{\eta})\big]^{-1}.\end{aligned}$$

If in the model (8) $\boldsymbol{\gamma} \in \mathcal{A}_{II,\beta_1,(c_{II})}$, where

$$\mathcal{A}_{II,\beta_1,(c_{II})} = \Big\{\boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{U}'(\mathbf{U}\mathbf{V}^{-1}\mathbf{U}')^{-}\mathbf{U}\boldsymbol{\gamma} \le 1\Big\},$$

then

$$\forall\big\{\mathbf{h} \in R^{k_1}\big\}\ \Big\{\mathbf{h}'\Big[E_{(c_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(a)}\Big) - \boldsymbol{\beta}_1\Big]\Big\}^2 \le \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(c_{II})}\Big) - \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_1^{(a_{II})}\Big).$$

The mean value of the estimator

$$\widehat{\widehat{\boldsymbol{\beta}}}_2^{(a_{II})} = -\Big[(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}\Big]'\big(\mathbf{B}_1\widehat{\boldsymbol{\beta}}_1 + \mathbf{b}\big)$$

(cf. Lemma 4.4) is

$$E_{(c_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_2^{(a_{II})}\Big) = \boldsymbol{\beta}_2 - \Big[(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}\Big]'\mathbf{L}\boldsymbol{\gamma}$$

in the model (8).

Lemma 4.13. *The covariance matrix of* $\widehat{\widehat{\boldsymbol{\beta}}}_2^{(c_{II})}$ *is*

$$\begin{aligned}\operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_2^{(c_{II})}\Big) &= \operatorname{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}}}_2^{(a_{II})}\Big) + \Big[(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}\Big]'\\ &\quad\times \mathbf{L}\Big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{S} + \mathbf{L}'\big(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2}\big)^{+}\mathbf{L}\Big]^{-1}\\ &\quad\times \mathbf{L}'(\mathbf{B}_2')^{-}_{m(B_1C^{-1}B_1')}.\end{aligned}$$

P r o o f. With respect to Lemma 4.4 the covariance matrix of $\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}$ is

$$\mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big) = \Big[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\Big]^{-1} - \mathbf{I}$$

Thus in the model (8)

$$\begin{aligned}
&\mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(c_{II})}}}\Big)\\
&= \left\{\mathbf{B}_2'\left[(\mathbf{B}_1, \mathbf{G})\begin{pmatrix} \mathbf{C}, \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S} & \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{X}, & \mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{B}_1' \\ \mathbf{G}\end{pmatrix} + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2\right\}^{-1} - \mathbf{I}\\
&= \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big) + \Big[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big]'\\
&\quad \times \mathbf{L}\Big\{\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^+\mathbf{S} + \mathbf{L}'(\mathbf{M}_2\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2})^+\mathbf{L}\Big\}^{-1}\mathbf{L}'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}.
\end{aligned}$$

□

Since

$$\begin{aligned}
E_{(c_{II})}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big) - \boldsymbol{\beta}_2 &\in \mathcal{M}\left\{\big[(\mathbf{B}_2')_{m(B_1C^{-1}B_1')}\big]'\mathbf{L}\right\}\\
&= \mathcal{M}\left[\mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(c_{II})}}}\Big) - \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}}\Big)\right],
\end{aligned}$$

the following statement is valid.

Theorem 4.14. *Let*

$$\begin{aligned}
\mathcal{A}^*_{II,\beta_2,(c_{II})} = \Bigg\{\boldsymbol{\gamma} : \boldsymbol{\gamma}'\mathbf{L}'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big\{\Big[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big]'\\
\times \mathbf{L}\Big[\mathbf{S}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^+\mathbf{S}\\
+ \mathbf{L}'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{B_2})^+\mathbf{L}\Big]^{-1}\mathbf{L}'(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big\}^-\\
\times \Big[(\mathbf{B}_2')^-_{m(B_1C^{-1}B_1')}\Big]'\mathbf{L}\boldsymbol{\gamma} \le 1\Bigg\}.
\end{aligned}$$

Then

$$\gamma \in \mathcal{A}^*_{II,\beta_2,(c_{II})} \;\Rightarrow\; \forall \left\{\mathbf{h} \in R^{k_2}\right\} \left\{ \mathbf{h}' \left[E_{(c_{II})} \left(\widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}} \right) - \boldsymbol{\beta}_2 \right] \right\}^2 + \operatorname{Var}\left(\mathbf{h}' \widehat{\widehat{\boldsymbol{\beta}_2^{(a_{II})}}} \right) \leq \operatorname{Var}\left(\mathbf{h}' \widehat{\widehat{\boldsymbol{\beta}_2^{(c_{II})}}} \right).$$

REFERENCES

[1] DRYGAS, H.: *Weak and strong consistency of the least squares estimators in regression models*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **34** (1975), 119–127.

[2] GROSS, J.—PUNTANEN, S.: *Estimation under a general partitioned linear model*, Linear Algebra Appl. **321** (2000), 131–144.

[3] KUBÁČEK, L., KUBÁČKOVÁ, L., VOLAUFOVÁ, J.: *Statistical Models with Linear Structures*. Veda, Bratislava, 1995.

[4] KUNDEROVÁ, P.: *Regular linear model with the nuisance parameters with constraints of the type I*, Acta Univ. Palacki Olomuc. Fac. Rerum Natur. Math. **40** (2001), 151–159.

[5] KUNDEROVÁ, P.: *Locally best and uniformly best estimators in linear model with nuisance parameters*, Tatra Mt. Math. Publ. **22** (2001), 27–36.

[6] KUNDEROVÁ, P.: *On certain classes of parametric functions in linear models with constraints*, Folia Fac. Sci. Natur. Univ. Masaryk. Brun. Math. (to appear).

[7] NORDSTRŐM, K.—FELLMAN, J.: *Characterizations and dispersion-matrix robustness of efficiently estimable parametric functionals in linear models with nuisance parameters*, Linear Algebra Appl. **127** (1990), 341–361.

[8] NURHONEN, M.—PUNTANEN, S.: *A property of partitioned generalized regression*, Comm. Statist. Theory Methods **21** (1992), 1579–1583.

[9] RAO, C. R.—MITRA, S. K.: *Generalized Inverse of Matrices and Its Applications*. John Wiley & Sons, New York, 1971.

[10] SCHEFFÉ, H.: *The Analysis of Variance*, (5th ed.), John Wiley & Sons, New York, 1967.

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