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ESTIMATION OF MA(1) MODEL BASED ON ROUNDED DATA

Meihui Guo — Gen-Liang Li

ABSTRACT. Most recorded data of continuous distributions are rounded to the nearest decimal place due to the precision of the recording mechanism. This rounding entails errors in estimation and measurement. In this study, we consider parameter estimation of time series models based on rounded data. The adjusted maximum likelihood estimates in [Stam, A.—Cogger, K. O.: *Rounding errors in autoregressive processes*, Internat. J. Forecast. **9** (1993), 487–508] are derived theoretically for the first order moving average MA(1) model. Simulations are performed to compare the efficiencies of the adjusted maximum likelihood estimators with other estimators.

1. Introduction

Most recorded data are rounded to the nearest decimal place due to precision of the recording mechanism. This rounding entails errors in estimation and measurement. For example, in 1982 the Vancouver stock exchange (VSE) initialized the stock market index at 1000. After 22 months of recomputing the index and truncating to three decimal places at each change in market value, the value of the index becomes 524.811, regardless of the fact that the "true" value is 1098.892 (McCullough and Vinod, 1999). Another example is the ineffectiveness of the Patriot missile defense system during the Gulf War (August, 1990–February, 1991) caused by the rounding error of its integer timing register (Skeel, 1992). The round-off error sometimes is a major source of measurement errors and distorts statistical inference. In the earlier days, the rounding errors were not regarded as a serious problem because the sample size was small and was tolerable

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relatively to the statistical problem. However, the technologies of nowadays develop rapidly, which makes it possible to collect, store and analyze data sets of huge size in fields such as biological sciences, finance and wireless communications, etc. The effect of rounding errors inevitably have to be considered in the statistical analysis of these huge data sets.

The earliest study on the rounding error for independent identically distributed (i.i.d.) sample is back to Sheppard (1897) who proposed the maximum-likelihood correction for the sample covariance estimator. See also Tricker (1984, 1990A,B, 1992), Dempster and Robin (1983), Hall (1982), Heitjan and Rubin (1991), Lee and Vardeman (2001, 2002). For dependent rounded sample, Stam and Cogger (1993) studied the adjusted maximum likelihood estimate for the autoregressive (AR) time series models. Bai, et al., (2009) proposed the approximate maximum likelihood estimation for time series models and proved strong consistency and asymptotic normality of the estimators. Zhang, et al., (2010) proposed the short overlapping series estimator for α-mixing models. In this study, we consider the parameter estimation of the first order moving average MA(1) time series models based on rounded data. The adjusted maximum likelihood estimates of the MA(1) models are derived theoretically. Simulation study is performed to compare the efficiencies of the adjusted maximum likelihood estimators with other estimators for the MA(1) models.

The paper is organized as follows. In Section 2, we introduce the adjusted maximum likelihood estimates for the AR models. In Section 3, we derive the adjusted maximum likelihood estimates for the MA(1) models. Simulation results and discussion are given in Section 4.

2. Adjusted MLE of the AR models

In this section, we introduce the adjusted maximum likelihood estimate of Stam and Cogger (1993). Consider the following AR(p) model

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + \varepsilon_t, \qquad (1)$$

where

$$\varepsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2) \qquad \text{for } t = 1, \ldots, n$$

and

$$\beta = (\mu, \Phi, \sigma^2), \qquad \Phi = (\phi_1, \ldots, \phi_p)$$

represent the parameter vector.

The rounded version of $\mathbf{X} = (X_1, \ldots, X_n)$ is denoted by

$$\mathbf{Y} = (Y_1, \ldots, Y_n),$$

where

$$Y_t = y_t \quad \text{if and only if} \quad X_t = x_t$$

and

$$y_t - \frac{h}{2} \le x_t < y_t + \frac{h}{2}, \qquad t = 1, \ldots, n,$$

and h is the width of the rounding interval. Herein, we assume

$$Y_t = X_t + U_t, \tag{2}$$

and the rounding errors U_t's are i.i.d $\mathrm{Unif}\left[-\frac{h}{2}, \frac{h}{2}\right]$ random variables. If the exact data $\mathbf{x} = (x_1, \ldots, x_n)$ are available, the maximum likelihood estimator (MLE) of β is obtained by maximizing the joint probability density function (pdf) $f(\mathbf{x}; \beta)$ of $\mathbf{x}$ with respect to β. In most situations, only the rounded data $\mathbf{y} = (y_1, \ldots, y_n)$ can be observed. The pseudo MLE (PMLE) $\hat{\beta}_0$ of β is obtained by ignoring the rounding effect and solving the following normal equations

$$\frac{\partial \ln f(\mathbf{y}; \beta)}{\partial \beta} = 0. \tag{3}$$

If the rounding effect is considered, then by (2) the likelihood function of β based on the rounded data $\mathbf{y}$ is

$$L(\beta; \mathbf{y}) = h^{-n} \int\limits_{y_n - h/2}^{y_n + h/2} \ldots \int\limits_{y_1 - h/2}^{y_1 + h/2} f(\mathbf{u}; \beta)\, \mathrm{d}u_1 \ldots \mathrm{d}u_n. \tag{4}$$

The MLE $\hat{\beta}$ of β based on the rounded data is obtained by maximizing the likelihood function $L(\beta; \mathbf{y})$ which is usually intractable due to its n-fold integrals. Lindley (1950) derived the Maclaurin expansion of the likelihood function $L(\beta; \mathbf{y})$ at $h = 0$ when f is a univariate distribution function. Tallis (1967) extended the Maclaurin expansion to the following multivariate case

$$L(\beta; \mathbf{y}) = f(\mathbf{y}; \beta) + \frac{h^2}{24} \sum_{t=1}^{n} \frac{\partial^2 f(\mathbf{y}; \beta)}{\partial y_t^2} + O\big(h^3\big). \tag{5}$$

The log-likelihood function can be approximated near $h = 0$ by

$$\ln L(\beta; \mathbf{y}) \approx \ln f(\mathbf{y}; \beta) + \frac{h^2}{24} \sum_{t=1}^{n} \left[\frac{\partial^2 f(\mathbf{y}; \beta)/\partial y_t^2}{f(\mathbf{y}; \beta)} \right] + O\big(h^3\big). \tag{6}$$

Using the PMLE $\hat{\beta}_0$ as the initial estimate, the estimate utilizing one iteration of the Newton-Raphson method is

$$\hat{\beta}_A = \hat{\beta}_0 - \mathbf{A}^{-1}\mathbf{b}, \tag{7}$$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{b} = [b_j]$ and

$$a_{ij} = \frac{\partial^2 \ln(L(\beta;\mathbf{y}))}{\partial\beta_i \partial\beta_j} \approx \left.\frac{\partial^2 \ln(f(\mathbf{y};\beta))}{\partial\beta_i \partial\beta_j}\right|_{\beta=\hat{\beta}_0}, \tag{8}$$

$$b_j = \frac{\partial \ln(L(\beta;\mathbf{y}))}{\partial\beta_j} \approx \left.\frac{h^2}{24}\frac{\partial}{\partial\beta_j}\sum_{t=1}^{n}\left[\frac{\partial^2 f(\mathbf{y};\beta)/\partial y_t^2}{f(\mathbf{y};\beta)}\right]\right|_{\beta=\hat{\beta}_0}, \tag{9}$$

where β_i denotes the ith element of $\beta, i, j = 1, \ldots, p+2$.
For the AR(1) model, the adjusted MLE's are

$$\hat{\mu}_A = \hat{\mu}_0, \quad \hat{\phi}_A = \hat{\phi}_0 + h^2\hat{\phi}_0\big(1-\hat{\phi}_0^2\big)/12\hat{\sigma}_0^2, \quad \hat{\sigma}_A^2 = \hat{\sigma}_0^2 - h^2\big(1+\hat{\phi}_0^2\big)/12\,,$$

where $\big(\hat{\mu}_0, \hat{\phi}_0, \hat{\sigma}_0^2\big)$ are the PMLE of $\big(\mu, \phi_1, \sigma^2\big)$.

3. Adjusted MLE of the MA(1) model

Assume $\mathbf{X} = (X_1, \ldots, X_n)$ follows the MA(1) model

$$X_t = \varepsilon_t - \theta\varepsilon_{t-1}, \qquad \varepsilon_t \overset{i.i.d.}{\sim} N\big(0, \sigma^2\big), \tag{10}$$

where $|\theta| < 1$. The joint pdf of $\mathbf{X} = \mathbf{x}$ is

$$f(\mathbf{x};\beta) = \big(2\pi\sigma^2\big)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{n}\left(\sum_{i=0}^{t-1}\theta^i x_{t-i}\right)^2\right\}, \tag{11}$$

where $\beta = \big(\theta, \sigma^2\big)$ is the parameter vector and assume $x_t = 0$, for $t \le 0$. The likelihood function of β based on the rounded data $\mathbf{y} = (y_1, \ldots, y_n)$ is the $L(\beta;\mathbf{y})$ defined in (4) with f given by (11), and the adjusted MLE's of β can be derived from (7), (8) and (9) with

$$\mathbf{A} \;=\; \begin{bmatrix} \dfrac{\partial^2 \ln f}{\partial\theta^2} & \dfrac{\partial^2 \ln f}{\partial\theta\partial\sigma^2} \\[2ex] \dfrac{\partial^2 \ln f}{\partial\theta\partial\sigma^2} & \dfrac{\partial^2 \ln f}{\partial(\sigma^2)^2} \end{bmatrix}_{\beta=\hat{\beta}_0}, \tag{12}$$

$$\mathbf{b}' \;= \frac{h^2}{24}\left(\sum_{t=1}^{n}\frac{\partial}{\partial\theta}\frac{\partial^2 f(\mathbf{y};\beta)/\partial y_t^2}{f(\mathbf{y};\beta)}, \sum_{t=1}^{n}\frac{\partial}{\partial\sigma^2}\frac{\partial^2 f(\mathbf{y};\beta)/\partial y_t^2}{f(\mathbf{y};\beta)}\right)_{\beta=\hat{\beta}_0}, \tag{13}$$

where $\hat{\beta}_0$ is the PMLE of β. The following Lemma 1–Lemma 2 are preliminaries of Theorem 3.

Lemma 1. *Let*

$$e_t=\sum_{i=0}^{t-1}\theta^i y_{t-i},\quad \dot e_t=\sum_{i=1}^{t-1} i\theta^{i-1}y_{t-i}\quad \textit{and}\quad \ddot e_t=\sum_{i=2}^{t-1} i(i-1)\theta^{i-2}y_{t-i}.$$

Then we have

(i) $\lim_{n\to\infty}\frac{1}{n}\sum_{t=2}^{n}\left(\dot e_t^2+e_t\ddot e_t\right)=\frac{(1+3\theta^2)\gamma_0^*+2\theta(3+\theta^2)\gamma_1}{(1-\theta^2)^3}$,

(ii) $\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sum_{i,j=0}^{n-t}\theta^{i+j}e_{t+i}\dot e_{t+j}=\frac{\theta(2+\theta^2)\gamma_0^*+(1+5\theta^2)\gamma_1}{(1-\theta^2)^4}$,

where

$$\gamma_0^*=\gamma_0+\frac{h^2}{12},\quad \gamma_0=\sigma^2\left(1+\theta^2\right)\quad \textit{and}\quad \gamma_1=-\theta\sigma^2.$$

P r o o f.

$$\frac{1}{n}\sum_{t=2}^{n}\left(\dot e_t^2+e_t\ddot e_t\right)=\frac{1}{n}\sum_{t=2}^{n}\sum_{l=1}^{t-1} l(2l-1)\theta^{2l-2}y_{t-l}^2 +\frac{2}{n}\sum_{t=2}^{n}\sum_{l=1}^{t-2} l(2l+1)\theta^{2l-1}y_{t-l}y_{t-l+1}+R_n,$$

where

$$R_n=\frac{1}{n}\sum_{k=2}^{n-1}\sum_{t=k}^{n}\sum_{l=1}^{t-k}a_{l,t}(\theta)y_{t-l}y_{t-l+k}$$

with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=k}^{n}\sum_{l=1}^{t-k}a_{l,t}(\theta)<\infty\qquad\text{for all }\ 2\le k\le n-1. \tag{14}$$

Since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=2}^{n}\sum_{l=1}^{t-1} l(2l-1)\theta^{2l-2}=\frac{1+3\theta^2}{(1-\theta^2)^3},$$

by Chebyshev's Theorem

$$\frac{1}{n}\sum_{t=2}^{n}\sum_{l=1}^{t-1} l(2l-1)\theta^{2l-2}y_{t-l}^2\ \underset{n\to\infty}{\longrightarrow}\ \frac{1+3\theta^2}{(1-\theta^2)^3}\gamma_0^*,\quad\text{in probability,} \tag{15}$$

where

$$\gamma_0^*=E\big(Y_i^2\big)=E(X_i+U_i)^2=\gamma_0+\frac{h^2}{12}$$

and

$$\gamma_0=E\big(X_i^2\big)=\sigma_0^2\big(1+\theta^2\big).$$

Similarly,

$$\frac{2}{n}\sum_{t=2}^{n}\sum_{l=1}^{t-2} l(2l+1)\theta^{2l-1} y_i y_{i+1} \underset{n\to\infty}{\longrightarrow} \frac{2\theta(3+\theta^2)}{(1-\theta^2)^3}\,\gamma_1^*, \quad \text{in probability}, \tag{16}$$

where

$$\gamma_1 = E(Y_t Y_{t+1}) = E(X_t X_{t+1}) = -\theta_0\sigma_0^2.$$

Finally, since

$$E(Y_t Y_{t+k}) = E(X_t + U_t)(X_{t+k} + U_{t+k}) = 0 \qquad \text{for} \quad k \ge 2,$$

by (14) and Chebyshev's Theorem we have

$$R_n \underset{n\to\infty}{\longrightarrow} 0, \quad \text{in probability}. \tag{17}$$

The result of (i) is obtained by (15), (16) and (17). By similar derivation, we have the result of (ii). $\square$

Lemma 2. *Let*

$$\hat\beta_0 = \big(\hat\theta_0, \hat\sigma_0^2\big)$$

be the PMLE of $\big(\theta, \sigma^2\big)$ *for the MA(1) model* (10), *then*

(i) $\lim_{n\to\infty} \frac{1}{n}\,\mathbf{A} = -\frac{1}{\hat\sigma_0^2}\operatorname{diag}\left(\frac{(1+3\hat\theta_0^2)(1+\hat\theta_0^2)\hat\sigma_0^2 + 2\hat\theta_0(3+\hat\theta_0^2)(-\hat\theta_0\hat\sigma_0^2)}{(1-\hat\theta_0^2)^3}, \frac{1}{2\hat\sigma_0^2}\right)$,

(ii) $\lim_{n\to\infty} \frac{1}{n}\,\mathbf{b}' = \frac{h^2}{12\hat\sigma_0^4}\left(\frac{\hat\theta_0}{(1-\hat\theta_0^2)^2}, -\frac{1}{2(1-\hat\theta_0^2)}\right)$.

Proof.

(i) By the normal equations (3), we have

$$-\frac{1}{\sigma^2}\sum_{t=2}^{n} e_t\dot e_t\bigg|_{\hat\beta_0} = 0, \qquad -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^{n} e_t^2\bigg|_{\hat\beta_0} = 0. \tag{18}$$

By (18), the second partial derivatives of $\ln f$ with respect to θ and σ^2 evaluated at $\hat\beta_0$ are

$$\frac{\partial^2 \ln f}{\partial(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{t=1}^{n} e_t^2 = -\frac{n}{2\hat\sigma_0^4}, \tag{19}$$

$$\frac{\partial^2 \ln f}{\partial\theta\partial\sigma^2} = \frac{1}{\sigma^4}\sum_{t=2}^{n} e_t\dot e_t = 0. \tag{20}$$

Furthermore, since

$$\partial^2 \ln f/\partial\theta^2 = -\frac{1}{\sigma^2}\sum_{t=2}^{n}\big(\dot e_t^2 + e_t\ddot e_t\big),$$

by Lemma 1 and (12), the result of (i) is obtained.

(ii) Note that

$$\frac{\partial}{\partial\theta}\frac{\partial^2\ln(f)}{\partial y_t^2} = -\frac{2\theta^{2n-2t+1}[\theta^{-2(n-t)} + (n-t)\theta^2 - (n-t+1)]}{\sigma^2(1-\theta^2)^2},$$

$$\frac{\partial}{\partial\sigma^2}\frac{\partial^2\ln(f)}{\partial y_t^2} = \frac{1}{\sigma^4}\sum_{i=0}^{n-t}\theta^{2i},$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\ln(f)}{\partial y_t}\right)^2 = \frac{2}{\sigma^4}\sum_{i=0}^{n-t}\theta^i e_{t+i}\left[\sum_{j=1}^{n-t} j\theta^{j-1}e_{t+j} + \sum_{k=0}^{n-t}\theta^k \dot{e}_{t+k}\right],$$

$$\frac{\partial}{\partial\sigma^2}\left(\frac{\partial\ln(f)}{\partial y_t}\right)^2 = -\frac{2}{\sigma^6}\sum_{i=0}^{n-t}\theta^i e_{t+i}\,.$$

And by (9),

$$b_1 = \frac{h^2}{12\hat{\sigma}_0^4}\sum_{t=1}^{n}\left[(1-\hat{\sigma}_0^2)\left(\sum_{i=1}^{n-t} i\hat{\theta}_0^{2i-1}\right) + \left(\sum_{i=t}^{n}\hat{\theta}_0^{i-t}e_i\right)\left(\sum_{j=t}^{n}\hat{\theta}_0^{j-t}\dot{e}_j\right)\right]\Bigg|_{\beta=\hat{\beta}_0}.$$

$$b_2 = -\frac{h^2}{24\hat{\sigma}_0^4}\frac{\hat{\theta}_0^{2+2n} - \hat{\theta}_0^2 + (1-\hat{\theta}_0^2)n}{(1-\hat{\theta}_0^2)^2}.$$

Then the limits of b_1/n and b_2/n can be obtained.

□

Finally, by Lemma 2 and the definition of the adjusted MLE given in (7), we have the following theorem.

Theorem 3. *The adjusted MLE of the MA(1) model* (10) *based on the rounded data are*

$$\hat{\theta}_A \approx \hat{\theta}_0 + \frac{\hat{\theta}_0 h^2}{12\hat{\sigma}_0^4}\left[\frac{1}{1-\hat{\theta}_0^2} - \frac{(1+\hat{\theta}_0^2)(1-\hat{\sigma}_0^2)}{n(1-\hat{\theta}_0^2)^2}\right],$$

$$\hat{\sigma}_A^2 \approx \hat{\sigma}_0^2 - \frac{h^2}{12}\left[\frac{1}{1-\hat{\theta}_0^2} - \frac{\hat{\theta}_0^2}{n(1-\hat{\theta}_0^2)^2}\right],$$

where $\hat{\theta}_0, \hat{\sigma}_0^2$ are the PMLE and h is the width of the rounding interval.

4. Discussion

Simulation study is performed to compare the efficiencies of the pseudo MLE (PMLE), the adjusted MLE (AD), the short overlapping series (SOS) estimator of Zhang, et al., (2010) and the approximate maximum likelihood estimation (AMLE) of Bai, et al., (2009). The root mean squared errors (RMSE) of the four estimators for (θ, σ^2) are given in Table 1 and Table 2, respectively. The results show that for both $\theta_0 = 0.5$ and -0.5, the adjusted MLE (AD) $\hat{\theta}$ has the smallest RMSE when $\sigma^2 > 0.5$ and $\hat{\sigma^2}$ has the smallest RMSE when $\sigma^2 \geq 0.25$.

TABLE 1. The RMSE of $\hat{\theta}$ for the MA(1) models.

	$\theta_0 = -0.5$				$\theta_0 = 0.5$			
σ_0^2	PMLE	AD	SOS	AMLE	PMLE	AD	SOS	AMLE
0.1	0.267	0.859	0.177	0.149	0.285	1.093	0.219	0.220
0.25	0.155	0.224	0.140	0.152	0.164	0.214	0.150	0.152
0.5	0.096	0.103	0.129	0.123	0.107	0.079	0.134	0.158
1	0.067	0.064	0.126	0.117	0.086	0.075	0.125	0.128
4	0.062	0.062	0.115	0.110	0.059	0.058	0.117	0.132

TABLE 2. The RMSE of $\hat{\sigma}^2$ for the MA(1) models.

	$\theta_0 = -0.5$				$\theta_0 = 0.5$			
σ_0^2	PMLE	AD	SOS	AMLE	PMLE	AD	SOS	AMLE
0.1	0.0565	0.0427	0.0178	0.0154	0.0530	0.0502	0.0221	0.0186
0.25	0.106	0.0332	0.0376	0.0400	0.105	0.0377	0.0408	0.0452
0.5	0.116	0.0542	0.0684	0.0649	0.115	0.0511	0.0679	0.0800
1	0.140	0.0885	0.120	0.116	0.142	0.0929	0.111	0.122
4	0.363	0.351	0.467	0.455	0.370	0.347	0.454	0.502

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Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 804
R.O.C.

E-mail: guomh@math.nsysu.edu.tw
xanadu922@gmail.com