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THE VELOCITY OF THE RIVER FLOW

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ABSTRACT. The aim of this paper is to contribute to a statistical solution of the estimation problem of a water flow.

The most essential objective of hydrometry still lies, according to the widely recognized opinion, in a precise determination of velocity of the water flow in a particular river basin.

In order to measure the water speed, a hydrometric propeller and Pitot's tube are frequently used. In the real situation we have measured the velocity of the points and the coordinates of these points in I places of the river basin.

The aim of the measurement is to find the estimators of the streamline, the estimator of the maximal velocity on the streamline, the parameter κ_1, κ_2 and κ_3, describing the decrease of the velocity according to the distance from the streamline in x, y and z direction.

The model described above could be studied by the theory of linear models.

1. Introduction

In hydrometric research we encounter the problems of determining the length and width of the river basin by planimetric (topographic) and altitudinal measurements, which then results in determination of transversal and longitudinal profile of the water system. Apart from this, however, the most essential objective of hydrometry is the determination of velocity of the river flow.

We have taken the truly mathematical problem and we have described it with the help of a general linear model with linear constraints on model parameters where constraints contain additional unknown parameters that are not distributional parameters.

The biggest advantage of this model is its possible modification to the case where the stream of the river will change a lot according to the distances from the measured points. We can use this model also with the assumption that the velocity of the streamline will be the function of the plane coordinates. All

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these results should be in the future compared with some of the interpolation geostatistical methods like Kriging or IDW.

The initial need for such a research has been proposed as one of the outcomes of the research project, taking place at Universität für Boden Kultur (BOKU) in Austria.

2. The model

The model of incomplete and indirect measurement of the vector parameter with the condition system of the Type II is given as:

$$\mathbf{Y} \sim (\mathbf{F}\boldsymbol{\Theta}, \boldsymbol{\Sigma}), \quad \mathbf{b} + \mathbf{C}\boldsymbol{\Theta} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}\,, \tag{1}$$

where $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_n)'$ is a random observation vector; $\theta \in R^{k_1}$ is a vector of the measured parameters; $\beta \in R^{k_2}$ is a vector of the additional unmeasured parameters.

If $h(\mathbf{F}_{n,k}) = k_1 < n, h\big(\mathbf{C}_{(q,k_1)}, \mathbf{B}_{(q,k_2)}\big) = q < k_1 + k_2, h(\mathbf{B}) = k_2 < q$, a $\boldsymbol{\Sigma}$ is positive definite, then the model is called regular.
In this paper we will work only with this regular model.

Theorem 1. *BLUE (Best Linear Unbiased Estimator) of the vector* $\binom{\boldsymbol{\Theta}}{\boldsymbol{\beta}}$ *is*

$$\begin{pmatrix} \hat{\hat{\boldsymbol{\Theta}}} \\ \hat{\hat{\boldsymbol{\beta}}} \end{pmatrix} = -\begin{pmatrix} \big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{B}'\mathbf{Q}_{1,1} \\ \mathbf{Q}_{2,1} \end{pmatrix}\mathbf{b} + \begin{pmatrix} \mathbf{I} - \big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{C}'\mathbf{Q}_{1,1}\mathbf{C} \\ -\mathbf{Q}_{2,1}\mathbf{B}_1 \end{pmatrix}\hat{\boldsymbol{\Theta}}, \tag{2}$$

where $\hat{\boldsymbol{\Theta}} = \big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$ *(the estimator does not respect the condition relating to the parameters of* $\boldsymbol{\Theta}, \boldsymbol{\beta}$*); its covariance matrix takes the form*

$$\operatorname{var}\begin{pmatrix} \hat{\hat{\boldsymbol{\Theta}}} \\ \hat{\hat{\boldsymbol{\beta}}}_2 \end{pmatrix} = \begin{pmatrix} \operatorname{var}\big(\hat{\hat{\boldsymbol{\Theta}}}\big), & \operatorname{cov}\big(\hat{\hat{\boldsymbol{\Theta}}}, \hat{\hat{\boldsymbol{\beta}}}\big) \\ \operatorname{cov}\big(\hat{\hat{\boldsymbol{\beta}}}, \hat{\hat{\boldsymbol{\Theta}}}\big), & \operatorname{var}\big(\hat{\hat{\boldsymbol{\beta}}}\big) \end{pmatrix}, \tag{3}$$

where

$$\operatorname{var}\Big(\hat{\hat{\boldsymbol{\Theta}}}\Big) = \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1} - \big(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{C}'\mathbf{Q}_{1,1}\mathbf{C}\big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1},$$

$$\operatorname{cov}\big(\hat{\hat{\boldsymbol{\Theta}}}, \hat{\hat{\boldsymbol{\beta}}}\big) = -\big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{C}'\mathbf{Q}_{1,2},$$

$$\operatorname{var}\big(\hat{\hat{\boldsymbol{\beta}}}_2\big) = -\mathbf{Q}_{2,2}$$

and

$$\begin{pmatrix} \mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2} \\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{C}\big(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\big)^{-1}\mathbf{C}', & \mathbf{B} \\ \mathbf{B}', & \mathbf{0} \end{pmatrix}^{-1}.$$

Proof: See [2] and pp. 129–131. □

The model of the river flow

We have measured the values $X^\times$, $Y^\times$, $Z^\times$ and $V^\times$ of the parameters X, Y, Z and V, which represent the velocity of the river flow in the point (x, y, z).

The aim of the measurement is to find the estimators of the real values v_0, κ_1, κ_2, κ_3, γ_1, γ_2, γ_3, γ_4, δ_1, δ_2, δ_3, δ_4, which represent the maximal velocity on the streamline, the parameter κ_1 describing the decrease of the velocity according to the distance from the streamline in the direction of x, the parameter κ_2 describing the decrease of the velocity according to the distance from the streamline in the direction of y, the parameter κ_3 describing the decrease of the velocity according to the distance from the streamline in the direction of z, parameters γ_i, $i = 1, 2, 3, 4$ describing the y coordinate of the streamline and parameters δ_i, $i = 1, 2, 3, 4$ describing z direction of the streamline.

Let the streamline of the given river be considered in the form

$$\begin{pmatrix} x \\ \beta_1(x) \\ \beta_2(x) \end{pmatrix} = \begin{pmatrix} x \\ \gamma_1 + \gamma_2 x + \gamma_3 x^2 + \gamma_4 x^3 \\ \delta_1 + \delta_2 x + \delta_3 x^2 + \delta_4 x^3 \end{pmatrix} . \tag{4}$$

The result of the measurement is represented by the vector of coordinates and velocities

$$\mathbf{Y}_{4 \cdot I} \sim N_{4 \cdot I} \begin{bmatrix} X_1^\times \\ Y_1^\times \\ Z_1^\times \\ V_1^\times \\ \vdots \\ X_I^\times \\ Y_I^\times \\ Z_I^\times \\ V_I^\times \end{bmatrix} + \epsilon , \quad \text{where } var(\epsilon) = \Sigma . \tag{5}$$

This vector exhibits a normal distribution with a covariance matrix Σ which is determined both by the error (uncertainty) of Pitot's measuring device and by the error (uncertainty) of position device.

The velocity is now represented by the equation

$$V_i^\times = V_0 - \left(X_i^\times - X_i^\triangle\right)^2 \kappa_1 - \left(Y_i^\times - Y_i^\triangle\right)^2 \kappa_2 - \left(Z_i^\times - Z_i^\triangle\right)^2 \kappa_3 , \tag{6}$$

where the symbol $\triangle$ is the symbol for the projection to the streamline (see Figure 1).

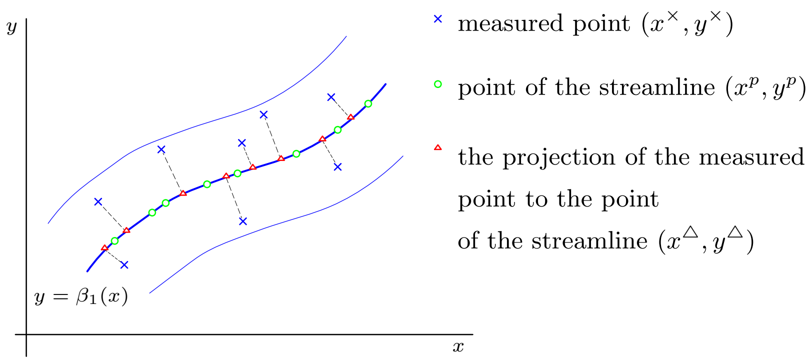

FIGURE 1. Measurements, projections and known points of the streamline

In our case

$$\begin{aligned} \Theta &= \left[X_1^{\times}, Y_1^{\times}, Z_1^{\times}, V_1^{\times}, \dots, X_I^{\times}, Y_I^{\times}, Z_I^{\times}, V_I^{\times}, V_0, \kappa_1, \kappa_2, \kappa_3\right]', && (7)\\ \beta &= \left[\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3, \delta_4\right]. && (8)\end{aligned}$$

Parameters Θ a β have to fulfill the following conditions:

$$\begin{aligned} g_1^i &= X_i^{\times} - X_i^{\triangle} + \left(Y_i^{\times} - Y_i^{\triangle}\right)\left(\gamma_2 + \gamma_3 X_i^{\triangle} + \gamma_4 \left(X_i^{\triangle}\right)^2\right) && (9)\\ &\quad + \left(Z_i^{\times} - Z_i^{\triangle}\right)\left(\delta_2 + \delta_3 X_i^{\triangle} + \delta_4 \left(X_i^{\triangle}\right)^2\right), \qquad i = 1, \dots I,\\ g_2^i &= \gamma_1 + \gamma_2 X_i^{\triangle} + \gamma_3 \left(X_i^{\triangle}\right)^2\Big) + \gamma_4 \left(X_i^{\triangle}\right)^3\Big) - Y_i^{\triangle} = 0, \quad i = 1, \dots I,\\ g_3^i &= \delta_1 + \delta_2 X_i^{\triangle} + \delta_3 \left(X_i^{\triangle}\right)^2\Big) + \delta_4 \left(X_i^{\triangle}\right)^3\Big) - Z_i^{\triangle} = 0, \quad i = 1, \dots I. \end{aligned}$$

According to the theory of nonlinear models, we will now construct the linear version of our model.

Let $\mathbf{F} = \frac{\partial E\mathbf{Y}}{\partial \Theta'}$, where $E\mathbf{Y} = \left[X_1^{\times}, Y_1^{\times}, Z_1^{\times}, V_1^{\times}, \dots, X_I^{\times}, Y_I^{\times}, Z_I^{\times}, V_I^{\times}\right]$.

The matrix $\mathbf{F}$ has the following structure

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_1^{\times}, & \mathbf{F}_1^{\triangle}, & \mathbf{0}, & \mathbf{0}, & \dots & \mathbf{0} & \mathbf{0} & \mathbf{F}_1\\ \mathbf{0}, & \mathbf{0}, & \mathbf{F}_2^{\times}, & \mathbf{F}_2^{\triangle}, & \dots & \mathbf{0} & \mathbf{0} & \mathbf{F}_2\\ \mathbf{0}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}, & \dots & \mathbf{F}_I^{\times}, & \mathbf{F}_I^{\triangle}, & \mathbf{F}_I \end{pmatrix}, \tag{10}$$

where

$$\mathbf{F}_i^{\times} = \begin{pmatrix} 1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 1\\ -2(X^{\times} - X^{\triangle})\kappa_1, & -2(Y^{\times} - Y^{\triangle})\kappa_2, & -2(Z^{\times} - Z^{\triangle})\kappa_3 \end{pmatrix},$$

$$\mathbf{F}_i^{\triangle} = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \\ 2(X^{\times} - X^{\triangle})\kappa_1, & 2(Y^{\times} - Y^{\triangle})\kappa_2, & 2(Z^{\times} - Z^{\triangle})\kappa_3 \end{pmatrix},$$

$$\mathbf{F}_i = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 1, & -(X^{\times} - X^{\triangle})^2, & -(Y^{\times} - Y^{\triangle})^2, & -(Z^{\times} - Z^{\triangle})^2 \end{pmatrix}.$$

The matrix $\mathbf{C}$ has the form of:

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_1^{\times}, & \mathbf{C}_1^{\triangle}, & \mathbf{0}, & \mathbf{0}, & \dots & \mathbf{0} & \mathbf{0} & \mathbf{C}_1 \\ \mathbf{0}, & \mathbf{0}, & \mathbf{C}_2^{\times}, & \mathbf{C}_2^{\triangle}, & \dots & \mathbf{0} & \mathbf{0} & \mathbf{C}_2 \\ \mathbf{0}, & \mathbf{0}, & \mathbf{0}, & \mathbf{0}, & \dots & \mathbf{C}_I^{\times}, & \mathbf{C}_I^{\triangle}, & \mathbf{C}_I \end{pmatrix}, \tag{11}$$

$$\mathbf{C}_i^{\times} = \begin{pmatrix} 1, & \gamma_2 + \gamma_3 X^{\triangle} + \gamma_4 (X^{\triangle})^2, & \delta_2 + \delta_3 X^{\triangle} + \delta_4 (X^{\triangle})^2 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

$$\mathbf{C}_i^{\triangle} = \begin{pmatrix} c_{11}, & c_{12}, & c_{13}, \\ c_{21}, & c_{22}, & c_{23} \\ c_{31}, & c_{32}, & c_{33} \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= -1 + (Y^{\times} - Y^{\triangle})(\gamma_3 + 2\gamma_4 X^{\triangle}) + (Z^{\times} - Z^{\triangle})(\delta_3 + 2\delta_4 X^{\triangle}), \\ c_{12} &= -\gamma_2 - \gamma_3 X^{\triangle} - \gamma^4 (X^{\triangle})^2, \qquad c_{13} = -\delta_2 - \delta_3 X^{\triangle} - \delta^4 (X^{\triangle})^2, \\ c_{21} &= \gamma_2 + 2\gamma_3 X^{\triangle} + 3\gamma^4 (X^{\triangle})^2, \qquad c_{31} = \delta_2 + 2\delta_3 X^{\triangle} + 3\delta^4 (X^{\triangle})^2, \end{aligned}$$

and

$$c_{22} = c_{23} = c_{32} = c_{33} = 0,$$
$$\mathbf{C}_i = \mathbf{0}_{3,4}\,.$$

The matrix $\mathbf{B}$ takes the form:

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1^{\gamma}, & \mathbf{B}_1^{\delta} \\ \mathbf{B}_2^{\gamma}, & \mathbf{B}_2^{\delta} \\ \dots & \\ \mathbf{B}_I^{\gamma}, & \mathbf{B}_I^{\delta} \end{pmatrix}, \tag{12}$$

$$\mathbf{B}_i^{\gamma} = \begin{pmatrix} 0, & Y^{\times} - Y^{\triangle}, & (Y^{\times} - Y^{\triangle})X^{\triangle}, & (Y^{\times} - Y^{\triangle})(X^{\triangle})^2 \\ 1, & X^{\triangle}, & (X^{\triangle})^2, & (X^{\triangle})^3 \\ 0, & 0, & 0, & 0 \end{pmatrix},$$

$$\mathbf{B}_i^{\delta} = \begin{pmatrix} 0, & Z^{\times} - Z^{\triangle}, & (Z^{\times} - Z^{\triangle})X^{\triangle}, & (Z^{\times} - Z^{\triangle})(X^{\triangle})^2 \\ 0, & 0, & 0, & 0 \\ 1, & X^{\triangle}, & (X^{\triangle})^2, & (X^{\triangle})^3 \end{pmatrix}.$$

3. The projection of the known point on the streamline

Let the streamline in the plane (X, Y) be given as an explicitly given function $y = \beta_1(x)$ (see Figure 1). Analogous is the situation $\big(z = \beta_2(x)\big)$ in the plane (X, Z) which we can leave without explanation.

In the real situation we have measured the velocity of the points and the coordinates of these points in I places of the river basin. We will denote these points in the graphical visualization by the symbol $\times$ and its coordinates by the symbol $x^{\times}, y^{\times}, z^{\times}$, we will denote the velocity in these points by $v^{\times}$. The observation vector Y in the above mentioned situation has the length $4 \cdot I$.

During the measurement that was part of the research project at Universität für Boden Kultur (BOKU) in Austria and which took place in the Danube basin the number I was approximately 2000.

The vectors of parameters θ and β have the length $4 \cdot I + 4$ and 8, respective. See (7) and (8).

The next assumption is that we know the shape of the streamline which is in the form (4). The parameters γ_i, $i = 1, 2, 3, 4$ describe y coordinates of the streamline and the parameters δ_i, $i = 1, 2, 3, 4$ describe z coordinates of the streamline. Our numerical study of the problem has shown that we can considered $\beta_1(x)$ and $\beta_2(x)$ as third order polynoms. In another cases the order of the polynom depends on the shape of the river basin. We consider a river which satisfies the assumptions for the laminar stream. This assumption leads to the decreasing changes in velocity according to the distance from the streamline.

Remark. It is possible to find the zero approximation of the streamline by the help of:

(1) Known points on the left bank and the right bank of a river,

(2) Points where the maximal velocity has been measured (it is less reliable, according to the different gradient of the river basin).

In the next step, which is connected with the construction of the model, we will work with the maximal velocity w and with the parameter κ_1, which determines how fast the decreasing of the velocity is in the points lying on the normal to the streamline (in the horizontal direction). The parameter κ_2 will determine how fast the decreasing of the velocity is in the points lying on the normal to the streamline (in the vertical direction).

We will denote the points of the streamline as $x_p, \beta_1(x_p)$, i.e., in Figure 1 by the symbol o.

The aim of this part is to find a projection of the arbitrary measured point $x^\times, y^\times, z^\times$ to the point of the streamline, which we will denote by $x^\triangle, y^\triangle, z^\triangle$.

We will determine the slope to the streamline (the streamline is now described by the explicit function in the form of $y_p = \beta_1(x_p)$. The differentiation of the streamline is

$$\beta_1'(x_p) = \frac{y^\triangle - \beta_1(x_p)}{x^\triangle - x_p}\,, \tag{13}$$

$$\beta_1'(x_p) = \frac{x - x^\triangle}{\beta_1(x_p) - y^\triangle}\,. \tag{14}$$

We can arrive, after the algebraic arrangement of the equations (13) and (14), to the next forms of

$$\beta_1'(x_p) \cdot x^\triangle - y^\triangle = \beta_1'(x_p) \cdot x^p - \beta_1(x_p)\,, \tag{15}$$

$$-x^\triangle + \beta_1'(x_p) \cdot y^\triangle = \beta_1'(x_p) \cdot \beta_1(x_p) - x^p\,. \tag{16}$$

The coordinates of the projection of the given point are now easy to find by the solution of this system equation

$$\begin{pmatrix} \beta_1'(x_p), & -1 \\ -1, & \beta_1'(x_p) \end{pmatrix} \begin{pmatrix} x^\triangle \\ y^\triangle \end{pmatrix} = \begin{pmatrix} \beta_1'(x_p) \cdot x_p - \beta_1(x_p) \\ \beta_1'(x_p) \cdot \beta_1(x_p) - x_p \end{pmatrix},$$

or after the algebraic arrangement

$$\begin{pmatrix} \beta_1'(x_p), & -1 \\ \frac{1}{\beta_1'(x_p)}, & -1 \end{pmatrix} \begin{pmatrix} x^\triangle \\ y^\triangle \end{pmatrix} = \begin{pmatrix} \beta_1'(x_p) \cdot x_p - \beta_1(x_p) \\ -\beta_1(x_p) + \frac{x_p}{\beta_1'(x_p)} \end{pmatrix}.$$

Firstly, we should compute the inverse matrix on the left side of the equation:

$$A^{-1} = \frac{1}{\big(\beta_1'(x_p)\big)^2 - 1} \cdot \begin{pmatrix} \beta_1'(x_p), & -\beta_1'(x_p) \\ 1, & -\big(\beta_1'(x_p)\big)^2 \end{pmatrix}.$$

We can now multiply the obtained matrix by the vector of the right side:

$$A^{-1} \cdot b = \frac{1}{\big(\beta_1'(x_p)\big)^2 - 1} \cdot \begin{pmatrix} \big(\beta_1'(x_p)\big)^2 \cdot x_p - \beta_1'(x_p) \cdot \beta_1(x_p) + \beta_1'(x_p) \cdot y^\times - x^\times \\ \beta_1'(x_p) \cdot x_p - \beta_1(x_p) + \big(\beta_1'(x_p)\big)^2 \cdot y^\times - \beta_1'(x_p) \cdot x^\times \end{pmatrix}$$

and the solution for the new coordinates is now given by:

$$\begin{pmatrix} x^{\triangle} \\ y^{\triangle} \end{pmatrix} = \frac{\beta_1'(x_p)}{1-\left(\beta_1'(x_p)\right)^2} \cdot \begin{pmatrix} \left(-\beta_1'(x_p)\right)^2 \cdot x_p + \beta_1(x_p) - y^{\times} + \frac{x^{\times}}{\beta_1'(x_p)} \\ -x_p + \frac{\beta_1(x_p)}{\beta_1'(x_p)} - \beta_1'(x_p) \cdot y^{\times} + x^{\times} \end{pmatrix}. \tag{17}$$

Conclusion Remarks

In this work we have computed the velocity of a river flow with the help of the measurement in the river basin. The exact model was constructed in order to explain the behaviour of the water in the river basin. The biggest advantage of this model is its possible modification to the case where the stream of the river will change a lot according to the distances from the measured points. The possible improvement of this work is in the assumption that the velocity of the streamline will be the function of the plane coordinates.

In the future we will spend space and time making numerical studies of our model and we will compare our results with other methods like Kriging or IDW.

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