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WEAK CONSISTENCY OF ESTIMATORS IN LINEAR REGRESSION MODEL

Petr Lachout

ABSTRACT. A linear regression model and M-estimator of its regression coefficients are considered. We present a derivation of a weak consistency of the M-estimator together with a rate. Derivation is made under general conditions set on the error term, say "asymptotic stationarity" property. The results are proved by means of L_2-convergence and cover the cases as the error term is ARMA, ARCH, GARCH process or it is attracted by an ARMA, ARCH, GARCH process. We do not separate random and deterministic covariates. Both cases are treated in one general setting.

1. Introduction

We will consider a linear regression model. We observe couples (Y_1, X_1), $(Y_2, X_2), \ldots, (Y_T, X_T)$, where Y_t are random variables and X_t are random vectors. We suppose linear regression model

$$Y_t = (X_t)^\top \beta_0 + \varepsilon_t \qquad \text{for all} \quad t = 1, 2, \ldots, T, \tag{1}$$

where $\beta_0 \in \mathbb{R}^m$ is vector of unknown regression parameters and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T$ are unknown random errors. We assume that the matrix

$$X \;=\; \begin{pmatrix} (X_1)^\top \\ (X_2)^\top \\ \cdots \\ (X_T)^\top \end{pmatrix}$$

possesses full column rank.

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Our aim is to estimate the unknown parameter β_0 in the best way. Considering several different criteria we receive several different estimators. Let us mention two of them.

OLS-estimator of regression coefficients possesses the form

$$\begin{aligned}\hat{\beta}_T &= \left(\sum_{t=1}^{T} X_t \left(X_t\right)^\top\right)^{-1} \sum_{t=1}^{t} X_t Y_t \\ &= \beta_0 + \left(\sum_{t=1}^{T} X_t \left(X_t\right)^\top\right)^{-1} \sum_{t=1}^{T} X_t \varepsilon_t.\end{aligned}$$

M-estimator $\hat{\beta}_T$ is a minimizer of

$$\Upsilon(b) = \sum_{t=1}^{T} \rho\left(Y_t - \left(X_t\right)^\top b\right) = \sum_{t=1}^{T} \rho\left(\varepsilon_t + \left(X_t\right)^\top \left(\beta_0 - b\right)\right), \tag{2}$$

where ρ is a given function. Having ρ twice continuously differentiable we obtain following formula

$$\begin{aligned}0 = \nabla_b \Upsilon(\hat{\beta}_T) &= -\sum_{t=1}^{T} \rho'\left(\varepsilon_t + \left(X_t\right)^\top \left(\beta_0 - \hat{\beta}_T\right)\right) X_t \\ &= -\sum_{t=1}^{T} \left(\rho'\left(\varepsilon_t\right) + \rho''\left(\varepsilon_t\right)\left(X_t\right)^\top \left(\beta_0 - \hat{\beta}_T\right) + R_t\right) X_t.\end{aligned}$$

Hence, we are receiving an expression

$$\begin{aligned}\hat{\beta}_T = \beta_0 &+ \left(\sum_{t=1}^{T} \rho''\left(\varepsilon_t\right) X_t \left(X_t\right)^\top\right)^{-1} \sum_{t=1}^{T} \rho'\left(\varepsilon_t\right) X_t \\ &+ \left(\sum_{t=1}^{T} \rho''\left(\varepsilon_t\right) X_t \left(X_t\right)^\top\right)^{-1} \sum_{t=1}^{T} R_t X_t.\end{aligned}$$

Finally, in the case of the reminder vanishing in probability, we have an asymptotic formula

$$\hat{\beta}_T = \beta_0 + \left(\sum_{t=1}^{T} \rho''\left(\varepsilon_t\right) X_t \left(X_t\right)^\top\right)^{-1} \sum_{t=1}^{T} \rho'\left(\varepsilon_t\right) X_t + o_P\left(1\right). \tag{3}$$

Consider that the reminder is precisely zero for OLS-estimator; i.e., particular case of M-estimator with $\rho\left(y\right) = y^2$.

In the case of the reminder vanishing in probability with a rate $\frac{1}{\sqrt{T}}$, we have an asymptotic formula

$$\hat{\beta}_T = \beta_0 + \left(\sum_{t=1}^{T} \rho'' (\varepsilon_t) X_t (X_t)^\top \right)^{-1} \sum_{t=1}^{T} \rho' (\varepsilon_t) X_t + O_P \left(\frac{1}{\sqrt{T}} \right). \tag{4}$$

Both types of estimators possess asymptotically similar structure: Inverse of a sum of random matrixes times another sum of random vectors. To control estimators quality we need to check the weak consistency of the estimator and weak rate of consistency. Having good weak consistency and weak rate of consistency, one could proceed to stronger estimator properties.

Linear regression model with errors which are i.i.d. or autoregressive processes are often considered in the literature. Let us mention some of them, e.g., [1]– [9].

Our paper is written with an intention to present and discuss a usage of L_2--convergence as a simple tool to derive weak consistency of M-estimators. Our result presents a general set of assumptions covering standard setting. Moreover, it covers the cases if error term is attracted by a stationary process, as ARMA, ARCH or GARCH process. The presented results do not separate random and deterministic covariates. Both cases are particular cases of our setting.

In the first section of the paper we introduce linear regression model. The second section introduces L_2-convergence, the tool used to derive our results. The next section presents results together with proof. We added Appendix as the last section of the paper. It recalls notion of Big O and Small o and contains an auxiliary lemma.

2. Proving tool for convergence

Consider a sequence $S_t \in \mathbb{R}^{\mathsf{d}}$, $t \in \mathbb{N}$ of random vectors.

Lemma 1. *If* $\mathsf{E}[S_t] \underset{t\to+\infty}{\longrightarrow} \theta$ *and* $\mathsf{tr}\big(\mathsf{Var}(S_t)\big) \underset{t\to+\infty}{\longrightarrow} 0$*, then* S_t *is tending to* θ *in* L_2 *and in probability.*

P r o o f. Applying Chebysheff inequality, we receive an estimation

$$\mathsf{P}\big(\|S_t - \mathsf{E}[S_t]\| \geq K\big) \leq \frac{\mathsf{tr}\left(\mathsf{Var}(S_t)\right)}{K^2} = \frac{1}{K^2} o(1).$$

Therefore, we have verified

$$S_t - \mathsf{E}[S_t] = o_P(1).$$

Finally,

$$\begin{aligned} S_t &= \theta + \big(\mathsf{E}[S_t] - \theta\big) + \big(S_t - \mathsf{E}[S_t]\big) \\ &= \theta + o(1) + o_P(1) \\ &= \theta + o_P(1). \end{aligned}$$

□

Lemma 2. *If* $\mathsf{E}[S_t] = \theta + \mathcal{O}(\tau_t)$ *and* $\mathsf{tr}\big(\mathsf{Var}(S_t)\big) = \mathcal{O}\left(\kappa_t^2\right)$, *where* $\tau_t > 0$, $\kappa_t > 0$ *for each* $t \in \mathbb{N}$ *and* $\tau_t \to 0$, $\kappa_t \to 0$, *then* S_t *is tending to* θ *in* L_2 *and in probability with rate* $\max\{\tau_t, \kappa_t\}$.

Thus, we can write $S_t = \theta + \mathcal{O}_P\big(\max\{\tau_t, \kappa_t\}\big)$.

P r o o f. Applying Chebysheff inequality, we receive an estimation

$$\mathsf{P}\big(\|S_t - \mathsf{E}[S_t]\| \geq K\kappa_t\big) \leq \frac{\mathsf{tr}\,(\mathsf{Var}\,(S_t))}{K^2\kappa_t^2} = \frac{1}{K^2}\mathcal{O}(1).$$

Therefore, we have verified

$$S_t - \mathsf{E}[S_t] = \mathcal{O}_P(\kappa_t).$$

Finally,

$$\begin{aligned} S_t &= \theta + \big(\mathsf{E}[S_t] - \theta\big) + \big(S_t - \mathsf{E}[S_t]\big) \\ &= \theta + \mathcal{O}(\tau_t) + \mathcal{O}_P(\kappa_t) \\ &= \theta + \mathcal{O}_P\big(\max\{\tau_t, \kappa_t\}\big). \end{aligned}$$

□

3. Consistency of M-estimators

Now, we return to the linear regression model (1) mentioned in the introductory section. We are interested in asymptotic of M-estimators fulfilling (3) or (4). To be able to do that we have to know something about regression error term and its relation to covariates.

3.1. Assumptions

Our crucial assumption is that

$$\text{errors } \varepsilon_t,\ t \in \mathbb{N} \text{ and covariates } X_t,\ t \in \mathbb{N} \text{ are independent.} \tag{5}$$

Set of assumptions on the error term:

$$\mathsf{E}\big[\rho'(\varepsilon_t)\big] \underset{t\to+\infty}{\longrightarrow} 0, \tag{6}$$

$$\mathsf{E}\big[\rho''(\varepsilon_t)\big] \underset{t\to+\infty}{\longrightarrow} \theta, \tag{7}$$

$$\mathsf{Cov}\big(\rho'(\varepsilon_t), \rho'(\varepsilon_{t+h})\big) \underset{t\to+\infty}{\longrightarrow} \phi(h), \tag{8}$$

$$\mathsf{Cov}\big(\rho''\left(\varepsilon_t\right), \rho''\left(\varepsilon_{t+h}\right)\big) \underset{t\to+\infty}{\longrightarrow} \psi(h), \tag{9}$$

$$\big|\mathsf{Cov}\big(\rho'\left(\varepsilon_t\right),\ \rho'\left(\varepsilon_{t+h}\right)\big)\big| \leq \Phi(h), \tag{10}$$

$$\big|\mathsf{Cov}\big(\rho''\left(\varepsilon_t\right), \rho''(\varepsilon_{t+h})\big)\big| \leq \Psi(h). \tag{11}$$

Set of assumptions on covariates:

$$\frac{1}{T}\sum_{t=1}^{T} \left|\mathsf{E}\left[X_{t;i}\right]\right| \leq M_i, \tag{12}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \mathsf{E}\left[X_{t;i}X_{t+h;j}\right] \underset{T\to+\infty}{\longrightarrow} \alpha_{i,j}(h), \tag{13}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \left|\mathsf{E}\left[X_{t;i}X_{t+h;j}\right]\right| \leq A_{i,j}(h), \tag{14}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \left|\mathsf{Cov}\left(X_{t;i}, X_{t+h;j}\right)\right| \leq \tilde{A}_{i,j}(h), \tag{15}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \mathsf{E}\left[X_{t;i}X_{t;j}X_{t+h;i}X_{t+h;j}\right] \underset{T\to+\infty}{\longrightarrow} \beta_{i,j}(h), \tag{16}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \mathsf{Cov}\left(X_{t;i}X_{t;j}, X_{t+h;i}X_{t+h;j}\right) \underset{T\to+\infty}{\longrightarrow} \gamma_{i,j}(h), \tag{17}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \left|\mathsf{E}\left[X_{t;i}X_{t;j}X_{t+h;i}X_{t+h;j}\right]\right| \leq B_{i,j}(h), \tag{18}$$

$$\frac{1}{T}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T} \left|\mathsf{Cov}\left(X_{t;i}X_{t;j}, X_{t+h;i}X_{t+h;j}\right)\right| \leq C_{i,j}(h). \tag{19}$$

Technical assumptions of summability:

$$\sum_{h=-\infty}^{+\infty} \Phi(h)A_{i,j}(h) < +\infty, \qquad \sum_{h=-\infty}^{+\infty} \tilde{A}_{i,j}(h) < +\infty, \tag{20}$$

$$\sum_{h=-\infty}^{+\infty} \Psi(h)B_{i,j}(h) < +\infty, \qquad \sum_{h=-\infty}^{+\infty} C_{i,j}(h) < +\infty. \tag{21}$$

3.2. Main result

Now, we can formulate and proof the main result of the paper.

Theorem 3. *Under assumptions* (5)–(21) $\theta \neq 0$, *matrix* $\left(\alpha_{i,j}(0)\right)_{i,j}$ *is regular, we have:*

- *Estimator fulfilling* (3) *possesses expression*
$$\hat{\beta}_T = \beta_0 + o_P(1). \tag{22}$$
- *Adding assumption* $\mathsf{E}\left[\rho'(\varepsilon_t)\right] = 0$, $t \in \mathbb{N}$, *estimator fulfilling* (4) *possesses expression*
$$\hat{\beta}_T = \beta_0 + \mathcal{O}_P\left(\frac{1}{\sqrt{T}}\right). \tag{23}$$

P r o o f. To proof convergence in probability for a given random vector or a given random matrix, it is necessary and sufficient to proof convergence in probability for each their member. Therefore, we fix indexes $i, j \in \{1, 2, \ldots, \mathsf{d}\}$ and consider separate members of treated objects.

(1) Assumptions (6), (12) and Lemma 4 give
$$\mathsf{E}\left[\frac{1}{T}\sum_{t=1}^{T}\rho'(\varepsilon_t)X_{t;i}\right] = \frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\left[\rho'(\varepsilon_t)\right]\mathsf{E}\left[X_{t;i}\right] \underset{T\to+\infty}{\longrightarrow} 0.$$

(2) Assumptions (7), (13), (14) together with Lemma 4 yield
$$\mathsf{E}\left[\frac{1}{T}\sum_{t=1}^{T}\rho''(\varepsilon_t)X_{t;i}X_{t;j}\right] = \frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\left[\rho''(\varepsilon_t)\right]\mathsf{E}\left[X_{t;i}X_{t;j}\right] \underset{T\to+\infty}{\longrightarrow} \theta\alpha_{i,j}(0).$$

(3) Let us consider variance of the first random sum.
$$\begin{aligned}
&\mathsf{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\rho'(\varepsilon_t)X_{t;i}\right)\\
&= \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathsf{Cov}\big(\rho'(\varepsilon_t)X_{t;i}, \rho'(\varepsilon_s)X_{s;i}\big)\\
&= \frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho'(\varepsilon_t)X_{t;i}, \rho'(\varepsilon_{t+h})X_{t+h;i}\big)\\
&= \frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho'(\varepsilon_t), \rho'(\varepsilon_{t+h})\big)\mathsf{E}\left[X_{t;i}X_{t+h;i}\right]\\
&\quad + \frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{E}\left[\rho'(\varepsilon_t)\right]\mathsf{E}\left[\rho'(\varepsilon_{t+h})\right]\mathsf{Cov}\left(X_{t;i}, X_{t+h;i}\right)\\
&= \frac{1}{T}\left(\sum_{h=-\infty}^{+\infty}\phi(h)\alpha_{i,i}(h) + o(1)\right),
\end{aligned}$$

since according to the assumptions (8), (10), (13), (14), (20), Lemma 4 gives

$$\frac{1}{T}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho'(\varepsilon_t),\rho'(\varepsilon_{t+h})\big)\mathsf{E}\left[X_{t;i}X_{t+h;i}\right] \underset{T\to+\infty}{\longrightarrow}\sum_{h=-\infty}^{+\infty}\phi(h)\alpha_{i,i}(h),$$

according to Assumptions (6), (15), (20), Lemma 4 gives

$$\frac{1}{T}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{E}\big[\rho'(\varepsilon_t)\big]\mathsf{E}\big[\rho'(\varepsilon_{t+h})\big]\mathsf{Cov}\left(X_{t;i},X_{t+h;i}\right)\underset{T\to+\infty}{\longrightarrow}0.$$

(4) Now, we compute the variance of the second random sum

$$\begin{aligned}
&\mathsf{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\rho''(\varepsilon_t)X_{t;i}X_{t;j}\right)\\
&=\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathsf{Cov}\big(\rho''(\varepsilon_t)X_{t;i}X_{t;j},\rho''(\varepsilon_s)X_{s;i}X_{s;j}\big)\\
&=\frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho''(\varepsilon_t)X_{t;i}X_{t;j},\rho''(\varepsilon_{t+h})X_{t+h;i}X_{t+h;j}\big)\\
&=\frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho''(\varepsilon_t),\rho''(\varepsilon_{t+h})\big)\mathsf{E}\left[X_{t;i}X_{t;j}X_{t+h;i}X_{t+h;j}\right]\\
&\quad+\frac{1}{T^2}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{E}\big[\rho''(\varepsilon_t)\big]\mathsf{E}\big[\rho''(\varepsilon_{t+h})\big]\mathsf{Cov}\left(X_{t;i}X_{t;j},X_{t+h;i}X_{t+h;j}\right)\\
&=\frac{1}{T}\left(\sum_{h=-\infty}^{+\infty}\psi(h)\beta_{i,j}(h)+\sum_{h=-\infty}^{+\infty}\theta^2\gamma_{i,j}(h)+o(1)\right),
\end{aligned}$$

since, according to the assumptions (9), (11), (16), (18), (21), we can apply Lemma 4 to receive

$$\frac{1}{T}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{Cov}\big(\rho''(\varepsilon_t),\rho''(\varepsilon_{t+h})\big)\mathsf{E}\left[X_{t;i}X_{t;j}X_{t+h;i}X_{t+h;j}\right] \underset{T\to+\infty}{\longrightarrow}\sum_{h=-\infty}^{+\infty}\psi(h)\beta_{i,j}(h),$$

according to Assumptions (7), (17), (19), (21), we can apply Lemma 4 to receive

$$\frac{1}{T}\sum_{h=1-T}^{T-1}\sum_{t=(1-h)\vee 1}^{(T-h)\wedge T}\mathsf{E}\left[\rho''\left(\varepsilon_t\right)\right]\mathsf{E}\left[\rho''\left(\varepsilon_{t+h}\right)\right]\mathsf{Cov}\left(X_{t;i}X_{t;j},X_{t+h;i}X_{t+h;j}\right)$$

$$\underset{T\to+\infty}{\longrightarrow}\sum_{h=-\infty}^{+\infty}\theta^2\gamma_{i,j}(h).$$

Applying Lemmas 1, 2 we have proved

$$\frac{1}{T}\sum_{t=1}^{T}\rho'\left(\varepsilon_t\right)X_{t;i}=o_P\left(1\right),$$

$$\frac{1}{T}\sum_{t=1}^{T}\rho'\left(\varepsilon_t\right)X_{t;i}=\mathcal{O}_P\left(\frac{1}{\sqrt{T}}\right)\quad\text{if}\;\;\mathsf{E}\left[\rho'\left(\varepsilon_t\right)\right]=0\qquad\text{for all}\quad t\in\mathbb{N},$$

$$\frac{1}{T}\sum_{t=1}^{T}\rho''\left(\varepsilon_t\right)X_{t;i}X_{t;j}=\theta\alpha_{i,j}(0)+o_P\left(1\right).$$

Plugging this observation into (3) and (4) we receive the theorem since $\theta\neq 0$ and matrix $\left(\alpha_{i,j}(0)\right)_{i,j}$ is regular. □

4. Appendix

Recall meaning of the symbols Big O and Small o. Consider a sequence of real vectors $a_t\in\mathbb{R}^{\mathsf{d}}$, $t\in\mathbb{N}$ and a sequence of positive reals τ_t, $t\in\mathbb{N}$. Then

$$a_t=o\left(\tau_t\right)\quad\Longleftrightarrow\quad\lim_{t\to+\infty}\frac{\|a_t\|}{\tau_t}=0,$$

$$a_t=\mathcal{O}\left(\tau_t\right)\quad\Longleftrightarrow\quad\limsup_{t\to+\infty}\frac{\|a_t\|}{\tau_t}<+\infty.$$

For a sequence of random vectors $X_t\in\mathbb{R}^{\mathsf{d}}$, $t\in\mathbb{N}$, we will denote

$$X_t=o_P\left(\tau_t\right)\quad\Longleftrightarrow\quad\frac{\|X_t\|}{\tau_t}\underset{t\to+\infty}{\overset{P}{\longrightarrow}}0,$$

$$X_t=\mathcal{O}_P\left(\tau_t\right)\quad\Longleftrightarrow\quad\forall\varepsilon>0\quad\exists\,0<K<+\infty\,:\;\limsup_{t\to+\infty}\mathsf{P}\left(\|X_t\|\geq K\tau_t\right)<\varepsilon,$$

Let us continue with an auxiliary lemma.

Lemma 4. *Consider collections of real numbers* $a_t, b_t,\ t \in \mathbb{N}$ *and* $A_{t,h}, B_{t,h}$, $t = 1, 2, \ldots, T,\ h = 0, 1, \ldots, T-1,\ T \in \mathbb{N}$.

$$a_t \to 0, \frac{1}{T}\sum_{t=1}^{T}|b_t| \le M \quad \Longrightarrow \quad \frac{1}{T}\sum_{t=1}^{T} a_t b_t \underset{T\to+\infty}{\longrightarrow} 0. \tag{24}$$

$$a_t \to \alpha, \frac{1}{T}\sum_{t=1}^{T} b_t \to \beta, \frac{1}{T}\sum_{t=1}^{T}|b_t| \le M \quad \Longrightarrow \quad \frac{1}{T}\sum_{t=1}^{T} a_t b_t \underset{T\to+\infty}{\longrightarrow} \alpha\beta. \tag{25}$$

$$\forall h\ A_{t,h} \to 0,\ |A_{t,h}| \le Q(h),\ \frac{1}{T}\sum_{t=1}^{T}|B_{t,h}| \le \Phi(h),$$

$$\sum_{h=0}^{+\infty} Q(h)\Phi(h) < +\infty \Longrightarrow \frac{1}{T}\sum_{h=0}^{T-1}\sum_{t=1}^{T-h} A_{t,h}B_{t,h} \underset{T\to+\infty}{\longrightarrow} 0. \tag{26}$$

$$\forall h\ A_{t,h} \to \alpha(h),\ |A_{t,h}| \le Q(h),\ \frac{1}{T}\sum_{t=1}^{T} B_{t,h} \to \beta(h),\ \frac{1}{T}\sum_{t=1}^{T}|B_{t,h}| \le \Phi(h),$$

$$\sum_{h=0}^{+\infty} Q(h)\Phi(h) < +\infty \Longrightarrow \frac{1}{T}\sum_{h=0}^{T-1}\sum_{t=1}^{T-h} A_{t,h}B_{t,h} \underset{T\to+\infty}{\longrightarrow} \alpha(h)\beta(h). \tag{27}$$

Proof. All properties follow by the application of Dominated Convergence Theorem. $\square$

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Department of Probability and
Mathematical Statistics
Faculty of Mathematics and Physics
Charles University in Prague
Sokolovská 83
CZ–186-75 Prague 8
CZECH REPUBLIC
E-mail: lachout@karlin.mff.cuni.cz