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ON VECTOR VALUED ERGODIC THEOREMS

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ABSTRACT. We show that certain pointwise ergodic theorems, known to be valid for scalar valued functions are also valid for Fréchet space valued functions, as a corollary of the scalar valued cases.

Suppose F is a Fréchet space with its topology generated by a family of seminorms $(p_k)_{k=1}^{\infty}$ and suppose that (X,β,μ) is a measure space. Following F. K. Schäfte [5] we describe F valued integration on (X,β,μ) as follows. Let

$$S = \left\{ s = \sum_{i=1}^{N} x_i \chi_{A_i} : x_n \in F;\ A_n \in \beta \right\},$$

denote the set of F valued simple functions on (X,β,μ). Here for a set A we denote its characteristic function by χ_A. We set the integral of the simple function s to be

$$I(s) = \sum_{i=1}^{N} x_i \mu(A_i) .$$

Also if $\mu(X) < \infty$ then set

$$\|h\| = \inf \left\{ \int_X m d\mu \colon h \leqslant m \colon \text{ with } m\,\beta\text{-measurable} \right\}$$

and if $(p_k)_{k=1}^{\infty}$ is a family of seminorms generating the topology of F set

$$\rho(a,b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|p_k(a,b)\|}{1 + \|p_k(a,b)\|} .$$

This defines a pseudo metric on the space of functions from X to F. We say $f\colon X \to F$ is μ integrable if there exists a sequence of simple functions $(s_n)_{n=1}^{\infty}$ from X to F such that $\lim_{n\to\infty} \rho(f, s_n) = 0$ and $I(f) = \lim_{n\to\infty} I(s_n)$, which must

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exist as a consequence is finite. We call $I(f)$ the integral of F. We let $L_F^1(X, \beta\mu)$ denote the set of μ integrable F valued functions on (X, β, μ). In [5] it is shown that f belongs to $L_F^1(X, \beta, \mu)$ if and only if there exists a sequence of simple functions $(s_n)_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} s_n = f \tag{1}$$

and given a natural number k and $\varepsilon > 0$ then there exists $N = N(\varepsilon)$ such that if $m, n > N$ we have

$$\left| \int_X p_k(s_m - s_n) d\mu \right| < \varepsilon . \tag{2}$$

We say $f \colon X \to F$ is in $L_F^p(X, \beta\mu)$ for p in $[1, \infty]$ if $\rho\big(f(x), 0\big) \in L^p(x, \beta, \mu)$. Suppose α is a sub σ-algebra of β and $f \colon X \to F$ is β measurable then we define $E(F|\alpha)$ by setting for simple

$$s = \sum_{i \in I} x_i \chi_{A_i} ,$$

$$E(s|\alpha) = \sum_{i \in I} x_i \mu(A_i|\alpha)$$

were $\mu(A|\alpha)$ denotes the standard μ measure of A conditioned with regard to the σ-algebra α. For general f in $L_F^p(C, \beta, \mu)$ set

$$E(f|\alpha) = \lim_{n \to \infty} E(s_n|\alpha) ,$$

where $(s_n)_{n=1}^{\infty}$ is a sequence of simple functions, taking values in F converging to f pointwise. This procedure is readily checked to be well defined. Here the limit has been defined via the pseudo-metric ρ. Before we can state our first theorem we need to introduce a class of sequences.

Suppose for a real number y we denote by $[y]$ its integer part and set $< y >= y - [y]$. Suppose $g(x)$ is a function from $[1, \infty)$ to itself whose derivative increases to infinity with its argument. Let k_n denote $\big[g(n)\big]$ $(n = 1, 2, \ldots)$ and let A_M denote the cardinality of $\{n \colon k_n \leqslant M\}$. Also for real x let $e(x) = e^{2\pi i x}$. Suppose for a function $a \colon [1, \infty) \to [1, \infty)$ increasing to infinity as it argument does, that we set

$$b(M) = \sup_{<\alpha> \in \left[\frac{1}{a(M)}, \frac{1}{2}\right)} \left| \sum_{n \colon k_n \leqslant M} e(\alpha k_n) \right| .$$

Suppose also for some decreasing function $c \colon [1, \infty) \to [1, \infty)$ such that

$$\frac{b(M) + A_{[a(M)]} + \frac{M}{a(M)}}{A_M} \leqslant C c(M) ,$$

and $\theta > 1$ that

$$\sum_{s=1}^{\infty} c(\theta^s) < \infty .$$

The sequences $\mathbf{k} = (k_n)_{n=1}^{\infty}$ we have just described is said to satisfied condition H. Examples of sequences that satisfy condition H include $g(n) = n^{\beta}$ where β is non-integer and greater than one; $g(n) = P(n)$, where $P(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$ with $\alpha_1, \ldots, \alpha_k$ not all rational multiples of the same real number and $g(n) = e^{(\log n)^k}$ with $k \in \left(1, \frac{3}{2}\right)$ [4];

Let (X, β, μ) be a probability space and let $T\colon X \to X$ be a measurable map that is also measure preserving that is, given $A \in \beta$, we have $\mu(T^{-1}A) = \mu(A)$, where $T^{-1}A$ denotes the set $\{x \in X : Tx \in A\}$.

Theorem 1. *Suppose $T\colon X \to X$ is measure preserving, f is in $L_F^p(X, \beta, \mu)$ for $p > 1$ and $(k_n)_{n=1}^{\infty}$ is a subsequence of the natural numbers satisfying condition H. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{k_n} x) = E(f|\mathcal{I})(x) ,$$

almost everywhere with respect to μ. Here $\mathcal{I}$ consist of the σ-algebra of elements $B \in \beta$ such that $T^{-1}B = B$.

The argument used to prove Theorem 1 and the other results in this paper has the virtue of giving results which are much more general than any of this kind proved to-date, while at the same time being simpler than any used previously. See for instance [6], where results of much more restricted scope are proved by much more technical means. The central ingredient in the proof of Theorem 1 is the following lemma, which appears as Theorem 4 in [4].

Lemma 2. *Suppose $T\colon X \to X$ is measure preserving, f is in $L^p(X, \beta, \mu)$ for $p > 1$ and $(k_n)_{n=1}^{\infty}$ is a subsequence of the natural numbers satisfying condition H. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{k_n} x) = E(f|\mathcal{I})(x) ,$$

almost everywhere with respect to μ.

Proof of Theorem 1. It is a straightforward consequence of Lemma 2 that Theorem 1 holds in the case that f in $L_F^p(X, \beta, \mu)$ is simple. For general f in $L_F^p(X, \beta, \mu)$ we can find a sequence of simple F valued functions $(s_j)_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \rho\big(s_j(x),\, f(x)\big) = 0 ,$$

pointwise. Then

$$\lim_{j\to\infty} E(s_j|\mathcal{I})(x) = E(f|\mathcal{I})(x)\,,$$

μ almost everywhere. For a fixed natural number k note that as a consequence of the triangle inequality for the pseudo-metric ρ

$$\begin{aligned} p_k&\left(\frac{1}{N}\sum_{n=1}^{N} f(T^{k_n}x) - E(F|\mathcal{I})(x)\right) \\ &\leqslant p_k\left(\frac{1}{N}\sum_{n=1}^{N}(f-s_j)(T^{k_n}x)\right) \\ &\quad + p_k\left(\frac{1}{N}\sum_{n=1}^{N} s_j(T^{k_n}x) - E(s_j|\mathcal{I})(x)\right) \\ &\quad + p_k\big(E(s_j|\mathcal{I}(x) - E(f|\mathcal{I})(x)\big)\,. \end{aligned}$$

Hence for each j and k

$$\begin{aligned} &\limsup_{N\to\infty} p_k\left(\frac{1}{N}\sum_{n=1}^{N} f(T^{k_n}x) - E(f|\mathcal{I})(x)\right) \\ &\leqslant \limsup_{N\to\infty} p_k\left(\frac{1}{N}\sum_{n=1}^{N}(f-s_j)(T^{k_n}x)\right) \\ &\quad + \limsup_{N} \to \infty p_k\left(\frac{1}{N}\sum_{n=1}^{N} s_j(T^{k_n}x) - E(s_j|\mathcal{I})(x)\right) \\ &\quad + p_k\big(E(s_j|\mathcal{I})(x) - E(f|\mathcal{I})(x)\big)\,. \end{aligned}$$

Let

$$f_{j,k}(x) = p_k\big((f-s_j)(x)\big)\,.$$

Because f is in $L^p_F(x,\beta,\mu)$ then $f_{j,k}$ is in $L^p(x,\beta,\mu)$. Hence if we set

$$g_{j,k}(x) = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_{j,k}(T^{k_n}x)\,,$$

Lemma 2 tells us that

$$g_{j,k}(x) = E(f_{j,k}|\mathcal{I})(x)\,,$$

μ almost everywhere. Also as $E\big(E(f|\alpha)\big) = E(f)$, for general f and α we see that

$$\limsup_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f(T^{k_n}x) - E(f|\mathcal{I})(x)\right) = 0\,,$$

except on a set of measure zero called N_k say. Theorem 1 now follows as $\mu\left(\bigcup_{k\geqslant 1} N_k\right) = 0$.

Some other general pointwise ergodic theorems taking values in a Fréchet space also follow from their scalar cases by similar argument. We describe some other examples.

For a set S let χ_S denote its characteristic function. Suppose $K = (n_k)_{n=1}^{\infty} \subseteq \mathbf{N}$ is a strictly increasing sequence of natural numbers. By identifying K with its characteristic function χ_K we may view it as a point in $\Lambda = \{0,1\}^{\mathbf{N}}$ the set of maps from $\mathbf{N}$ to $\{0,1\}$. Conversely we can also view Λ as the space of strictly increasing sequences of natural numbers like k. We endow Λ with a probability measure by viewing it as the Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$ where for each n $(n = 1,2\dots)$ we have $X_n = \{0,1\}$ and specify the probability π_n on X_n by $\pi_n(\{1\}) = q_n$ with $0 \leqslant q_n \leqslant 1$ and $\pi_n(\{0\}) = 1 - q_n$ where the sequence $(q_n)_{n=1}^{\infty}$ satisfies $\lim_{N\to\infty} q_n n(\log\log n)^{-1} = 0$. The desired probability measure on Λ is the corresponding product measure $\pi = \prod_{n=1}^{\infty} \pi_n$. The underlying σ-algebra β is that generated by the "cylinders"

$$\{\lambda - (\lambda_n)_{n=1}^{\infty} \in \Lambda \colon \lambda_{i_i} = a_{i_r}, \dots \lambda_{i_r} = a_{i_r}\}$$

for all possible choices of $i_1, \dots, i_r$ and $\alpha_{i_1}, \dots \alpha_{i_r}$. Then almost every K in (Λ, β, π) is also L^p good universal for each p in $(1, \infty]$. $\square$

Theorem 3. *Suppose $T \colon X \to X$ is measure preserving, f is in $L^p_F(X, \beta, \mu)$ for $p > 1$ and $(k_n)_{n=1}^{\infty}$ is almost any strictly increasing sequence of natural numbers with respect to the measure π. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{k_n} x) = E(f|\mathcal{I})(x),$$

almost everywhere with respect to μ.

See [2] for the scalar case of Theorem 3.

Let (X, β, μ) and $(S, \mathcal{A}, \gamma)$ be complete measure spaces with μ σ-finite. Assume S is a semigroup with product indicated by juxtaposition and that there is a map $(x,g) \to T_g(x)$ from $X \times S$ to X, measurable with respect to $\beta \otimes \mathcal{A}$ such that $T_{g_1}(T_{g_2}(x)) = T_{g_1 g_2}(x)$ for all g_1, g_2 in S and x in X. Assume $\mu(T_g^{-1} A) = \mu(A)$ for all g is S and A in β. Also assume that gE and Eg are measurable, $\gamma(gE) = \gamma(E) = \gamma(Eg)$ and that $g^{-1}E$ and Eg^{-1} are measurable for all $g \in G$ and $E \in \mathcal{A}$. Here gE denotes $\{gx \colon x \in E\}$, Eg denotes $\{xg \colon x \in E\}$, $g^{-1}E$ denotes $\{y \in S \colon y = gx \colon x \in E\}$ and Eg^{-1} denotes $\{y \in S \colon y = xg \colon x \in E\}$.

Let $(A_n)_{n=1}^{\infty}$ be a sequence of measurable subsets of S such that:

(i) $0 < \gamma(A_n) < \infty$;

(ii) $A_n \subseteq A_m$ if $n \leqslant m$;

(iii) for each k and n $A_k A_n$ is measurable and

$$\frac{\gamma(A_k A_n \triangle A_n)}{\gamma(A_n)} = 0\,;$$

and

(iv) there exists $K > 1$ such that $\gamma(A_n^{-1} A_n) \leqslant \gamma(A_n)$ for all n.

For $f \in L_F^1(X, \beta, \mu)$ let $f_g(x) = f(T_g x)$ and one checks easily that $x \to f_g(x)$ from A_n to $L_F^1(X, \beta, \mu)$ is integrable almost everywhere. Setting

$$\delta_n(f) = \frac{1}{\gamma(A_n)} \int\limits_{A_n} f_g \, d\gamma(g)\,.$$

Our theorem is the following.

THEOREM 4. *If (X, β, μ), and $(S, \mathcal{A}, \gamma)$ are as described after the statement of Theorem 3, $(A_n)_{n=1}^{\infty}$ satisfies* (i)–(iv) *and f is in $L_F^1(X, \beta, \mu)$. Then*

$$\lim_{n\to\infty} \delta_n(f)(x) = E(f, \mathcal{I})(x)$$

μ almost everywhere, where $\mathcal{I}$ is the sub σ-algebra of β of all sets, invariant under the semi-group $(T_g)_{g\in G}$.

Theorem 4 in the scalar case and in fact the Banach valued case appear in [1]. See also [7] and [3] for other special cases.

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