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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2019-0001
Tatra Mt. Math. Publ. **73** (2019), 1–8

DEDEKIND'S CRITERION AND INTEGRAL BASES

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ABSTRACT. Let R be a principal ideal domain with quotient field K, and $L = K(\alpha)$, where α is a root of a monic irreducible polynomial $F(x) \in R[x]$. Let $\mathbb{Z}_L$ be the integral closure of R in L. In this paper, for every prime p of R, we give a new efficient version of Dedekind's criterion in R, i.e., necessary and sufficient conditions on $F(x)$ to have p not dividing the index $\left[\mathbb{Z}_L : R[\alpha]\right]$, for every prime p of R. Some computational examples are given for $R = \mathbb{Z}$.

1. Introduction

Throughout this paper unless otherwise stated, R is a principal ideal domain with quotient field K. For every prime element p of R, let ν_p be the p-adic discrete valuation on R and $k(p) = \frac{R}{(p)}$ the residue field associated to p. The Gaussian valuation of $K(x)$ which extends ν_p and defined by

$$\nu_p\left(\sum_{i=0}^{l} a_i X^{l-i}\right) = \min\{\nu_p(a_i),\, 0 \le i \le l\} \quad \text{is also denoted by } \nu_p.$$

Let $L = K(\alpha)$, where α is a root of a monic irreducible polynomial $F(x) \in R[x]$. Let $\operatorname{disc}(F)$ be the discriminant of F, $\mathbb{Z}_L$ the integral closure of R in L, and $\operatorname{ind}(\alpha) = \left[\mathbb{Z}_L : R[\alpha]\right]$ the index of $R[\alpha]$ in $\mathbb{Z}_L$. A natural question is: when does $\mathbb{Z}_L = R[\alpha]$? If $R = \mathbb{Z}$, then for every prime integer p, Dedekind gave a criterion to test whether or not p divides $\operatorname{ind}(\alpha)$; more precisely, he proved that p does not divide $\operatorname{ind}(\alpha)$ if and only if for every $i = 1, \dots, r$, either $e_i = 1$ or $e_i \ge 2$ and $\overline{\phi_i}(x)$ does not divide $\overline{M}(x)$, where

$$M(x) = \frac{F(x) - \prod_{j=1}^{r} \phi_j^{l_j}(x)}{p} \quad \text{and} \quad \overline{F}(x) = \prod_{j=1}^{r} \overline{\phi_j}^{l_j}(x) \pmod{p}$$

is the factorization of $\overline{F}(x)$ in $\mathbb{F}_p[x]$ (see [5, Theorem 6.1.4] and [8]).

2010 Mathematics Subject Classification: 13A18, 11Y40, 11S05.
Keywords: Dedekind's criterion,integral bases, Power integral bases.

This criterion was also proved over any valuation ring R and any algebraic field extension $L = K(\alpha)$ of K, where L/K is not necessarily separable [7]. In this paper, we give a more efficient version of this criterion for any principal ideal domain R with no separability assumption on the extension L/K. We further give some computational examples in the case $R = \mathbb{Z}$.

2. Main results

We recall here the definition of the index $\mathrm{ind}(\alpha) = \big[\mathbb{Z}_L : R[\alpha]\big]$. Since R is a principal ideal domain, $\mathbb{Z}_L$ is a free R-module of rank $n = \deg(F)$. Let $\mathbf{B} = \{u_1, \dots, u_n\}$ be an R-basis of $\mathbb{Z}_L$ and P_B^F the transition matrix from $\mathbf{B}$ to the R-basis $\mathbf{F} = \{1, \alpha, \dots, \alpha^{n-1}\}$ of $R[\alpha]$. The index $\big[\mathbb{Z}_L : R[\alpha]\big]$ is the principal ideal of R generated by the determinant of P_B^F. It is well known that this principal ideal $\big[\mathbb{Z}_L : R[\alpha]\big]$ is well defined and is independent on the choice of the bases $\mathbf{B}$ and $\mathbf{F}$ of $\mathbb{Z}_L$ and $R[\alpha]$, respectively. Since R is a principal ideal domain, it follows from the invariant factor Theorem that there exists $\mathbf{B} = \{u_1, \dots, u_n\}$ an R-basis of $\mathbb{Z}_L$ and $(q_1, \dots, q_n) \in R^n$ such that for every $i = 1, \dots, n-1$, q_i divides q_{i+1}, and $\mathbf{F} = \{q_1u_1, \dots, q_nu_n\}$ is an R-basis of $R[\alpha]$. Since P_B^F is the diagonal matrix with diagonal elements: $q_1, \dots, q_n$, the index $\big[\mathbb{Z}_L : R[\alpha]\big]$ is then precisely the principal ideal of R generated by $\prod_{i=1}^n q_i$. If $R = \mathbb{Z}$, then $\mathrm{ind}(\alpha)$ is the cardinal order of the finite group $\mathbb{Z}_L/\mathbb{Z}[\alpha]$.

In this section, let

$$F(x) \equiv \prod_{i=1}^{r} \phi_i^{l_i}(x) \pmod{p}$$

be the factorization of $\overline{F}(x)$ in $k(p)[x]$, where for every $i := 1, \dots, r$, ϕ_i is a monic polynomial in $R[x]$. For every $i := 1, \dots, r$, let $Q_i(x)$ and $R_i(x)$ be the quotient and the remainder of the Euclidean division of $F(x)$ by $\phi_i(x)$, respectively.

Our next Theorem computationally improves the well known Dedekind's criterion.

Theorem 2.1. *Under the above hypotheses, p does not divide the index $\big[\mathbb{Z}_L : R[\alpha]\big]$ if and only if for every $i := 1, \dots, r$, either $l_i = 1$ or $l_i \geq 2$ and $\nu_p\big(R_i(x)\big) = 1$.*

Proof. If for every $i := 1, \dots, r$, $l_i = 1$, then by the generalized Dedekind's criterion p does not divide $\mathrm{ind}(\alpha)$ (see for example [7]). Otherwise, let

$$M(x) = \frac{F(x) - \prod_{j=1}^{r} \phi_j^{l_j}(x)}{p}$$

as defined in the Dedekind's criterion and let us show that for every $i = 1, \dots, r$, with $l_i \geq 2$, $\nu_p\big(R_i(x)\big) = 1$ if and only if $\overline{\phi_i}$ does not divide $\overline{M}(x)$ in $k(p)[x]$.

Indeed, as

$$F(x) \equiv \prod_{j=1}^{r} \phi_j^{l_j}(x) \pmod{p},$$

then $\overline{\phi_i}(x)$ divides

$$\overline{F}(x), \quad \overline{R_i}(x) = \overline{0} \pmod{p} \quad \text{and} \quad \overline{Q_i}(x) = \overline{\phi_i^{l_i-1}(x)\prod_{j\neq i}\phi_j^{l_j}} \pmod{p}.$$

Thus there exists some $H_i(x) \in R[x]$ such that

$$Q_i(x) = \phi_i^{l_i-1}(x)\prod_{j\neq i}\phi_j^{l_j}(x) + pH_i(x).$$

Therefore,

$$F(x) = \Big(\phi_i^{l_i-1}(x)\prod_{j\neq i}\phi_j^{l_j} + pH_i(x)\Big)\phi_i(x) + R_i(x)$$

and

$$M(x) = \frac{F(x) - \prod_{j=1}^{r}\phi_j^{l_j}(x)}{p} = H_i(x)\phi_i(x) + \frac{R_i(x)}{p}.$$

It follows that $\overline{\phi}_i$ does not divide $\overline{M}(x)$ in $k(p)[x]$ if and only if $\frac{R_i(x)}{p} \not\equiv 0 \pmod{p}$. That is $\nu_p\big(R_i(x)\big) = 1$. □

Corollary 2.2. *If R is a discrete valuation ring with maximal ideal (p), then the equality $\mathbb{Z}_L = R[\alpha]$ holds if and only if for every $i := 1, \ldots, r$, either $l_i = 1$ or $l_i \geq 2$ and $\nu_p\big(R_i(x)\big) = 1$.*

Corollary 2.3. *Under the hypotheses of theorem 2.1, if R is a Dedekind domain, then for every prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ does not divide the index $\big[\mathbb{Z}_L : R[\alpha]\big]$ if and only if for every $i := 1, \ldots, r$, either $l_i = 1$ or $l_i \geq 2$ and $R_i(x) \in \mathfrak{p}[X] - \mathfrak{p}^2[X]$.*

Remark. A similar result holds with applications in more general rings, namely Prüfer domains (cf. [9]). In This work, we are interested in another way, namely computation of integral bases.

Theorem 2.4. *Let $L = K(\alpha)$, where α is any root of $F(x) = \phi(x)^n - a \in R[x]$ such that $\nu_p(a)$ and n are coprime and $\phi(x) \in R[x]$ is a monic polynomial whose reduction modulo p is irreducible. Then $\{\alpha^i\theta^j,\ 0 \leq i < m-1$ and $0 \leq j < n-1\}$ is a p-integral basis of $\mathbb{Z}_L$, where $m = \deg(\phi)$, $\theta = \frac{\phi(\alpha)^u}{p^v}$, u and v are non-negative integers satisfying $\nu_p(a)u - nv = 1$ such that $0 \leq u < n$.*

Proof. First, $L = F(\alpha)$, where $F = K\big(\phi(\alpha)\big)$, $g(x) = x^n - a$ is the minimal polynomial of $\phi(\alpha)$ over K, and $h(x) = \phi(x) - \phi(\alpha)$ is the minimal polynomial of α over F. As $\nu_p(a)$ and n are coprime, using the Euclid's algorithm, there exists a unique solution of non-negative integers (u, v) of $\nu_p(a)u - nv = 1$

such that $0 \le u < n$. Consider $g_1(x) = x^n - \frac{a^u}{p^{nv}}$. Then $g_1(x) \in R[x]$ and $g_1(\theta) = 0$. Since $\nu_p(\frac{a^u}{p^{nv}}) = \nu_p(a)u - nv = 1$, by Eisenstein's criterion $g_1(x)$ is irreducible in $R[x]$. By Theorem 2.1, p does not divide the index $\left[\mathbb{Z}_F = R[\theta]\right]$; $\{\theta^j, 0 \le j < n-1\}$ is a p-integral basis of $\mathbb{Z}_F$ over R. Thus $p\mathbb{Z}_F = \mathfrak{p}^n$, where $\mathfrak{p} = (p, \theta)$. As $\overline{h}(x) = \overline{\phi}(x) \pmod{\mathfrak{p}}$, $\overline{\phi}(x)$ is irreducible over $k(p)$, and $f(\mathfrak{p}/p) = 1$; $k(\mathfrak{p}) = k(p)$, we have $\overline{h}(x) = \overline{\phi}(x)$ is irreducible over $k(\mathfrak{p})$. Again by Theorem 2.1,

$$\left[\mathbb{Z}_L = \mathbb{Z}_F[\alpha]\right] \not\subset \mathfrak{p}; \qquad \{\alpha^i, 0 \le i < m-1\}$$

is a $\mathfrak{p}$-integral basis of $\mathbb{Z}_L$ over $\mathbb{Z}_F$, where $m = \deg(\phi)$. Finally,

$$\{\alpha^i\theta^j, 0 \le i < m-1 \quad \text{and} \quad 0 \le j < n-1\}$$

is a p-integral basis of $\mathbb{Z}_L$ over R. □

In particular, if $\phi(x) = x$, then we have the following corollaries:

Corollary 2.5. *Let p be a prime of R, $L = K(\alpha)$, where α is a root of an irreducible polynomial $F(x) = x^n - a \in R[x]$ such that $\nu_p(a$ and n are co-prime. Let $\theta = \frac{\alpha^u}{p^v}$, where u and v are the unique non-negative integers satisfying $\nu_p(a)u - nv = 1$ and $0 \le u < n$. Then p does not divide the index $\left[\mathbb{Z}_L : R[\theta]\right]$.*

For any element $\theta \in \mathbb{Z}_L$, we say that θ generates a power integral basis of $\mathbb{Z}_L$ over R if $(1, \theta, \ldots, \theta^{n-1})$ is a R-basis of $\mathbb{Z}_L$, where n is the degree $[L : K]$; $\mathbb{Z}_L = R[\theta]$. When a field L has a power integral basis, the field L is said to be monogenic. It is called a problem of Hasse to characterize whether the ring of integers in an algebraic number field has a power integral basis or does not [1–3]. The following corollaries give a condition on a in order to have the monogeneses of any field L defined by $F(x) = x^n - a$.

Corollary 2.6. *Keep the assumptions and notations of Corollary 2.5, if R is a discrete valuation ring with maximal ideal (p), then $\mathbb{Z}_L = R[\theta]$, where $\theta = \frac{\alpha^u}{p^v}$ and α is a root of $F(x) = x^n - a$. We say that θ generates a power integral basis of $\mathbb{Z}_L$ over R.*

Corollary 2.7. *Let $L = \mathbb{Q}(\alpha)$ be a pure prime number field; α a complex root of an irreducible polynomial $F(x) = x^p - a \in \mathbb{Z}[x]$, where p is an odd prime. Assume that $a = m^e$ with $0 < e < p$ and let $\theta = \frac{\alpha^u}{m^v}$, where u and v are the unique non-negative integers satisfying $eu - nv = 1$ and $0 \le u < n$. Then we have the following:*

1. *If p divides m or p does not divide m and $\nu_p(m^{p-1} - 1) = 1$, then $\mathbb{Z}_L$ is monogenic. Especially $\mathbb{Z}_L = \mathbb{Z}[\theta]$.*
2. *If p does not divide m and $\nu_p(m^{p-1} - 1) \ge 2$, then $(1, \theta, \ldots, \theta^{p-2}, \frac{\theta^{p-1}}{p})$ is an integral basis of $\mathbb{Z}_L$.*

Theorem 2.8. *Let $L = \mathbb{Q}(\alpha)$ be a pure prime number field; α is a complex root of an irreducible polynomial $F(x) = x^p - a \in \mathbb{Z}[x]$, where p is an odd prime. We can assume that $\nu_q(a) < p$ for every prime integer q; set $a = \mp\prod_{i=1}^r p_i^{e_i}$ the factorization of a into powers of positive prime integers such that for every $i = 1, \dots, r$, $e_i < p$. Then we have the following:*

(1) *If p divides a or p does not divide a and $\nu_p(a^{p-1} - 1) = 1$, then*

$$\left(\frac{\alpha^k}{\prod_{i=1}^r p_i^{\lfloor \frac{ke_i}{p} \rfloor}}, \ 0 \le k < p\right) \quad \text{is a } \mathbb{Z}\text{-integral basis of } \mathbb{Z}_L.$$

(2) *If p does not divide a and $\nu_p(a^{p-1} - 1) \ge 2$, then*

$$\left\{\frac{\alpha^k}{\prod_{i=1}^r p_i^{\lfloor \frac{ke_i}{p} \rfloor}}, \ 0 \le k < p-1\right\} \cup \left\{\frac{\alpha^{p-1}}{p\prod_{i=1}^r p_i^{\lfloor \frac{(p-1)e_i}{p} \rfloor}}\right\}$$

is a $\mathbb{Z}$-integral basis of $\mathbb{Z}_L$.

Proof. (1) We have to check that for every positive prime integer q dividing $\mathrm{disc}(F) = \mp p^p \cdot a^{p-1}$, if q does not divide the index $\big[\mathbb{Z}_L : \mathbb{Z}[\theta]\big]$. Let p_i be a prime integer dividing a. Then $\overline{F}(x) = x^p \pmod{p_i}$, the Newton polygon $N_x(F) = S$ is one sided of slope e_i/p, and its attached residual polynomial is $F_S(y)$ is of degree 1 (because $\gcd\big(l(S), h(S)\big) = 1$, where $l(S)$ and $h(S)$ are the length and the height of S, respectively). Thus, by [6, Prop 2.1],

$$\left(\left\{\frac{\alpha^k}{p_i^{\lfloor \frac{ke_i}{p} \rfloor}}\right\}, \ 0 \le k < p\right) \quad \text{is a } p_i\text{-integral basis of } \mathbb{Z}_L.$$

If p does not divide a and $\nu_p(a^{p-1} - 1) = 1$, then $\overline{F}(x) = (x - a)^p \pmod p$. Let

$$H(x) = F(x + a) = x^p + \cdots + pa^{p-1}x + a^p - a.$$

Then

$$\overline{H}(x) = x^p \pmod p.$$

As $a^p - a$ is the remainder of $H(x)$ by x and $\nu_p(a^{p-1} - 1) = 1$, by Theorem 2.1, q does not divide the index $\big[\mathbb{Z}_L : \mathbb{Z}[\alpha]\big]$.

(2) If p does not divide a and $\nu_p(a^{p-1} - 1) \ge 2$, then $\overline{H}(x) = x^p \pmod p$ and its x-Newton polygon

$$N_x(H) = S_1 + S_2,$$

where S_1 is of height 1 and S_2 is of length 1. Thus [6, Prop 2.1],

$$\left(1, \alpha, \dots, \alpha^{p-2}, \frac{\alpha^{p-1}}{p}\right) \quad \text{is a } p\text{-integral basis of } \mathbb{Z}_L. \qquad \square$$

3. Examples

(1) Let

$$F(x) = x^{16} + 8x^{15} + 20x^{14} - 70x^{12} - 56x^{11} + 112x^{10} + 120x^{9} - 125x^{8} - 120x^{7} + 112x^{6} + 56x^{5} - 70x^{4} + 20x^{2} - 8x - 7.$$

Since

$$F(x) = (x^2 + x - 1)^8 - 8, \quad \overline{\phi}(x) = \overline{x^2 + x - 1} \pmod 2$$

is irreducible in $\mathbb{F}_2[x]$, and $\nu_2(8)$ is coprime to 8, by [4, Theorem 1.6], $F(x)$ is irreducible over $\mathbb{Q}$. Let α be a complex root of $F(x)$ and $L = \mathbb{Q}(\alpha)$. Since

$$\operatorname{disc}\big(F(x)\big) = \mp 2^{90}.73.1831, \quad \text{for every prime integer } p \neq 2,$$

p does not divide $\operatorname{ind}(\alpha)$. For $p = 2$, let $\theta = \frac{\alpha^3}{2}$. Then

$$\big(1, \theta, \dots, \theta^7, \alpha, \alpha\theta, \dots, \alpha\theta^7\big) \quad \text{is an integral basis of } \mathbb{Z}_L.$$

(2) Let

$$F(x) = x^{16} + 8x^{15} + 20x^{14} - 70x^{12} - 56x^{11} + 112x^{10} + 120x^{9} - 125x^{8} - 120x^{7} + 112x^{6} + 56x^{5} - 70x^{4} + 20x^{2} - 8x - 23.$$

Since

$$F(x) = (x^2 + x - 1)^8 - 24,$$

it is a 3-Eisenstein polynomial. So it is irreducible over $\mathbb{Q}$. Let α be a complex root of $F(x)$ and $L = \mathbb{Q}(\alpha)$. Since $\operatorname{disc}\big(F(x)\big) = \mp 2^{90}.3^{14}.163.7253$, for every prime integer $q \notin \{2, 3\}$, q does not divide $\operatorname{ind}(\alpha)$. For $p = 2$, let $\theta = \frac{\alpha^3}{2}$. Then 2 does not divide $\big[\mathbb{Z}_L : \mathbb{Z}[\theta]\big]$. For $p = 3$, since $\nu_3(24) = 1$, 3 does not divide $\big[\mathbb{Z}_L : \mathbb{Z}[\alpha]\big]$.

(3) Let p be a non-negative prime integer, $F(x) = x^p - a \in \mathbb{Z}[x]$ an irreducible polynomial, α a complex root of $F(x)$, and $L = \mathbb{Q}(\alpha)$.

  (a) For $p = 5$ and $a = 22^2$, let $\theta = \frac{\alpha^4}{22^3}$. Then for every prime integer $q \notin \{2, 5, 11\}$, q does not divide $\operatorname{ind}(\alpha)$. For $q \in \{2, 11\}$, q does not divide $\operatorname{ind}(\theta)$, too. For $p = 5$, since $v_5(22^4 - 1) = 1$, $v_5\big(\operatorname{ind}(\theta)\big) = 0$. Thus $\mathbb{Z}_L$ is monogenic, with $\theta = \frac{\alpha^4}{22^3}$ generating a power integral basis.

  (b) Let $p = 11$ and $a = 23^6.11^5$. Since $\operatorname{disc}(F) = \pm 11^{11} a^{10}$, for every prime $q \notin \{2, 3, 11\}$, p does not divide $\operatorname{disc}(F)$.

$$E=\left\{1,\alpha,\alpha^2,\frac{\alpha^3}{11},\frac{\alpha^4}{11},\frac{\alpha^5}{11^2},\frac{\alpha^6}{11^2},\frac{\alpha^7}{11^3},\frac{\alpha^8}{11^3},\frac{\alpha^9}{11^4},\frac{\alpha^{10}}{11^4}\right\}$$

is an 11-integral basis of $\mathbb{Z}_L$, i.e., 11 does not divide the index $[\mathbb{Z}_L : S]$, where S is the $\mathbb{Z}$-order generated by E. Similarly, by using $\theta = \frac{\alpha^2}{3}$, we get

$$T=\left\{1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3},\frac{\alpha^4}{3^2},\frac{\alpha^5}{3^2},\frac{\alpha^6}{3^3},\frac{\alpha^7}{3^3},\frac{\alpha^8}{3^4},\frac{\alpha^9}{3^4},\frac{\alpha^{10}}{3^5}\right\}$$

as a 3-integral basis of $\mathbb{Z}_L$.

Thus,

$$B=\left\{1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3.11},\frac{\alpha^4}{3^2.11},\frac{\alpha^5}{3^2.11^2},\frac{\alpha^6}{3^3.11^2},\frac{\alpha^7}{3^3.11^3},\frac{\alpha^8}{3^4.11^3},\frac{\alpha^9}{3^4.11^4},\frac{\alpha^{10}}{3^5.11^4}\right\}$$

is an integral basis of $\mathbb{Z}_L$.

(c) Let $p = 11$ and $a = 3^6$. Then $\operatorname{disc}(F) = \pm 11^{11}3^{60}$. For $q \notin \{3, 11\}$, q does not divide $\operatorname{ind}(\alpha)$. Then

$$\left\{1,\alpha,\frac{\alpha^2}{3},\frac{\alpha^3}{3},\frac{\alpha^4}{3^2},\frac{\alpha^5}{3^2},\frac{\alpha^6}{3^3},\frac{\alpha^7}{3^3},\frac{\alpha^8}{3^4},\frac{\alpha^9}{3^4},\frac{(\alpha-3^6)^{10}}{11.3^5}\right\}$$

is an integral basis of $\mathbb{Z}_L$.

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Received September 8 2018

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