sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2025-0011
Tatra Mt. Math. Publ. **89** (2025), 103–126

DISCRETE BOURGAIN-MORREY SPACES

Martha Guzmán-Partida — Luis San Martín —
Alejandro Villegas-Acuña

Universidad de Sonora, Hermosillo, MÉXICO

ABSTRACT. We address discrete versions of Bourgain-Morrey spaces. We prove some basic properties and introduce several equivalent norms on these spaces. Finally, we analyze the action of a dyadic version of the Hardy-Littlewood maximal operator on such spaces.

1. Introduction

In this paper, we study discrete versions of Bourgain-Morrey spaces introduced and studied by J. Bourgain in [3], S. Masaki and J. Segata in [14], N. Hatano et al. in [12] and N. Diarra in [4], just to mention a few.

Discrete Bourgain-Morrey spaces generalize the discrete Morrey spaces introduced by H. Gunawan, E. Kikianty and C. Schwanke in [6] and have been studied in several articles such as [1], [2], [5], [7], [8], [9], [10], [11], and [13].

For $1 \leq p \leq q < \infty$, discrete Morrey spaces for dimension $n = 1$, were defined by Gunawan et al. (see [6]) in the following way:

The set

$$l_q^p = \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \|x\|_{l_q^p} < \infty \right\},$$

where

$$\|x\|_{l_q^p} = \sup_{m\in\mathbb{Z},\ N\in\mathbb{N}_0} \frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{k=m-N}^{m+N} |x_k|^p \right)^{1/p},$$

is called a discrete Morrey space. Here, $\mathbb{N}_0$ denotes $\mathbb{N}\cup\{0\}$. Moreover, $\left(l_q^p, \|\cdot\|_{l_q^p}\right)$ is a Banach space such that $l_p^p = l^p$.

2020 Mathematics Subject Classification: 42B35, 46B45, 42B25.
Keywords: Bourgain-Morrey space, equivalent norms, dyadic maximal function.
The third author was supported by CONAHCYT, Grant no. 921835.

As in [6], for $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$, we denote by $S_{m,N}$ the set

$$\{m-N, m-N+1, \ldots, m+N\}.$$

Clearly, the cardinality of $S_{m,N}$ is $2N+1$ and it will be denoted by $|S_{m,N}|$.

Let $1 \le r < \infty$. The discrete Bourgain-Morrey space denoted by $l^p_{q,r}$ is defined as

$$l^p_{q,r} = \left\{ x = (x_k)_{k\in\mathbb{Z}} : \|x\|_{l^p_{q,r}} < \infty \right\},$$

where

$$\begin{aligned} \|x\|_{l^p_{q,r}} &= \left\| |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{1/p} \right\|_{l^r(\mathbb{Z}\times\mathbb{N}_0)} \\ &= \left[\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{r/p} \right]^{1/r}. \end{aligned}$$

For $r = \infty$, the discrete Bourgain-Morrey space $l^p_{q,\infty}$ is the space of sequences x such that

$$\|x\|_{l^p_{q,\infty}} = \sup_{m\in\mathbb{Z}, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k=m-N}^{m+N} |x_k|^p \right)^{1/p} = \|x\|_{l^p_q} < \infty.$$

Thus, in this case, $l^p_{q,\infty}$ coincide with the discrete Morrey space l^p_q.

Moreover, it is easy to see that: for

$$1 \le p < q < \infty \quad \text{and} \quad 1 < r \le \infty,$$

$$l^p_{q,r} \subset l^p_q \subset l^\infty.$$

In the following sections of this paper, we will study some important properties of these discrete spaces, such as completeness, inclusion relations and equivalent norms. Moreover, we will study the action of a dyadic version of the classical Hardy-Littlewood maximal operator on these spaces.

We will use standard notation, and as usual, we shall denote by C a constant that could be changing line by line. Sometimes we will denote a constant by $C_{a,b}$ to indicate that the constant depends on a and b.

2. Some properties of the spaces $l^p_{q,r}$

Along this section, we will be assuming that p, q and r are real numbers such that $1 \le p \le q < \infty$, and $1 \le r \le \infty$.

Proposition 2.1. *The following assertions hold:*

a): $\left(l^p_{q,r}, \|\cdot\|_{l^p_{q,r}}\right)$ *is a Banach space.*

b): *If* $1 \le r_1 \le r_2 \le \infty$, *then* $l^p_{q,r_1} \subset l^p_{q,r_2}$ *with* $\|\cdot\|_{l^p_{q,r_2}} \le \|\cdot\|_{l^p_{q,r_1}}$.

c): *If* $1 \le p_2 \le p_1 \le q$, *then* $l^{p_1}_{q,r} \subset l^{p_2}_{q,r}$ *with* $\|\cdot\|_{l^{p_2}_{q,r}} \le \|\cdot\|_{l^{p_1}_{q,r}}$.

Proof. To prove b), we recall that $l^{r_1} \subset l^{r_2}$ with $\|\cdot\|_{l^{r_2}} \le \|\cdot\|_{l^{r_1}}$, and this immediately implies that $l^p_{q,r_1} \subset l^p_{q,r_2}$ with $\|\cdot\|_{l^p_{q,r_2}} \le \|\cdot\|_{l^p_{q,r_1}}$.

Now, according to [6, Proposition 2.4], we have that $l^{p_1}_q \subset l^{p_2}_q$ with $\|\cdot\|_{l^{p_2}_q} \le \|\cdot\|_{l^{p_1}_q}$, which implies c).

Finally, to show a) we proceed as follows.

For $r<\infty$, let us consider a Cauchy sequence $(x^{(j)})_{j=1}^\infty \in l^p_{q,r}$. Then, given $\varepsilon>0$, we can find $j_\varepsilon \in \mathbb{N}$ such that for every $j, l \in \mathbb{N}$ with $j, l \ge j_\varepsilon$ we have

$$\left\| x^{(j)} - x^{(l)} \right\|_{l^p_{q,r}} < \varepsilon. \tag{1}$$

Using b) with $r_1 = r$ and $r_2 = \infty$, we see that

$$l^p_{q,r} \subset l^p_q \quad \text{and} \quad \|\cdot\|_{l^p_q} \le \|\cdot\|_{l^p_{q,r}},$$

hence $(x^{(j)})_{j=1}^\infty$ is a Cauchy sequence in $(l^p_q, \|\cdot\|_{l^p_q})$, and since the latter is a Banach space, there exists $x \in l^p_q$ such that $x^{(j)} \to x$ in l^p_q as $j \to \infty$. We need to prove that $x \in l^p_{q,r}$ and $x^{(j)} \to x$ in $\|\cdot\|_{l^p_{q,r}}$ as $j \to \infty$.

Indeed, since $x^{(j)} \to x$ in l^p_q, for $\varepsilon > 0$ given above, we can find $l_\varepsilon \ge j_\varepsilon$ such that for each $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$,

$$|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left[\sum_{k \in S_{m,N}} \left| x^{(j)}_k - x_k \right|^p \right]^{1/p} < \varepsilon \quad \text{if } j \ge l_\varepsilon. \tag{2}$$

Taking $N = 0$ in (2), we obtain

$$x^{(j)}_k \to x_k \quad \text{as } j \to \infty \quad \text{for every } k \in \mathbb{Z}.$$

Now, using (1) and Fatou's lemma, we have for every $j, l \in \mathbb{N}$ such that j, $l \ge l_\varepsilon$

$$\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} \left| x^{(j)}_k - x_k \right|^p \right)^{r/p} \le$$

$$\liminf_{l\to\infty} \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} \left| x^{(j)}_k - x^{(l)}_k \right|^p \right)^{r/p} \le \varepsilon^r$$

and this implies that $x \in l^p_{q,r}$ and $x^{(j)} \to x$ in $\|\cdot\|_{l^p_{q,r}}$.

The case $r = \infty$ is also true because $l^p_{q,\infty}$ is a discrete Morrey space l^p_q.

This concludes the proof. $\square$

The following proposition examines dilation, translation, and convolution properties for $l^p_{q,r}$.

Proposition 2.2. *Let $1 \le p \le q < \infty$, $1 \le r \le \infty$, and $x \in l^p_{q,r}$.*

a): *If $t \in \mathbb{N}$, then $\|x(t\cdot)\|_{l^p_{q,r}} \le t^{\frac{1}{p}-\frac{1}{q}} \|x\|_{l^p_{q,r}}$.*

b): *If $s \in \mathbb{Z}$, then $\|x(\cdot - s)\|_{l^p_{q,r}} = \|x\|_{l^p_{q,r}}$.*

c): *If $y \in l^1$, then $\|x * y\|_{l^p_{q,r}} \le \|y\|_{l^1} \|x\|_{l^p_{q,r}}$.*

Proof. To see a), let us consider first the case $r < \infty$.

$$\begin{aligned}
\|x(t\cdot)\|^r_{l^p_{q,r}} &= \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} |x_{tk}|^p\right)^{r/p} \\
&\le \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{l\in S_{tm,tN}} |x_l|^p\right)^{r/p} \\
&= \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \left[t^{\frac{1}{p}-\frac{1}{q}} (2tN+t)^{\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{l\in S_{tm,tN}} |x_l|^p\right)^{1/p}\right]^r \\
&\le t^{r\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \left[(2tN+1)^{\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{l\in S_{tm,tN}} |x_l|^p\right)^{1/p}\right]^r \\
&\le t^{r\left(\frac{1}{p}-\frac{1}{q}\right)} \|x\|^r_{l^p_{q,r}}.
\end{aligned}$$

The case $r = \infty$ is easier and we omit it.

To prove b), consider the case $r = \infty$.

$$\begin{aligned}
\|x(\cdot - s)\|_{l^p_{q,\infty}} &= \|x(\cdot - s)\|_{l^p_q} \\
&= \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_{k-s}|^p\right)^{1/p} \\
&= \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{l\in S_{m-s,N}} |x_l|^p\right)^{1/p} \\
&= \|x\|_{l^p_q} = \|x\|_{l^p_{q,\infty}}.
\end{aligned}$$

The case $r < \infty$ is similar.

Finally, let us prove c). Using b), we have for every $N \in \mathbb{N}$

$$\left\| \sum_{k=-N}^{N} y(k)\, x(\cdot - k) \right\|_{l^p_{q,r}} \le \sum_{k=-N}^{N} |y(k)| \, \|x(\cdot - k)\|_{l^p_{q,r}}$$
$$= \sum_{k=-N}^{N} |y(k)| \, \|x\|_{l^p_{q,r}} ,$$

and taking limit as $N \to \infty$ in both sides of the inequality, we obtain

$$\|y * x\|_{l^p_{q,r}} \le \|y\|_{l^1} \, \|x\|_{l^p_{q,r}} ,$$

as we wanted to show. □

We can also prove Young's inequality for discrete Bourgain-Morrey spaces.

Theorem 2.3. *Let p, p_0, p_1, q, q_0, q_1, r, r_0, r_1 be real numbers such that*

$$1 \le p \le q < \infty, \quad 1 \le p_0 < q_0 < \infty, \quad 1 \le p_1 < q_1 < \infty, \quad 1 < r, r_0, r_1 < \infty$$

and

$$\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p} + 1, \qquad \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q} + 1, \qquad \frac{1}{r_0} + \frac{1}{r_1} = \frac{1}{r} + 1.$$

Then, for all

$$x \in l^{p_0}_{q_0,r_0} \quad \text{and} \quad y \in l^{p_1}_{q_1,r_1}$$

we have

$$\|x * y\|_{l^p_{q,r}} \le 2 \, \|x\|_{l^{p_0}_{q_0,r_0}} \, \|y\|_{l^{p_1}_{q_1,r_1}} .$$

Proof. Without loss of generality, we can assume that the sequences x and y are non-negative.

Let m, $\overline{m} \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. First, observe that by the properties of characteristic functions and the definition of convolution, we can rewrite the following expression as

$$\left\| (x * y)\, \chi_{S_{m,N}} \right\|_{l^p} = \left(\sum_{k \in \mathbb{Z}} \left| (x * y)(k)\, \chi_{S_{m,N}}(k) \right|^p \right)^{1/p}$$
$$= \left(\sum_{k \in S_{m,N}} |(x * y)(k)|^p \right)^{1/p}$$
$$= \left(\sum_{k \in S_{m,N}} \left| \sum_{\overline{m} \in \mathbb{Z}} x(\overline{m})\, y(k - \overline{m}) \right|^p \right)^{1/p} . \qquad (3)$$

Then, by (3) and by Minkowski's inequality for integrals, we obtain

$$\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p} = \left(\sum_{k\in S_{m,N}}\left|\sum_{\overline{m}\in\mathbb{Z}} x(\overline{m})\,y(k-\overline{m})\right|^p\right)^{1/p}$$

$$\leq \sum_{\overline{m}\in\mathbb{Z}}\left(\sum_{k\in S_{m,N}}\left|x(\overline{m})\,y(k-\overline{m})\right|^p\right)^{1/p}. \quad (4)$$

Next, since

$$\sum_{\overline{m}\in\mathbb{Z}}\left(\sum_{k\in S_{m,N}}\left|x(\overline{m})\,y(k-\overline{m})\right|^p\right)^{1/p} = \sum_{\overline{m}\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\left|x(\overline{m})\,y(k-\overline{m})\,\chi_{S_{m,N}}(k)\right|^p\right)^{1/p}$$

$$= \sum_{\overline{m}\in\mathbb{Z}}\left\|x(\overline{m})\,y(\cdot-\overline{m})\,\chi_{S_{m,N}}(\cdot)\right\|_{l^p}, \quad (5)$$

from (4) and (5), we get that

$$\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p} \leq \sum_{\overline{m}\in\mathbb{Z}}\left\|x(\overline{m})\,y(\cdot-\overline{m})\,\chi_{S_{m,N}}(\cdot)\right\|_{l^p}. \quad (6)$$

Moreover, since $x, y \geq 0$, it follows that

$$x(\overline{m})\,y(\cdot-\overline{m})\,\chi_{S_{m,N}}(\cdot) \leq \sum_{t\in S_{\overline{m},N}} x(t)\,y(\cdot-t)\,\chi_{S_{m,N}}(\cdot)$$

for every $\overline{m}\in\mathbb{Z}$, and consequently

$$\sum_{\overline{m}\in\mathbb{Z}}\left\|x(\overline{m})\,y(\cdot-\overline{m})\,\chi_{S_{m,N}}(\cdot)\right\|_{l^p}$$

$$\leq \sum_{\overline{m}\in\mathbb{Z}}\left\|\sum_{t\in S_{\overline{m},N}} x(t)\,y(\cdot-t)\,\chi_{S_{m,N}}(\cdot)\right\|_{l^p}$$

$$= \sum_{\overline{m}\in\mathbb{Z}}\left\|\sum_{t\in\mathbb{Z}} x(t)\,y(\cdot-t)\,\chi_{S_{m,N}}(\cdot)\,\chi_{S_{\overline{m},N}}(t)\right\|_{l^p}. \quad (7)$$

Now, denoting by

$$S_{m,N}-S_{\overline{m},N} = \{j-k : j\in S_{m,N},\ k\in S_{\overline{m},N}\} = S_{m-\overline{m},2N},$$

and noticing that for $s, t\in\mathbb{Z}$

$$\chi_{S_{m,N}}(s)\,\chi_{S_{\overline{m},N}}(t) \leq \chi_{S_{m,N}-S_{\overline{m},N}}(s-t)\,\chi_{S_{\overline{m},N}}(t)$$

$$= \chi_{S_{m-\overline{m},2N}}(s-t)\,\chi_{S_{\overline{m},N}}(t),$$

we obtain

$$\sum_{\overline{m}\in\mathbb{Z}} \left\| \sum_{t\in\mathbb{Z}} x(t)\, y(\cdot - t)\, \chi_{S_{m,N}}(\cdot)\, \chi_{S_{\overline{m},N}}(t) \right\|_{l^p} \leq \sum_{\overline{m}\in\mathbb{Z}} \left\| \sum_{t\in\mathbb{Z}} x(t)\, y(\cdot - t)\, \chi_{S_{m-\overline{m},2N}}(\cdot - t)\, \chi_{S_{\overline{m},N}}(t) \right\|_{l^p}. \quad (8)$$

Thus, combining (6), (7) and (8), we get

$$\left\| (x * y)\, \chi_{S_{m,N}} \right\|_{l^p} \leq \sum_{\overline{m}\in\mathbb{Z}} \left\| \sum_{t\in\mathbb{Z}} x(t)\, y(\cdot - t)\, \chi_{S_{m-\overline{m},2N}}(\cdot - t)\, \chi_{S_{\overline{m},N}}(t) \right\|_{l^p}. \quad (9)$$

Furthermore, we can rewrite the right-hand side of (9) as

$$\sum_{\overline{m}\in\mathbb{Z}} \left\| \sum_{t\in\mathbb{Z}} x(t)\, y(\cdot - t)\, \chi_{S_{m-\overline{m},2N}}(\cdot - t)\, \chi_{S_{\overline{m},N}}(t) \right\|_{l^p} = \sum_{\overline{m}\in\mathbb{Z}} \left\| \left(x\chi_{S_{\overline{m},N}}\right) * \left(y\chi_{S_{m-\overline{m},2N}}\right) \right\|_{l^p}. \quad (10)$$

Now, using (9), (10) and Young's inequality for l^p spaces with $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p} + 1$, we get

$$\begin{aligned} \left\| (x * y)\, \chi_{S_{m,N}} \right\|_{l^p} &\leq \sum_{\overline{m}\in\mathbb{Z}} \left\| \left(x\chi_{S_{\overline{m},N}}\right) * \left(y\chi_{S_{m-\overline{m},2N}}\right) \right\|_{l^p} \\ &\leq \sum_{\overline{m}\in\mathbb{Z}} \left\| x\chi_{S_{\overline{m},N}} \right\|_{l^{p_0}} \left\| y\chi_{S_{m-\overline{m},2N}} \right\|_{l^{p_1}}. \end{aligned} \quad (11)$$

Hence, by (11) and the fact that $\frac{1}{q} - \frac{1}{p} = \left(\frac{1}{q_0} - \frac{1}{p_0}\right) + \left(\frac{1}{q_1} - \frac{1}{p_1}\right)$, we have

$$\begin{aligned} &\left(\sum_{m\in\mathbb{Z}} (2N+1)^{\frac{r}{q} - \frac{r}{p}} \left\| (x * y)\, \chi_{S_{m,N}} \right\|_{l^p}^r \right)^{1/r} \\ &\leq \left(\sum_{m\in\mathbb{Z}} (2N+1)^{\frac{r}{q} - \frac{r}{p}} \left(\sum_{\overline{m}\in\mathbb{Z}} \left\| x\chi_{S_{\overline{m},N}} \right\|_{l^{p_0}} \left\| y\chi_{S_{m-\overline{m},2N}} \right\|_{l^{p_1}} \right)^r \right)^{1/r} \\ &= \left(\sum_{m\in\mathbb{Z}} \left(\sum_{\overline{m}\in\mathbb{Z}} (2N+1)^{\frac{1}{q_0} - \frac{1}{p_0}} \left\| x\chi_{S_{\overline{m},N}} \right\|_{l^{p_0}} (2N+1)^{\frac{1}{q_1} - \frac{1}{p_1}} \left\| y\chi_{S_{m-\overline{m},2N}} \right\|_{l^{p_1}} \right)^r \right)^{1/r}. \end{aligned} \quad (12)$$

In addition, defining for $\overline{m} \in \mathbb{Z}$

$$\begin{aligned} a(\overline{m}) &:= (2N+1)^{\frac{1}{q_0} - \frac{1}{p_0}} \left\| x\chi_{S_{\overline{m},N}} \right\|_{l^{p_0}}, \\ b(\overline{m}) &:= (2N+1)^{\frac{1}{q_1} - \frac{1}{p_1}} \left\| y\chi_{S_{\overline{m},2N}} \right\|_{l^{p_1}}, \end{aligned}$$

the right-hand side of(12) can be written as

$$\left(\sum_{m\in\mathbb{Z}}\left(\sum_{\overline{m}\in\mathbb{Z}}a\left(\overline{m}\right)b\left(m-\overline{m}\right)\right)^{r}\right)^{1/r}=\left(\sum_{m\in\mathbb{Z}}\left|\left(a*b\right)\left(m\right)\right|^{r}\right)^{1/r},$$

then, using Young's inequality for l^r , we have

$$\left(\sum_{m\in\mathbb{Z}}\left|\left(a*b\right)\left(m\right)\right|^{r}\right)^{1/r}\leq\ \|a\|_{l^{r_0}}\|b\|_{l^{r_1}}=$$

$$\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{1/r_0}\times$$

$$\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,2N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}. \quad (13)$$

Thus, by (12) and (13), we obtain

$$\left(\sum_{m\in\mathbb{Z}}(2N+1)^{\frac{r}{q}-\frac{r}{p}}\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p}^{r}\right)^{1/r}\leq$$

$$\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{1/r_0}\times$$

$$\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,2N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}. \quad (14)$$

Now, since

$$S_{m,2N}=S_{m-N,N}\cup S_{m+N,N},$$

then

$$\chi_{S_{m,2N}}\leq\chi_{S_{m-N,N}}+\chi_{S_{m+N,N}}. \quad (15)$$

Moreover, for each $l\in\mathbb{Z}$ we have

$$\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m+l,N}}\right\|_{l^{p_1}}\right)^{r_1}=$$

$$\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}. \quad (16)$$

Taking $l = N, -N$, from (15) and (16) we obtain that

$$\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,2N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1} \leq \left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left(\left\|y\chi_{S_{m-N,N}}\right\|_{l^{p_1}}+\left\|y\chi_{S_{m+N,N}}\right\|_{l^{p_1}}\right)\right)^{r_1}\right)^{1/r_1} = 2\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}. \quad (17)$$

Therefore, by (14) and (17)

$$\left(\sum_{m\in\mathbb{Z}}(2N+1)^{\frac{r}{q}-\frac{r}{p}}\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p}^r\right)^{1/r} \leq 2\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{1/r_0} \times \left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}. \quad (18)$$

Now, raising to the power r, adding over $N \in \mathbb{N}_0$ and taking rth root on both sides of the above inequality, it follows that

$$\left(\sum_{N\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}}(2N+1)^{\frac{r}{q}-\frac{r}{p}}\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p}^r\right)^{1/r} \leq 2\left[\sum_{N\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{r/r_0} \times \left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{r/r_1}\right]^{1/r}. \quad (19)$$

Thus, according to the definition of the $l^p_{q,r}$-norm and (19),

$$\|x*y\|_{l^p_{q,r}} = \left(\sum_{N\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}}(2N+1)^{\frac{r}{q}-\frac{r}{p}}\left\|(x*y)\,\chi_{S_{m,N}}\right\|_{l^p}^r\right)^{1/r} \leq 2\left[\sum_{N\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{r/r_0} \times \left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{r/r_1}\right]^{1/r}. \quad (20)$$

Now, observing that

$$
\begin{aligned}
&2\left[\sum_{N\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{r/r_0}\right.\\
&\quad\left.\times\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{r/r_1}\right]^{1/r}\\
&\le 2\sup_{N\in\mathbb{N}_0}\left[\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{r/r_1}\right]^{1/r}\\
&\quad\times\left[\sum_{N\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{r/r_0}\right]^{1/r}, \qquad (21)
\end{aligned}
$$

and noticing that

$$
\begin{aligned}
&2\sup_{N\in\mathbb{N}_0}\left[\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{r/r_1}\right]^{1/r}\\
&\quad\times\left[\sum_{N\in\mathbb{N}_0}\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{r/r_0}\right]^{1/r}\\
&= 2\left\|\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}\right\|_{l^\infty(\mathbb{N}_0)}\\
&\quad\times\left\|\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{1/r_0}\right\|_{l^r(\mathbb{N}_0)}, \qquad (22)
\end{aligned}
$$

we conclude, combining (20), (21) and (22), that

$$
\begin{aligned}
\|x*y\|_{l^p_{q,r}} &\le 2\left\|\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}}\left\|y\chi_{S_{m,N}}\right\|_{l^{p_1}}\right)^{r_1}\right)^{1/r_1}\right\|_{l^\infty(\mathbb{N}_0)}\\
&\quad\times\left\|\left(\sum_{m\in\mathbb{Z}}\left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}}\left\|x\chi_{S_{m,N}}\right\|_{l^{p_0}}\right)^{r_0}\right)^{1/r_0}\right\|_{l^r(\mathbb{N}_0)}. \qquad (23)
\end{aligned}
$$

Finally, using the fact that $r_1 < r$, we obtain

$$\left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}} \left\| x\chi_{S_{m,N}} \right\|_{l^{p_0}} \right)^{r_0} \right)^{1/r_0} \right\|_{l^r(\mathbb{N}_0)} \leq$$

$$\left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}} \left\| y\chi_{S_{m,N}} \right\|_{l^{p_1}} \right)^{r_1} \right)^{1/r_1} \right\|_{l^{r_1}(\mathbb{N}_0)}, \quad (24)$$

and since

$$\left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}} \left\| y\chi_{S_{m,N}} \right\|_{l^{p_1}} \right)^{r_1} \right)^{1/r_1} \right\|_{l^\infty(\mathbb{N}_0)} \leq$$

$$\left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}} \left\| x\chi_{S_{m,N}} \right\|_{l^{p_0}} \right)^{r_0} \right)^{1/r_0} \right\|_{l^r(\mathbb{N}_0)}, \quad (25)$$

we conclude by (23), (24) and (25) that

$$\begin{aligned}
\|x*y\|_{l^p_{q,r}} \leq\ & 2 \left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}} \left\| y\chi_{S_{m,N}} \right\|_{l^{p_1}} \right)^{r_1} \right)^{1/r_1} \right\|_{l^\infty(\mathbb{N}_0)} \\
& \times \left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}} \left\| x\chi_{S_{m,N}} \right\|_{l^{p_0}} \right)^{r_0} \right)^{1/r_0} \right\|_{l^r(\mathbb{N}_0)} \\
\leq\ & 2 \left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_0}-\frac{1}{p_0}} \left\| x\chi_{S_{m,N}} \right\|_{l^{p_0}} \right)^{r_0} \right)^{1/r_0} \right\|_{l^{r_0}(\mathbb{N}_0)} \\
& \times \left\| \left(\sum_{m\in\mathbb{Z}} \left((2N+1)^{\frac{1}{q_1}-\frac{1}{p_1}} \left\| y\chi_{S_{m,N}} \right\|_{l^{p_1}} \right)^{r_1} \right)^{1/r_1} \right\|_{l^{r_1}(\mathbb{N}_0)} \\
=\ & 2 \|x\|_{l^{p_0}_{q_0,r_0}} \|y\|_{l^{p_1}_{q_1,r_1}}. \qquad (26)
\end{aligned}$$

This completes the proof. □

Next, we give some examples of elements in discrete Bourgain-Morrey spaces.

Example. Let $1 \leq p < q < \infty$, and $\frac{2}{\frac{1}{p}-\frac{1}{q}} < r < \infty$. Define

$$x = (x_k)_{k\in\mathbb{Z}} \quad \text{by} \quad x_k = \delta_{k0}.$$

Then, $x \in l^p_{q,r}$.

Indeed,

$$\|x\|_{l^p_{q,r}}^r = \sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}\cap\{0\}} |x_k|^p \right)^{r/p}. \tag{27}$$

Notice that $k \in S_{m,N} \cap \{0\}$ if and only if $m \in \{-N, \cdots, N\}$. Thus, the right--hand side of (27) can be expressed as

$$\sum_{N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \sum_{m=-N}^{N} 1 = \sum_{N=0}^{\infty} (2N+1)^{1+r\left(\frac{1}{q}-\frac{1}{p}\right)}$$

and this series converges if $1 + r\left(\frac{1}{q} - \frac{1}{p}\right) < -1$, that is, if $r > \dfrac{2}{\frac{1}{p} - \frac{1}{q}}$.

Example. Let $1 < r \le p < q < \infty$. Define $x = (x_k)_{k\in\mathbb{Z}}$ by

$$x_k = \begin{cases} |k|^{-1/q} & \text{if} \quad k \ne 0, \\ 1 & \text{if} \quad k = 0. \end{cases}$$

Then,

$$x \in l^p_q \setminus l^p_{q,r}.$$

In [6], it is proved that $x \in l^p_q$. Now, let us fix $N \in \mathbb{N}_0$ and observe that

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{r/p} \\ &\ge (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k=-N}^{N} |x_k|^p \right)^{r/p} \\ &= (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in\{-N,\cdots,N\}-\{0\}} |k|^{-p/q} \right)^{r/p} \\ &\ge (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} N^{-r/q} \sim N^{-r/p}, \end{aligned}$$

this implies that

$$\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} (2N+1)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{r/p} \ge C \sum_{N=0}^{\infty} \frac{1}{N^{r/p}} = \infty,$$

or, $x \notin l^p_{q,r}$.

3. Equivalent norms

In this section, we introduce a couple of norms in the discrete Bourgain--Morrey space that will be equivalent to the original norm given in $l^p_{q,r}$. As a consequence, we will also obtain equivalent norms in the discrete Morrey space l^p_q.

For $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ we define the dyadic interval of integers

$$\begin{aligned} I(j,k) &= \left[2^j k, 2^j (k+1)\right) \cap \mathbb{Z} \\ &= \left\{2^j k, 2^j k+1, \cdots, 2^j (k+1) - 1\right\}. \end{aligned}$$

The set of all dyadic intervals as defined above is denoted by $\mathcal{I}$.

The family of sets $S_{m,N}$, varying $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ will be denoted by $\mathcal{S}$, and the subfamily of $\mathcal{S}$ whose elements are the sets $S_{m,2^N}$, with $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ will be denoted by $\mathcal{S}_{\mathcal{I}}$.

Let $1 \le p \le q < \infty$, $1 \le r < \infty$, and $x \in l^p_{q,r}$. We define

$$\|x\|^{\mathcal{I}}_{l^p_{q,r}} = \left[\sum_{k\in\mathbb{Z},\, j\in\mathbb{N}_0} \left(2^j\right)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{l\in I(j,k)} |x_l|^p\right)^{r/p}\right]^{1/r},$$

and

$$\|x\|^{\mathcal{S}_{\mathcal{I}}}_{l^p_{q,r}} = \left[\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \left(2^{N+1}+1\right)^{r\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\sum_{l\in S_{m,2^N}} |x_l|^p\right)^{r/p}\right]^{1/r}.$$

Theorem 3.1. *The norms $\|x\|_{l^p_{q,r}}$, $\|x\|^{\mathcal{I}}_{l^p_{q,r}}$ and $\|x\|^{\mathcal{S}_{\mathcal{I}}}_{l^p_{q,r}}$ are equivalent.*

Proof. Let us first show that $\|x\|_{l^p_{q,r}}$ and $\|x\|^{\mathcal{S}_{\mathcal{I}}}_{l^p_{q,r}}$ are equivalent.

Let $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$. If $N = 0$, then

$$|x_m| = \frac{1}{|S_{m,0}|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,0}} |x_l|^p\right)^{1/p} \le \left(\frac{3}{|S_{m,2^0}|}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{l\in S_{m,2^0}} |x_l|^p\right)^{1/p}. \qquad (28)$$

For $N \ge 1$, let $f(N) = \lceil \log_2 N \rceil$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. Thus,

$$\log_2 N \le f(N) \le \log_2 2N \quad \text{or} \quad N \le 2^{f(N)} \le 2N.$$

This implies that

$$S_{m,N} \subset S_{m,2^{f(N)}}$$

and

$$2N+1 \le 2\cdot 2^{f(N)} + 1 \le 4N+1.$$

Hence,

$$\left(\frac{2N+1}{2N+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,N}} |x_l|^p\right)^{1/p} \leq \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,2^{f(N)}}} |x_l|^p\right)^{1/p}$$

or

$$\left(\frac{2N+1}{\left|S_{m,2^{f(N)}}\right|}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,N}} |x_l|^p\right)^{1/p} \leq \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,2^{f(N)}}} |x_l|^p\right)^{1/p},$$

then,

$$\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,N}} |x_l|^p\right)^{1/p} \leq \left(\frac{\left|S_{m,2^{f(N)}}\right|}{2N+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,2^{f(N)}}} |x_l|^p\right)^{1/p}.$$

Since for every $N \in \mathbb{N}$

$$1 = \frac{2N+1}{2N+1} \leq \frac{\left|S_{m,2^{f(N)}}\right|}{2N+1} \leq \frac{4N+1}{2N+1} \leq 2,$$

we have

$$1 \leq \left(\frac{\left|S_{m,2^{f(N)}}\right|}{2N+1}\right)^{\frac{1}{p}-\frac{1}{q}} \leq 2^{\frac{1}{p}-\frac{1}{q}},$$

and so,

$$\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,N}} |x_l|^p\right)^{1/p} \leq 2^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{m,2^{f(N)}}} |x_l|^p\right)^{1/p}. \quad (29)$$

Using (28) and (29), we get

$$\begin{aligned} \|x\|_{l^p_{q,r}} &= \left(\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{(2N+1)^{\frac{r}{p}-\frac{r}{q}}} \left(\sum_{l\in S_{m,N}} |x_l|^p\right)^{r/p}\right)^{1/r} \\ &\leq 3^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{\left|S_{m,2^{f(N)}}\right|^{\frac{r}{p}-\frac{r}{q}}} \left(\sum_{l\in S_{m,2^{f(N)}}} |x_l|^p\right)^{r/p}\right)^{1/r} \qquad (30) \\ &\leq 3^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} \frac{1}{\left|S_{m,2^N}\right|^{\frac{r}{p}-\frac{r}{q}}} \left(\sum_{l\in S_{m,2^N}} |x_l|^p\right)^{r/p}\right)^{1/r} = 3^{\frac{1}{p}-\frac{1}{q}} \|x\|^{\mathcal{S}_\mathcal{I}}_{l^p_{q,r}}. \end{aligned}$$

Clearly,

$$\|x\|_{l^p_{q,r}}^{\mathcal{S}_{\mathcal{I}}} \leq \|x\|_{l^p_{q,r}} ,$$

thus, we have proved the desired equivalence.

Next, we show the equivalence of the norms $\|x\|_{l^p_{q,r}}^{\mathcal{S}_{\mathcal{I}}}$ and $\|x\|_{l^p_{q,r}}^{\mathcal{I}}$. For $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, notice that

$$\left(2^j k + 2^{j-1}\right) - 2^{j-1} = 2^j k \quad \text{and} \quad \left(2^j k + 2^{j-1}\right) + 2^{j-1} = 2^j \left(k+1\right),$$

so,

$$I\left(j,k\right) \subset S_{2^j k+2^{j-1}, 2^{j-1}},$$

and

$$2^j = \left|I\left(j,k\right)\right| \leq \left|S_{2^j k+2^{j-1}, 2^{j-1}}\right| = 2^j + 1.$$

Thus,

$$\left(\frac{2^j}{2^j}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left(2^j+1\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in I(j,k)} |x_l|^p\right)^{1/p} \leq \frac{1}{\left(2^j+1\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{2^j k+2^{j-1}, 2^{j-1}}} |x_l|^p\right)^{1/p},$$

or

$$\left(\frac{2^j}{2^j+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left(2^j\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in I(j,k)} |x_l|^p\right)^{1/p} \leq \frac{1}{\left(2^j+1\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{2^j k+2^{j-1}, 2^{j-1}}} |x_l|^p\right)^{1/p}. \tag{31}$$

On the other hand, since for each $j \in \mathbb{N}_0$

$$\frac{1}{2} \leq \frac{2^j}{2^j+1} \leq 1, \quad \text{we obtain} \quad \left(\frac{2^j+1}{2^j}\right)^{\frac{1}{p}-\frac{1}{q}} \leq 2^{\frac{1}{p}-\frac{1}{q}},$$

and therefore, for every $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$, by (31) we obtain

$$\frac{1}{\left(2^j\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in I(j,k)} |x_l|^p\right)^{1/p} \leq 2^{\frac{1}{p}-\frac{1}{q}} \frac{1}{\left(2^j+1\right)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l\in S_{2^j k+2^{j-1}, 2^{j-1}}} |x_l|^p\right)^{1/p}. \tag{32}$$

This implies that

$$\begin{aligned}
\|x\|_{l^p_{q,r}}^{\mathcal{I}} &= \left[\sum_{k\in\mathbb{Z},\, j\in\mathbb{N}_0} \frac{1}{\left(2^j\right)^{\frac{r}{p}-\frac{r}{q}}} \left(\sum_{l\in I(j,k)} |x_l|^p\right)^{r/p}\right]^{1/r} \\
&\leq 2^{\frac{1}{p}-\frac{1}{q}} \left[\sum_{k\in\mathbb{Z},\, j\in\mathbb{N}_0} \frac{1}{\left(2^j+1\right)^{\frac{r}{p}-\frac{r}{q}}} \left(\sum_{l\in S_{2^j k+2^{j-1}, 2^{j-1}}} |x_l|^p\right)^{r/p}\right]^{1/r} \\
&\leq 2^{\frac{1}{p}-\frac{1}{q}} \|x\|_{l^p_{q,r}}^{\mathcal{S}_{\mathcal{I}}}.
\end{aligned}$$

Now, let

$$m \in \mathbb{Z} \quad \text{and} \quad N \in \mathbb{N}_0.$$

Observe that

$$\left| S_{m,2^N} \right| = 2^{N+1} + 1,$$

and the family of sets $\{I(N+1,k) : k \in \mathbb{Z}\}$ partitions $\mathbb{Z}$ into groups of 2^{N+1} consecutive integers. This implies that there exists at least one integer s, and at most two such that

$$s = 2^{N+1} l \in S_{m,2^N}, \quad \text{for some} \quad l \in \mathbb{Z}.$$

If we denote by $s(m,N) = 2^{N+1} l(m,N) \in S_{m,2^N}$ the largest integer satisfying this condition, we will have

$$S_{m,2^N} \subset I\big(N+1, l(m,N)\big) \cup I\big(N+1, l(m,N)-1\big) =: A(m,N),$$

and

$$|A(m,N)| = 2^{N+2}.$$

Therefore, for any $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$

$$\left(\frac{2^{N+1}+1}{2^{N+1}+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{|A(m,N)|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in S_{m,2^N}} |x_l|^p \right)^{1/p} \leq \frac{1}{|A(m,N)|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in A(m,N)} |x_l|^p \right)^{1/p}$$

$$= \frac{1}{|A(m,N)|^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in I(N+1,l(m,N)-1)} |x_l|^p + \sum_{l \in I(N+1,l(m,N))} |x_l|^p \right)^{1/p}$$

$$\leq \frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in I(N+1,l(m,N)-1)} |x_l|^p \right)^{1/p} + \frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in I(N+1,l(m,N))} |x_l|^p \right)^{1/p},$$

which implies

$$\frac{1}{(2^{N+1}+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in S_{m,2^N}} |x_l|^p \right)^{1/p}$$

$$\leq \left(\frac{|A(m,N)|}{2^{N+1}+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in I(N+1,l(m,N)-1)} |x_l|^p \right)^{1/p}$$

$$+ \left(\frac{|A(m,N)|}{2^{N+1}+1}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{l \in I(N+1,l(m,N))} |x_l|^p \right)^{1/p}.$$

Observing that

$$1 \leq \frac{|A(m,N)|}{2^{N+1}+1} \leq 2 \quad \text{or} \quad 1 \leq \left(\frac{|A(m,N)|}{2^{N+1}+1}\right)^{\frac{1}{p}-\frac{1}{q}} \leq 2^{\frac{1}{p}-\frac{1}{q}},$$

we conclude that

$$\begin{aligned}
&\frac{1}{(2^{N+1}+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in S_{m,2^N}}|x_l|^p\right)^{1/p} \\
&\le 2^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in I(N+1,l(m,N)-1)}|x_l|^p\right)^{1/p}\right) \\
&\quad + 2^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in I(N+1,l(m,N))}|x_l|^p\right)^{1/p}\right). \qquad (33)
\end{aligned}$$

Using (33), we obtain

$$\begin{aligned}
\|x\|_{l^p_{q,r}}^{\mathcal{S}_\mathcal{I}} &= \left(\sum_{m\in\mathbb{Z},\,N\in\mathbb{N}_0}\frac{1}{(2^{N+1}+1)^{\frac{r}{p}-\frac{r}{q}}}\left(\sum_{l\in S_{m,2^N}}|x_l|^p\right)^{r/p}\right)^{1/r} \\
&\le \left[\sum_{m\in\mathbb{Z},\,N\in\mathbb{N}_0}\left[2^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in I(N+1,l(m,N)-1)}|x_l|^p\right)^{1/p}\right)\right.\right. \\
&\quad \left.\left. + 2^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{(2^{N+1})^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in I(N+1,l(m,N))}|x_l|^p\right)^{1/p}\right)\right]^r\right]^{1/r}. \qquad (34)
\end{aligned}$$

Now, according to Minkowski's inequality, expression (34) can be estimated by

$$\begin{aligned}
&2^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{m\in\mathbb{Z},\,N\in\mathbb{N}_0}\frac{1}{(2^{N+1})^{\frac{r}{p}-\frac{r}{q}}}\left(\sum_{l\in I(N+1,l(m,N)-1)}|x_l|^p\right)^{r/p}\right)^{1/r} \\
&\quad + 2^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{m\in\mathbb{Z},\,N\in\mathbb{N}_0}\frac{1}{(2^{N+1})^{\frac{r}{p}-\frac{r}{q}}}\left(\sum_{l\in I(N+1,l(m,N))}|x_l|^p\right)^{r/p}\right)^{1/r} \\
&\le 2^{\frac{1}{p}-\frac{1}{q}}\|x\|_{l^p_{q,r}}^{\mathcal{I}} + 2^{\frac{1}{p}-\frac{1}{q}}\|x\|_{l^p_{q,r}}^{\mathcal{I}} \\
&= 2^{1+\frac{1}{p}-\frac{1}{q}}\|x\|_{l^p_{q,r}}^{\mathcal{I}},
\end{aligned}$$

as we wanted to prove. □

Reminding that for $r=\infty$, $l^p_{q,r}=l^p_q$ and defining

$$\|x\|^{\mathcal{I}}_{l^p_q}=\sup_{k\in\mathbb{Z},\, j\in\mathbb{N}_0}\left(2^j\right)^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{l\in I(j,k)}|x_l|^p\right)^{1/p},$$

$$\|x\|^{\mathcal{S}_\mathcal{I}}_{l^p_q}=\sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0}\left(2^{N+1}+1\right)^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{l\in S_{m,2^N}}|x_l|^p\right)^{1/p},$$

we have the following result.

Corollary 3.2. *The norms $\|x\|_{l^p_q}$, $\|x\|^{\mathcal{I}}_{l^p_q}$ and $\|x\|^{\mathcal{S}_\mathcal{I}}_{l^p_q}$ are equivalent on the discrete Morrey spaces.*

P r o o f. From (28) and (29) we get

$$\|x\|_{l^p_q}\le 3^{\frac{1}{p}-\frac{1}{q}}\|x\|^{\mathcal{S}_\mathcal{I}}_{l^p_q}\le 3^{\frac{1}{p}-\frac{1}{q}}\|x\|_{l^p_q}.$$

Now, using (32), and (33), we arrive at

$$\|x\|^{\mathcal{I}}_{l^p_q}\le 2^{\frac{1}{p}-\frac{1}{q}}\|x\|^{\mathcal{S}_\mathcal{I}}_{l^p_q}\le 2^{\frac{1}{p}-\frac{1}{q}}2^{1+\frac{1}{p}-\frac{1}{q}}\|x\|^{\mathcal{I}}_{l^p_q},$$

and this ends our proof. □

Remark 1. Using the equivalence of norms on $l^p_{q,r}$, we can show that the sequence given in Example 2 belongs to $l^p_{q,r}$ for every $1\le r<\infty$.

Indeed,

$$\|x\|^{\mathcal{I}}_{l^p_{q,r}}=\left[\sum_{k\in\mathbb{Z},\, j\in\mathbb{N}_0}\frac{1}{|I(j,k)|^{\frac{r}{p}-\frac{r}{q}}}\left(\sum_{l\in I(j,k)}|x_l|^p\right)^{r/p}\right]^{1/r}$$

$$=\left[\sum_{j\in\mathbb{N}_0}\frac{1}{|I(j,0)|^{\frac{r}{p}-\frac{r}{q}}}\left(\sum_{l\in I(j,0)}|x_l|^p\right)^{r/p}\right]^{1/r}$$

$$=\left[\sum_{j\in\mathbb{N}_0}\left(2^j\right)^{\frac{r}{q}-\frac{r}{p}}(1)^{r/p}\right]^{1/r}=\left[\sum_{j=0}^{\infty}(2^{\frac{r}{q}-\frac{r}{p}})^j\right]^{1/r}$$

and this series converges if $p<q$.

4. The dyadic maximal operator

In this section, we will examine the behavior of the dyadic version of Hardy-Littlewood maximal operator on the spaces $l^p_{q,r}$. We will consider the following dyadic version of the maximal operator: for a sequence $x = (x_k)_{k\in\mathbb{Z}}$ we define

$$M_{\mathcal{I}}(x)(k) = \sup_{k\in I\in\mathcal{I}} \frac{1}{|I|}\sum_{l\in I}|x_l|$$

for each $k \in \mathbb{Z}$. We will prove the boundedness of this operator on the spaces $l^p_{q,r}$. We will take advantage of the fact that we can use the dyadic norm $\|\cdot\|^{\mathcal{I}}_{l^p_{q,r}}$ on $l^p_{q,r}$. We also closely follow the proof given by [12].

Theorem 4.1. *Let $1 < p < q < r < \infty$. Then, the maximal operator $M_{\mathcal{I}}$ is bounded on $l^p_{q,r}$.*

Proof. Let $x \in l^p_{q,r}$. For each $I \in \mathcal{I}$ define

$$x_I^{(1)} = x\chi_I \quad \text{and} \quad x_I^{(2)} = x\chi_{\mathbb{Z}-I}.$$

Clearly, $x_I^{(1)} \in l^p$ and

$$\left(\frac{1}{|I|}\sum_{m\in I}\left|M_{\mathcal{I}}x_I^{(1)}(m)\right|^p\right)^{1/p} \le C_p\left(\frac{1}{|I|}\sum_{m\in I}|x(m)|^p\right)^{1/p}.$$

Then,

$$\left(\sum_{I\in\mathcal{I}}|I|^{\frac{r}{q}-\frac{r}{p}}\left(\sum_{m\in I}\left|M_{\mathcal{I}}\left(x_I^{(1)}\right)(m)\right|^p\right)^{r/p}\right)^{1/r} \le C_p\left(\sum_{I\in\mathcal{I}}|I|^{\frac{r}{q}-\frac{r}{p}}\left(\sum_{m\in I}|x(m)|^p\right)^{r/p}\right)^{1/r}. \tag{35}$$

On the other hand, for $k \in \mathbb{N}_0$ let $P_k(I) \in \mathcal{I}$ be the kth parent of I, that is,

$$P_k(I) \supset I \quad \text{and} \quad \left|P_k(I)\right| = 2^k|I|.$$

Now, take an arbitrary $m \in I$. Notice that

$$M_{\mathcal{I}}x_I^{(2)}(m) = \sup_{m\in R\in\mathcal{I}}\frac{1}{|R|}\sum_{s\in R}\left|x_I^{(2)}(s)\right| = \sup_{k\in\mathbb{N}}\frac{1}{|P_k(I)|}\sum_{s\in P_k(I)}\left|x_I^{(2)}(s)\right|$$

because for any $P \in \mathcal{I}$ such that $P \subset I$ we have

$$\frac{1}{|P|}\sum_{s\in P}\left|x_I^{(2)}(s)\right| = 0.$$

Thus, for any $m \in I$

$$\begin{aligned} M_{\mathcal{I}} x_I^{(2)}(m) &= \sup_{k\in\mathbb{N}} \frac{1}{|P_k(I)|} \sum_{s\in P_k(I)} \left| x_I^{(2)}(s) \right| \\ &\le \sup_{k\in\mathbb{N}} \left(\frac{1}{|P_k(I)|} \sum_{s\in P_k(I)} \left| x_I^{(2)}(s) \right|^p \right)^{1/p} \\ &\le \sum_{k=1}^{\infty} \left(\frac{1}{|P_k(I)|} \sum_{s\in P_k(I)} |x(s)|^p \right)^{1/p} =: R(I), \end{aligned}$$

where $R(I) \in [0, \infty]$. This implies that

$$\left(\frac{1}{|I|} \sum_{m\in I} \left| M_{\mathcal{I}} x_I^{(2)}(m) \right|^p \right)^{1/p} \le R(I)$$

and so,

$$\begin{aligned} &\left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} \left| M_{\mathcal{I}} \left(x_I^{(2)} \right)(m) \right|^p \right)^{r/p} \right)^{1/r} \\ &\le \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} R(I) \right)^r \right)^{1/r} \\ &= \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{|P_k(I)|} \sum_{m\in P_k(I)} |x(m)|^p \right)^{1/p} \right)^r \right)^{1/r}. \end{aligned} \tag{36}$$

Next, for an arbitrary and fixed $k \in \mathbb{N}$, let us consider any interval $R \in \mathcal{I}$. If R turns out to be the kth parent of some interval $I^* \in \mathcal{I}$, then there exist additionally $2^k - 1$ intervals $I \in \mathcal{I}$ such that R is also the kth parent, that is, $R = P_k(I)$, and hence

$$\sum_{I\in\mathcal{I},\ R=P_k(I)} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} = 2^k |R|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in R} |x(m)|^p \right)^{r/p}.$$

If R is not the kth parent of some interval in $\mathcal{I}$, then

$$\sum_{I\in\mathcal{I},\ R=P_k(I)} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} = 0 \le 2^k |R|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in R} |x(m)|^p \right)^{r/p}.$$

In any case, we obtain by summing over all $R \in \mathcal{I}$

$$\begin{aligned} \sum_{I\in\mathcal{I}} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} &\le 2^k \sum_{R\in\mathcal{I}} |R|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in R} |x(m)|^p \right)^{r/p} \\ &= 2^k \left(\|x\|_{l^p_{q,r}}^{\mathcal{I}} \right)^r. \end{aligned} \tag{37}$$

Now, since $x = x_I^{(1)} + x_I^{(2)}$, we get

$$\|M_{\mathcal{I}}x\|_{l^p_{q,r}}^{\mathcal{I}} = \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} |M_{\mathcal{I}}x\,(m)|^p\right)^{r/p}\right)^{1/r}$$

$$= \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} \left|M_{\mathcal{I}}\left(x_I^{(1)} + x_I^{(2)}\right)(m)\right|^p\right)^{r/p}\right)^{1/r}. \tag{38}$$

Next, by subadditivity of the maximal function and the triangle inequality

$$\left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} \left|M_{\mathcal{I}}\left(x_I^{(1)} + x_I^{(2)}\right)(m)\right|^p\right)^{r/p}\right)^{1/r}$$

$$\leq \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} \left|M_{\mathcal{I}}\left(x_I^{(1)}\right)(m)\right|^p\right)^{r/p}\right)^{1/r}$$

$$+ \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} \left|M_{\mathcal{I}}\left(x_I^{(2)}\right)(m)\right|^p\right)^{r/p}\right)^{1/r}, \tag{39}$$

then, by (35) and (36), the right-hand side of the last inequality can be estimated by

$$C_p \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in I} |x\,(m)|^p\right)^{r/p}\right)^{1/r}$$

$$+ \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{|P_k\,(I)|} \sum_{m\in P_k(I)} |x\,(m)|^p\right)^{1/p}\right)^r\right)^{1/r}$$

$$= C_p \|x\|_{l^p_{q,r}}^{\mathcal{I}}$$

$$+ \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{|P_k\,(I)|} \sum_{m\in P_k(I)} |x\,(m)|^p\right)^{1/p}\right)^r\right)^{1/r}. \tag{40}$$

Therefore, by (38), (39) and (40) we obtain

$$\|M_{\mathcal{I}}x\|_{l^p_{q,r}}^{\mathcal{I}} \leq C_p \|x\|_{l^p_{q,r}}^{\mathcal{I}} + \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{|P_k\,(I)|} \sum_{m\in P_k(I)} |x\,(m)|^p\right)^{1/p}\right)^r\right)^{1/r}. \tag{41}$$

Furthermore, applying Minkowski's inequality, we get

$$C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \left(\sum_{I\in\mathcal{I}} \left(|I|^{\frac{1}{q}} \sum_{k=1}^{\infty} \left(\frac{1}{|P_k(I)|} \sum_{m\in P_k(I)} |x(m)|^p \right)^{1/p} \right)^r \right)^{1/r}$$

$$\leq C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}} \left(\frac{1}{|P_k(I)|} \sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} \right)^{1/r} . \tag{42}$$

Moreover, since $|I| = 2^{-k} |P_k(I)|$, it holds

$$\sum_{k=1}^{\infty} \left(\sum_{I\in\mathcal{I}} |I|^{\frac{r}{q}} \left(\frac{1}{|P_k(I)|} \sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} \right)^{1/r}$$

$$= \sum_{k=1}^{\infty} 2^{-\frac{k}{q}} \left(\sum_{I\in\mathcal{I}} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} \right)^{1/r},$$

and using the last equality, we obtain that expression (42) can be written as

$$C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{-\frac{k}{q}} \left(\sum_{I\in\mathcal{I}} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} \right)^{1/r} . \tag{43}$$

In addition, by (37),

$$C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{-\frac{k}{q}} \left(\sum_{I\in\mathcal{I}} |P_k(I)|^{\frac{r}{q}-\frac{r}{p}} \left(\sum_{m\in P_k(I)} |x(m)|^p \right)^{r/p} \right)^{1/r}$$

$$\leq C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{-\frac{k}{q}} \left[2^k \left(\|x\|^{\mathcal{I}}_{l^p_{q,r}} \right)^r \right]^{1/r} . \tag{44}$$

Therefore, combining (41), (42), (43) and (44) we obtain

$$\|M_{\mathcal{I}} x\|^{\mathcal{I}}_{l^p_{q,r}} \leq C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{-\frac{k}{q}} \left[2^k \left(\|x\|^{\mathcal{I}}_{l^p_{q,r}} \right)^r \right]^{1/r}$$

$$= C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{\frac{k}{r}-\frac{k}{q}} \|x\|^{\mathcal{I}}_{l^p_{q,r}} . \tag{45}$$

Finally, since $q < r$, we conclude

$$\begin{aligned} \|M_{\mathcal{I}} x\|^{\mathcal{I}}_{l^p_{q,r}} &\leq C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \sum_{k=1}^{\infty} 2^{\frac{k}{r} - \frac{k}{q}} \|x\|^{\mathcal{I}}_{l^p_{q,r}} \\ &= C_p \|x\|^{\mathcal{I}}_{l^p_{q,r}} + \|x\|^{\mathcal{I}}_{l^p_{q,r}} \sum_{k=1}^{\infty} \left(2^{\frac{1}{r} - \frac{1}{q}}\right)^k \\ &\leq C_{p,q,r} \|x\|^{\mathcal{I}}_{l^p_{q,r}} . \end{aligned} \tag{46}$$

This completes the proof. □

REFERENCES

[1] ABE, Y.—SAWANO, Y.: *Littlewood-Paley characterization of discrete Morrey spaces and its applications to the discrete martingale transform*, Matematiche **LXXVIII** (2023), no. 2, 337–358.

[2] ALIEV, R. A.—AHMADOVA, A. N.: *Boundedness of discrete Hilbert transform on discrete Morrey spaces*, Ufa Math. J. **13** (2021), 98–109.

[3] BOURGAIN, J.: *On the restriction and multiplier problems in* $\mathbb{R}^3$, in: Geometric Aspects of Functional Analysis (1989–1990), in: Lecture Notes in Math. Vol. 1469, Springer, Berlin, 1991, pp. 179–191.

[4] DIARRA, N: *Hardy-Littlewood-Sobolev Theorem for Bourgain-Morrey Spaces and Approximation*, European J. Math. Anal. **4** (2024) 16, doi: 10.28924/ada/ma.4.16.

[5] GUNAWAN, H.—KIKIANTY, E.—SAWANO, Y.—SCHWANKE, C.: *Three geometric constants for Morrey spaces*, Bull. Korean Math. Soc. **56** (2019), 1569–1575

[6] GUNAWAN, H.—KIKIANTY, E.—SCHWANKE, C.: *Discrete Morrey spaces and their inclusion properties*, Math. Nachr. **291** (2018), 1283–1296.

[7] GUNAWAN, H.—SCHWANKE, C.: *The Hardy-Littlewood maximal operator on discrete Morrey spaces*, Mediterr. J. Math. 16 (2019), paper. no. 24, 12 pp.

[8] GUZMÁN-PARTIDA, M.: *Boundedness and compactness of some operators on discrete Morrey spaces*, Comment. Math. Univ. Carolin. **62** (2021), 151–158.

[9] GUZMÁN-PARTIDA, M.—SAN MARTÍN, L.—VILLEGAS-ACUÑA, A.: *Vainikko operator on discrete Morrey spaces*, Rev. Colombiana Mat. **57** (2023), no. 2, 179–191.

[10] HAROSKE, D.—SKRZYPCZAK, L.: *Morrey sequence spaces: Pitt's theorem and compact embeddings*, Constr. Approx. **51** (2020), no. 3, 505–535.

[11] HAROSKE, D.—SKRZYPCZAK, L.: *Nuclear embeddings of Morrey sequence spaces and smoothness Morrey spaces*, Bull. Malays. Math. Sci. Soc. **47** (2024), paper no. 111, 34 pp.

[12] HATANO, N.—NOGAYAMA, T.—SAWANO, Y.—HAKIM, D.: *Bourgain-Morrey spaces and their applications to boundedness of operators*, J. Funct. Anal. **284** (2023), paper no. 109720; `https://doi.org/10.1016/j.jfa.2022.109720`.

[13] KIKIANTY, E.—SCHWANKE, C.: *Discrete Morrey spaces are closed subspaces of their continuous counterparts*, Function Spaces **XII**, Banach Center Publ. **119** (2019), 223–231.

[14] MASAKI, S.—SEGATA, J.: *Existence of a minimal non-scatering solution to the mass-subcritical generalized Korteweg-de Vries equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **35** (2018), no. 2, 283–326; DOI 10.1016/J.ANIHPC.2017.04.003

Received October 29, 2024
Revised December 17, 2024
Accepted April 13, 2025
Publ. online September 20, 2025

M. Guzmán-Partida
Luis San Martín
Alejandro Villegas-Acuña
Departamento de Matemáticas
Universidad de Sonora
Hermosillo, Sonora 83000
MÉXICO

E-mail: martha@mat.uson.mx
luisrene.sanmartin@unison.mx
a213202325@unison.mx