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TESTING PROCEDURES BASED ON THE EMPIRICAL CHARACTERISTIC FUNCTIONS I: GOODNESS-OF-FIT, TESTING FOR SYMMETRY AND INDEPENDENCE

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ABSTRACT. We review the most recent developments in the area of testing by the empirical characteristic function. Our main focus is on goodness-of-fit tests based on i.i.d. observations, but we also refer to testing for symmetry and testing for independence.

1. Introduction

Let X be a random variable with unspecified distribution function (DF) $F(x) = \mathrm{P}(X \le x)$, and consider a specific parametric class $\mathrm{F}_\Theta = \{F_\vartheta, \vartheta \in \Theta\}$, of distributions indexed by $\vartheta \in \Theta$, where Θ is an open subset of arbitrary dimension. Because the Glivenko-Cantelli theorem asserts that $\sup_x |F_n(x) - F(x)| \to 0$ almost surely, in classical consistent tests for the null hypothesis,

$$H_0 : F \in \mathrm{F}_\Theta, \qquad \text{for some} \quad \vartheta \in \Theta, \tag{1.1}$$

the empirical DF, $F_n(x)$ is employed. An alternative way to tackle the composite goodness-of-fit problem is to consider a transform of the DF of the type

$$K(z) = \int_{-\infty}^{\infty} k(z, x) \mathrm{d}F(x), \qquad z = s + it$$

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and the corresponding empirical version,

$$K_n(z) = \int_{-\infty}^{\infty} k(z,x)\mathrm{d}F_n(x),$$

computed from the observations. Hence, assuming that the transform $K(\cdot)$ uniquely determines the law of X, such an approach typically yields consistent tests for the null hypothesis in (1.1). Although in principle the kernel $k(\cdot, x)$ may be any arbitrary function defined over the entire complex plane, certain restrictions on $k(\cdot, x)$ and z, have gained prominent position in Statistics. For instance, if we let $k(z,x) = \exp(zx)$, then the characteristic function results for purely imaginary z. With the same exponential kernel, $t = 0, s \in R^1$ corresponds to the moment generating function, while for $z = (s, 0), s < 0$, the Laplace transform emerges. For z real, the kernel $k(z,x) = z^x, 0 < z < 1$, is called the probability generating function and is typically employed with discrete measures, and we finally mention the Mellin transform with $k(z,x) = x^z, z > 0$.

Although all these transforms (along with their empirical counterparts), possibly lead to efficient statistical procedures, and in fact have appeared in the literature, certain properties make the CF the most appropriate transform to work with. Amongst these properties is of course the one-to-one correspondence between CF's and DF's, but equally important is the fact that the CF exists (it is finite), for all $t \in R$. Nevertheless, some unexpected phenomena occurs in the richness of complex function theory. For instance, and unlike other transforms, uniqueness holds only if two CF's coincide over the entire imaginary axis. For counterexamples of two different DF's with corresponding CF's being identical over compact intervals the reader is referred to Ushakov [27].

In this paper we review methods for testing certain hypotheses that make use of the empirical CF. In Section 2, goodness-of-fit procedures with estimated parameters are surveyed and a general framework is suggested under which asymptotic properties of the resulting test statistics may be studied. In Section 3, special emphasis is attached to inference procedures for the properties of symmetry and independence that make use of the specific way that these properties are reflected upon the population CF.

2. Goodness-of-fit tests

Let $X_1, X_2, \ldots, X_n$, be independent observations on the random variable X, on the basis of which we wish to test the null hypothesis in (1.1). Assume further that $\vartheta = (\theta_1, \theta_2) \in (-\infty, \infty) \times (0, \infty)$ and the DF of X may be written as $F(x; \vartheta) = G\big((x - \theta_1)/\theta_2\big)$, for some function G. A general test statistic may be constructed by considering the analog in the frequency domain, of integrated empirical DF-based methods. In this approach the quantity of interest is $D_n(t) = |\varphi_n(t) - \varphi(t)|$, where $\varphi(t) = \varphi(t; \theta_1, \theta_2)$ denotes the CF under H_0, and $\varphi_n(t) = n^{-1} \sum_{j=1}^{n} e^{itX_j}$ is the empirical CF. As pointed out earlier, considering the behavior $D_n(\cdot)$ over the entire real line—say via an integrated distance based on $D_n(\cdot)$—renders the test statistic consistent against all possible deviations from H_0, whereas additional assumptions are needed to yield a consistent test if one considers $D_n(\cdot)$ only over finite intervals of t-values. On the other hand, considering the CF and the empirical CF over the entire real line introduces some extra technical difficulties. For example, periodic components often dominate the behavior of $\varphi(t)$, as well as that of $\varphi_n(t)$, for large values of t. Hence, some sort of damping in $D_n(\cdot)$ is necessary, in order to make the integrals involved convergent. Also, as our method is aiming at the widest applicability possible, the parameter $\vartheta = (\theta_1, \theta_2)$ is treated as nuisance, to be consistently estimated by $(\hat{\theta}_1, \hat{\theta}_2)$ from $X_j,\ j = 1, 2 \dots, n$. In particular, we propose to employ not $D_n(\cdot)$, but an estimated version of the distance, namely $\hat{D}_n(t) = |\hat{\varphi}_n(t) - \varphi_0(t)|$, where $\varphi_0(t) = \varphi(t; 0, 1)$, and $\hat{\varphi}_n(t) = n^{-1} \sum_{j=1}^{n} e^{itY_j}$ is the empirical CF of the standardized observations $Y_j = (X_j - \hat{\theta}_1)/\hat{\theta}_2, j = 1, 2 \dots, n$. Moreover, since we are considering location-scale families we wish our test statistic, say $T = T(X_1, X_2, \dots, X_n)$, to satisfy $T(cX_1 + \delta, cX_2 + d, \dots, cX_n + \delta) = T(X_1, X_2, \dots, X_n)$, for each $\delta \in R^1$, and $c > 0$. To this end we require that our estimators $\hat{\theta}_m := \hat{\theta}_m(X_1, X_2, \dots, X_n),\ m = 1, 2$, satisfy certain equivariance-invariance properties. Namely, we assume that $\hat{\theta}_1(cX_1 + \delta, cX_2 + d, \dots, cX_n + \delta) = c\hat{\theta}_1(X_1, X_2, \dots, X_n) + \delta$, and that $\hat{\theta}_2(cX_1 + \delta, cX_2 + d, \dots, cX_n + \delta) = c\hat{\theta}_2(X_1, X_2, \dots, X_n)$. Then, any test statistic which depends on $X_j,\ j = 1, 2, \dots, n$, solely via $Y_j,\ j = 1, 2, \dots, n$, satisfies $T(Y_1, Y_2, \dots, Y_n) = T(X_1, X_2, \dots, X_n)$, i.e., it is location-scale invariant.

With these considerations in mind we suggest to reject the null hypothesis H_0 in (1.1) for large values of

$$T_{n,\beta} = n \int_{-\infty}^{\infty} \hat{D}_n^2(t) \beta(t) \, \mathrm{d}t, \tag{2.1}$$

where β denotes a damping measure (often termed weight function). To capture almost complete analogy with the Anderson-Darling statistic, Epps [3] proposed to employ $\beta(t) = |\varphi_0(t)|^2 / \int |\varphi_0(u)|^2 \mathrm{d}u$, but in what follows, we reserve the liberty of assigning to the damping function any non-negative integrable function satisfying $\beta(t) = \beta(-t)$. This approach was followed by Epps and Pulley [4], Gürtler and Henze [8], and Meintanis [20], [21], in testing for the normal, the Cauchy, the Laplace and the logistic distribution, respectively.

The following theorem, in which the asymptotic behavior of the test statistic is investigated, holds under mild restrictions on $F(\cdot)$, the estimators of the parameters and the damping measure. A convenient setting for asymptotic distribution theory is the Hilbert space $\mathcal{H} = L^2(R^1, \mathcal{B}, \beta)$ of (equivalence classes of) measurable functions $f\colon R^1 \to R^1$ satisfying $\int_{-\infty}^{\infty} f^2(t)\beta(t)\,\mathrm{d}t < \infty$ with inner product and norm in $\mathcal{H}$ defined by

$$< f, g >= \int_{-\infty}^{\infty} f(t)g(t)\beta(t)\,\mathrm{d}t \quad \text{and} \quad ||f|| = \left(\int_{-\infty}^{\infty} f^2(t)\beta(t)\,\mathrm{d}t \right)^{1/2},$$

respectively. In what follows, $\to^D$ denotes convergence in distribution, $\to^P$ denotes convergence in probability, while $o_P(1)$ stands for convergence in probability to zero.

Theorem 2.1. *Let $X_1, \dots, X_n, \dots$, be i.i.d. observations with DF, $F(\cdot)$. Assume that the weight function is non-negative and satisfies*

$$\beta(t) = \beta(-t), \quad t \in R^1, \qquad 0 < \int_{-\infty}^{\infty} \beta(t)\,\mathrm{d}t < \infty.$$

Assume further that the estimator of θ_m, $m = 1, 2$, admits a first order asymptotic representation (see Jurečková *and* Sen [16]*) of the type*

$$\sqrt{n}(\hat{\theta}_m - \theta_{m0}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \psi_m(X_j) + o_P(1),$$

where $\theta_{10} = 0, \theta_{20} = 1$, and $\mathrm{E}\big[\psi_m(X_1)\big] = 0$, $\mathrm{E}\big[\psi_m^2(X_1)\big] < \infty$, $m = 1, 2$.

Then the test statistic in (2.1) *admits the following representation*

$$T_{n,\beta} = \int_{-\infty}^{\infty} Z_n^2(t)\beta(\hat{\theta}_2 t)\,\mathrm{d}t,$$

where $Z_n(t) = (\hat{\theta}_2/n)^{1/2} \sum_{j=1}^{n} \big\{\cos tX_j + \sin tX_j - \varphi(t; \hat{\theta}_1, \hat{\theta}_2)\big\}$.

Moreover, under the null hypothesis H_0, there is a zero-mean Gaussian process $Z = \big\{Z(t);\ t \in R^1\big\}$ such that $Z_n \to^D Z$ and,

$$T_{n,\beta} \to^D \int_{-\infty}^{\infty} Z^2(t)\beta(t)\,\mathrm{d}t := T_\beta,$$

while under fixed alternatives we have $T_{n,\beta} \to^P \infty$.

Remark 1. The proof of a more extended version of Theorem 2.1 is given in Meintanis and Swanepoel [24].

Remark 2. The covariance kernel, say $\omega(s,t)$, of Z depends on the distribution being tested as well as on the type of estimators employed, but not on the true values of the parameters θ_1 and θ_2. Although the distribution of T_β is complicated, its moments may in principle be calculated via $\omega(s,t)$. For example,

$$\mathrm{E}(T_\beta) = \int_{-\infty}^{\infty} \omega(t,t)\beta(t)\,\mathrm{d}t,$$

and

$$\mathrm{Var}(T_\beta) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2(s,t)\beta(s)\beta(t)\,\mathrm{d}s\,\mathrm{d}t,$$

are the limit expectation and limit variance of the test statistic, respectively.

Remark 3. The law of T_β coincides with that of $\sum_{j\geq 1}\lambda_j N_j^2$, where $N_1, N_2, \ldots,$ are i.i.d standard normal variates, and $\lambda_j,\ j=1,2,\ldots,$ are the nonzero eigenvalues of the integral equation $\int \omega(s,t)\Lambda(t)\beta(t)\,\mathrm{d}t = \lambda\Lambda(s)$. The typical eigenvalue may be represented as $\lambda := \lambda_\beta(F,\vartheta)$, i.e., it depends both on the damping measure $\beta(\cdot)$, as well as on the family being tested, and the value of the parameter ϑ. In pure location-scale families however we have $\lambda := \lambda_\beta^{(0)}(F)$, where $\lambda_\beta^{(0)}(F)$, denotes the eigenvalue computed at $\vartheta = (0,1)$, with $\vartheta = (\theta_1,\theta_2)$. Matsui and Takemura [19], attack the computational sophisticated problem of approximating the limit null distribution of $T_{n,\beta}$, by computing the eigenvalues. An alternative approach, which in fact has been followed by the majority of researchers working in this area (see also Wong and Sim [28]), is to perform the test by Monte Carlo approximation of the critical points of $T_{n,\beta}$.

Remark 4. Although it is the most natural and straightforward to employ $\hat{D}_n(t) = |\hat{\varphi}_n(t) - \varphi_0(t)|$ in (2.1), there exist other variations that make use of the specific structure of the CF under the null hypothesis. In testing for symmetric stability for instance, and faced with certain computational difficulties in applying the above "direct approach", Meintanis [22], motivated by a differential equation satisfied by the CF of the standard symmetric stable distribution, employs in (2.1), instead of $\hat{D}_n(t)$, $\tilde{D}_n(t) = |\hat{\varphi}_n'(t) + \hat{\alpha}|t|^{\hat{\alpha}-1}\mathrm{sign}(t)\hat{\varphi}_n(t)|$. Here, and apart from the standard location-scale parameter, there exists an extra shape parameter (often called "characteristic exponent") denoted by α, $1 < \alpha \leq 2$, and estimated by $\hat{\alpha}$. Other variants are those of Henze and Meintanis [11], [12], utilizing certain characterizations of the exponential distribution based on the CF. Namely, under unit exponentiality the equations $\mathrm{Im}\varphi_0(t) = t\mathrm{Re}\varphi_0(t)$, and $|\varphi_0(t)|^2 = \mathrm{Re}\varphi_0(t)$, hold true for each $t \in R^1$. The natural approach is then to employ in (2), instead of $\hat{D}_n(t)$, in [11], $\tilde{D}_n(t) = |\mathrm{Im}\hat{\varphi}_n(t) - t\mathrm{Re}\hat{\varphi}_n(t)|$, and in [12], $\tilde{D}_n(t) = ||\hat{\varphi}_n(t)|^2 - \mathrm{Re}\hat{\varphi}_n(t)|$. However it should be noted that in both cases θ_1 is considered known (and therefore without loss of generality it is set equal to zero), i.e., tests for the one-parameter and not the two-parameter exponential distribution are constructed. Asymptotic results similar to those obtained in Theorem 2.1, hold also for modifications of the test statistic, such as those in Henze and Meintanis [12] and Meintanis [22]. However assumptions and techniques of proof should be accordingly modified. The approach described herein could be extended to cover the case of testing goodness-of-fit in the regression context. Naturally, then there exist additional nuisance parameters, the regression parameters. Hence, this extension requires extra assumptions on the regression estimators (see Hušková and Meintanis [15]).

Several multivariate goodness-of-fit methods may easily be included in the general framework of this section, by considering the empirical CF, $\varphi_n(\mathbf{t}) = n^{-1}\sum_{j=1}^{n} e^{i\mathbf{t}'\mathbf{X}_j}$, where $\mathbf{t}$ and $\mathbf{X}_j$ denote vectors of arbitrary dimension d. Such are the tests for multivariate normality of Naito [25], Henze and Zinkler [14], and Henze and Wagner [13]. Naito [25] employs the function $D_n(\mathbf{t}) = |\hat{\varphi}_n(\mathbf{t})|^2 - e^{-||\mathbf{t}||^2/2}$, which is of course characteristic of the standard multivariate normal law, in the test statistic $\sqrt{n}\int D_n(\mathbf{t})\beta(\mathbf{t})d\mathbf{t}$. The empirical CF $\hat{\varphi}_n(\mathbf{t})$ is calculated from the standardized observations $\mathbf{Y}_j = \mathbf{S}^{-1/2}\mathbf{X}_j,\ j = 1, 2, \ldots, n$, where $\mathbf{S}$ denotes the sample covariance matrix of $\mathbf{X}_j,\ j = 1, 2, \ldots, n$. The test statistic has a normal limit null distribution, which is of course an advantage over the methods presented earlier. On the other hand, and although Naito [25] provides formulae for the test statistic with $d = 1$ and $d = 2$, these formulas do not seem to lend themselves to extension in arbitrary dimension easily. This shortcoming is remedied by the test of Henze and Wagner [13] (see also Henze and Zinkler [14]), in which the affine invariant test statistic $n\int D_n^2(\mathbf{t})\beta(\mathbf{t})d\mathbf{t}$, where $D_n(\mathbf{t}) = |\hat{\varphi}_n(\mathbf{t}) - e^{-||\mathbf{t}||^2/2}|$, but with the empirical CF computed from the fully standardized observations $\mathbf{Y}_j = \mathbf{S}^{-1/2}(\mathbf{X}_j - \bar{\mathbf{X}}),\ j = 1, 2, \ldots, n$, where $\bar{\mathbf{X}}$ denotes the sample mean of $\mathbf{X}_j,\ j = 1, 2, \ldots, n$. Despite the fact that squaring the distance function brings in asymptotics as those encountered earlier in Remark 2, this test admits a convenient closed form expression in arbitrary dimension for specific choices of the damping measure $\beta(\mathbf{t})$.

In order to dispense with complicated asymptotics, Fan [5] and Koutrouvelis and Meintanis [18] suggested to (essentially) assign a discrete measure to $\beta(t)$, and thereby obtain a chi-squared limit law for the test statistic. This method is appropriate for the general multivariate goodness-of-fit problem (and not just for multivariate normality), with a damping scheme related to the limit covariance matrix of $D_n(\cdot)$. The drawback is the loss of consistency against general alternatives, but the finite-sample documentation in [18] is encouraging.

3. Other applications of the empirical CF

In this section procedures are reviewed for multivariate symmetry and independence based on the multivariate CF $\varphi(\mathbf{t};\mathbf{X}) = \mathrm{E}(\mathbf{t}'\mathbf{X})$. It should be mentioned beforehand that all these procedures are, unless otherwise stated, consistent, and that finite sample results indicate their competitiveness with respect to more established methods.

3.1. Tests for symmetry

The point of departure in this case is that $\mathrm{Im}\varphi(\mathbf{t};\mathbf{X}-\mu) = 0$, identically in $\mathbf{t}$, if the distribution of $\mathbf{X}$ is symmetric around μ. All procedures employ the distance function $D_n(\mathbf{t};\mu) = \left|n^{-1}\sum_{j=1}^{n}\sin[\mathbf{t}'(\mathbf{X}_j-\mu)]\right|$, in the normalized version either of the supremum type test $\sup_{\mathbf{t}} D_n(\mathbf{t};\mu)$ or of the integrated type statistic, $\int D_n^2(\mathbf{t};\mu)\beta(\mathbf{t})d\mathbf{t}$. Ghosh and Ruymgaart [7] regard μ as fixed and obtain the asymptotic distribution of the integrated test statistic under the more restrictive hypothesis of spherical symmetry. On the other hand, Henze *et al.* [10] consider the more general problem of testing for (reflective) symmetry around an unknown center μ. The test utilizes the integrated type statistic based on a fully standardized version $D_n(\mathbf{S}^{-1/2}\mathbf{t};\bar{\mathbf{X}})$ of the distance function, and it is therefore affine invariant. The method is implemented as a permutation procedure along the lines suggested by Neuhaus and Zhu [26] who also investigated the performance of the supremum statistic initially proposed by Heathcote *et al.* [9].

3.2. Tests for independence

Under the restrictive setting of normal marginals, Bilodeau and Lafaye de Micheaux [1] developed a method for testing independence between the normal components of a random vector based on the empirical CF. However, the most general test statistic was proposed by Kankainen and Ushakov [17], by extending to arbitrary dimension the method suggested by Feuerverger [6] for $d = 2$. The distance function characteristic of independence is $D_n(\mathbf{t}) = |\varphi_n(\mathbf{t}) - \prod_{l=1}^{d}\varphi_{nl}(t_l)|$, where $\varphi_{nl}(t_l) = n^{-1}\sum_{j=1}^{n} e^{it_lX_{jl}}$, is the (marginal) empirical CF computed from the observation $X_{jl}, j = 1,2,\ldots,n$, on $X_l,\ l = 1,2,\ldots,d$. The point of departure in [17] is an earlier test of Csörgő [2] based on $D_n(\mathbf{t})$ calculated at a single point $\mathbf{t} = \mathbf{t}_n$, where $\mathbf{t}_n$ denotes an estimate of a specific value $\mathbf{t} = \mathbf{t}_0$. They showed by counterexample that even when $\mathbf{t}_0$ is specially (optimally) chosen, the resulting test statistic fails to be consistent. Although the test statistic $nD_n^2(\mathbf{t})\beta(\mathbf{t})d\mathbf{t}$ of Kankainen and Ushakov [17] may appear to depend on the underlying distribution of $\mathbf{X}$, it can be made entirely non-parametric by replacing the observations X_{jl} by the values $F_n(X_{jl})$ of the corresponding empirical DF. (For simulation results on this integrated test the reader is referred to Meintanis and Iliopoulos [23]).

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