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ON EXTENDED VERSION OF $\aleph_0$-COVERING SETS AND THEIR APPLICATIONS

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ABSTRACT. We prove that

- There is a partition of 2^ω into an F_σ meager set M and a G_δ negligible set N such that every two countable, disjoint sets can be moved into M and N, respectively, by the same translation.
- Under CH there exists a Luzin (Sierpiński) set L and a meager (negligible) set X such that each countable subset of L can be obtained by an intersection of L and some translation of X.
- Under CH for every meager (negligible) set X there exists a Luzin (Sierpiński) set L such that each countable subset of X can be obtained by an intersection of X and some translation of L.
- The property of being an $\aleph_0$-covering set is not preserved by homeomorphisms defined on the whole space.

1. Notation

In this article we consider subsets of two topological groups, namely the real line: $\langle\mathbb{R},+\rangle$ and the Cantor set $\langle 2^\omega,+\rangle$, where in this case $(x+y)(i) = \bigl(x(i)+y(i)\bigr) \mod 2$. All the definitions below can be considered in these two structures.

Assume that $\mathcal{I}$ is an arbitrary σ-ideal with a Borel base, which means that if $X \in \mathcal{I}$ then there exists a Borel set $B \in \mathcal{I}$ such that $X \subseteq B$. Let $\mathcal{N}, \mathcal{M}$ and $\mathcal{E}$ denote the σ-ideal of negligible sets of reals, the σ-ideal of meager sets, and the σ-ideal generated by closed negligible sets, respectively. A set L is $\mathcal{I}$-Luzin (generalized $\mathcal{I}$-Luzin, respectively) if the intersection of L with any set from $\mathcal{I}$ is countable (of size $< 2^\omega$, respectively). Hence Luzin set is $\mathcal{M}$-Luzin and Sierpiński set is $\mathcal{N}$-Luzin.

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Recall that $cov(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \cup\mathcal{A} = \mathbb{R}(2^\omega)\}$. Denote: $\mathcal{I}^+ = \{X : X \notin \mathcal{I}\}$. The notation $G \leqslant \mathbb{R}(2^\omega)$ will mean that G is an additive subgroup of the real line (Cantor set, respectively). If $C \subseteq \mathbb{R}(2^\omega)$ then $\langle C\rangle$ denotes the additive subgroup generated by C.

With the customary abuse of notation, we often identify any equivalence relation R with a family $\mathcal{R}_R$ of its equivalence classes.

If $\mathcal{A}$ and $\mathcal{B}$ are partitions then we write $\mathcal{A} \prec \mathcal{B}$ iff $\mathcal{B}$ is a refinement of $\mathcal{A}$, which means that $\forall_{X\in\mathcal{B}}\exists_{Y\in\mathcal{A}} X \subseteq Y$. It is well-known that the family $Part$ of all partitions of ω equipped with the relation $\prec$ is a lattice. So let $\mathcal{A}\wedge\mathcal{B}$ denote the partition which is an infimum of $\{\mathcal{A}, \mathcal{B}\}$ in the lattice $\langle Part, \prec\rangle$

We say that X is a *selector* (*partial selector*, respectively) of a partition $\mathcal{A}$ iff $\forall_{A\in\mathcal{A}} |A \cap X| = 1$ ($|A\cap X| \leqslant 1$, respectively).

Recall also the following abbreviations: $\forall^\infty_{n\in\omega}$ means $\exists_{k\in\omega}\forall_{n>k}$, while $\exists^\infty_{n\in\omega}$ means $\forall_{k\in\omega}\exists_{n>k}$.

The complement of a set A is denoted by A^c.

We will use this combinatorial fact: Let $(I_n)_{n\in\omega}$ be a partition of ω into finite disjoint intervals, let $h(n) \leqslant 2^{|I_n|}$ for $n \in \omega$, for each $r \in 2^{I_0\cup\ldots\cup I_{n-1}}$ let $H_n^{(r)}$ be a subset of 2^{I_n} of size $h(n)$, and let

$$S = \left\{x \in 2^\omega : \exists^\infty_n x \restriction I_n \in H^{(x\restriction(I_0\cup\ldots\cup I_{n-1}))}\right\}.$$

Then S is a G_δ set, S has measure zero if and only if $\sum_{n=0}^{\infty} \frac{h(n)}{2^{|I_n|}} < \infty$, and the complement S^c of S is meager if and only if $\exists^\infty_n h(n) > 0$.

2. Borel $\aleph_0$-covering sets

The notion of an $\aleph_0$-covering set was defined and investigated by many authors, for example in [1], [2], [5] and [6]. Namely, a set $X \subseteq \mathbb{R}$ (or 2^ω) is $\aleph_0$-covering iff for every countable $D \subseteq \mathbb{R}$ (or 2^ω) there exists an $x \in \mathbb{R}$ (or 2^ω) such that $x + D \subseteq X$.

It is well-known that for many standard σ-ideals $\mathcal{I}$ (for example the σ-ideal $\mathcal{E}$) there exists an $\aleph_0$-covering X from $\mathcal{I}$.

A disjoint system $\{X_i : i \in I\}$ of $\aleph_0$-covering sets is said to have *the color property* if for every countable set D of reals and for every color: $X \to I$ there exists a real t such that $\{x \in D : \mathrm{color}(x) = i\} + t \subseteq X_i$ for every $i \in I$.

Let us recall this result ([6], Theorem 1):

Theorem 2.1. *If X is an $\aleph_0$-covering set and A is a set of cardinality $< 2^\omega$ then the set $X \setminus A$ is an $\aleph_0$-covering set, too. Therefore for every countable set C we have $\left|\{t : C + t \subseteq X\}\right| = 2^\omega$.*

In the next part of the paper we will use the following easy result:

Observation 2.2. *For every $\aleph_0$-covering set X there exists a partition $\{X_n : n \in \omega\}$ of X having the color property.*

Proof. Enumerate all disjoint sequences of countable sets and define the sets X_n, $n \in \omega$, simultaneously by induction up to the continuum by virtue of the fact that for each countable set C we have $\left|\{t : C + t \subseteq X\}\right| = 2^\omega$. $\square$

Let us notice that if $M \cup N = \mathbb{R}(2^\omega)$ is any partition into a meager set and a negligible set then both M and N are $\aleph_0$-covering. However, we have the following strengthening:

Theorem 2.3. *There exists a partition of 2^ω into a meager F_σ set M and negligible G_δ set N such that if X, Y are disjoint, countable, then there exists t such that $t + X \subseteq M$ and $t + Y \subseteq N$.*

Proof. Before starting the proof we show that if we do not require M, N to be Borel then the existence of the partition follows from Observation (2.2). Indeed, if $\mathbb{R} = M_1 \cup N_1$ is any partition of the real line into a meager and negligible set and Z is an $\aleph_0$-covering set which is both meager and negligible, then by virtue of Observation (2.2) there exist $X_0, X_1 \subseteq Z$ such that $X_0 \cap X_1 = \emptyset$ and for each countable, disjoint sets C_0, C_1 there exists t such that $C_0 + t \subseteq X_0$ and $C_1 + t \subseteq X_1$. Then define $M = (M_1 \setminus X_0) \cup X_1$ and $N = (N_1 \setminus X_1) \cup X_0$. It is easy to conclude that such defined sets M, N satisfy the required properties from Theorem 2.3, of course, except the property of M, N being Borel.

However, in the case of Borel sets the proof uses the structure of a Cantor set: First, let us choose a sequence of natural numbers $(s_n)_{n\in\omega}$ such that:

$s_0 = 1$,

$s_n = 4^{s_0 + \cdots + s_{n-1}}$.

Let $l_n = \sqrt{s_n}$. Next, choose a sequence of intervals $(I_n)_{n\in\omega}$ such that $\bigcup_{n\in\omega} I_n = \omega$, $\max I_n + 1 = \min I_{n+1}$ and moreover $|I_n| = s_n$. Let us divide I_n into subintervals: $I_n = I_n^{(0)} \cup \ldots \cup I_n^{(l_n - 1)}$, where each subinterval is of size l_n. For $j \in I_n$ let us define a function $e_n^{(j)} : I_n \to 2$ by $e_n^{(j)}(i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$

For $k < l_n$ let us define $H_n^{(k)} = \{e_n^{(j)} : j \in I_n^{(k)}\}$.

Let us observe that for every distinct $k_0, k_1 \in \{0, \ldots, l_n - 1\}$ and for every $t_0, t_1 \in 2^{I_n}$ we have $\left|(H_n^{(k_0)} + t_0) \cap (H_n^{(k_1)} + t_1)\right| \leqslant 1$. Therefore, for each $k_0 \in \{0, \ldots, l_n - 1\}$ and for every $t_0, \ldots, t_{l_n - 1} \in 2^{I_n}$ we have

$$\left(H_n^{(k_0)} + t_{k_0}\right) \setminus \bigcup_{\substack{k=0,\ldots,l_n-1 \\ k \neq k_0}} \left(H_n^{(k)} + t_k\right) \neq \emptyset. \tag{1}$$

Denote $J_n = I_0 \cup \dots \cup I_{n-1}$ and for each $n \in \omega$ choose any bijection $u_n \colon 2^{J_n} \to l_n$. Also, for the sake of simplicity let us denote for $r \in 2^{J_n}$: $\tilde{H}_n^{(r)} = H_n^{(u_n(r))}$, then $\left|\tilde{H}_n^{(r)}\right| = l_n$.

Next, define

$$M = \left\{x \in 2^\omega \colon \forall_n^\infty x \restriction I_n \notin \tilde{H}_n^{(x \restriction J_n)}\right\}$$

and

$$N = M^c = \left\{x \in 2^\omega \colon \exists_n^\infty x \restriction I_n \in \tilde{H}_n^{(x \restriction J_n)}\right\}.$$

It is clear that M is a meager F_σ set and N is a G_δ set. To verify that N is negligible we compute:

$$\sum_{n=0}^{\infty} \frac{\left|\tilde{H}_n^{(x \restriction J_n)}\right|}{2^{|I_n|}} = \sum_{n=0}^{\infty} \frac{l_n}{2^{|I_n|}} = \sum_{n=0}^{\infty} \frac{\sqrt{s_n}}{2^{s_n}} < \infty .$$

Now suppose that C and D are disjoint, infinite countable sets. Let $\{c_k\}_{k\in\omega}$ and $\{d_k\}_{k\in\omega}$ be a one-to-one enumeration of elements of sets C and D.

Let $r \colon \omega \to \omega$ be a function such that $\forall_{k\in\omega}\exists_n^\infty r(n) = k$. At first, choose a sequence $N_0 < N_1 < N_2 \dots$ such that for each $k \in \omega$ all the elements of

$$\left\{c_i \restriction J_{N_k} \colon i = 0, \dots, k-1\right\} \cup \left\{d_{r(k)} \restriction J_{N_k}\right\}$$

are distinct.

We will construct a sequence $x \in 2^\omega$ by induction. Suppose that we have defined $x \restriction I_0$, $x \restriction I_1$, $\dots$, $x \restriction I_{n-1}$. Choose $k \in \omega$ such that $n \in [N_k + 1; N_{k+1} + 1)$. Then the elements $\left\{c_i \restriction J_n \colon i = 0, \dots, k-1\right\} \cup \left\{d_{r(k)} \restriction J_n\right\}$ are all distinct. By the equation (1) we obtain that

$$\left(\tilde{H}_n^{((x+d_{r(k)})\restriction J_n)} + \left(d_{r(k)} \restriction I_n\right)\right) \setminus \bigcup_{i=0,\dots,k-1} \left(\tilde{H}_n^{((x+c_i)\restriction J_n)} + \left(c_i \restriction I_n\right)\right) \neq \emptyset . \quad (2)$$

Next, take as $x \restriction I_n$ any element of this nonempty set (2).

Finally, we obtain that $\forall_{k\in\omega} x + c_k \in M$ and $\forall_{k\in\omega} x + d_k \in N$. $\square$

3. Countable intersections

In this section we assume that $\mathcal{I}$ is moreover a transitive invariant σ-ideal (i.e., $\forall_{A\in\mathcal{I}}\forall_{t\in\mathbb{R}} A + t \in \mathcal{I}$).

For every $\mathcal{I}$-Luzin set L and $X \in \mathcal{I}$ we have $\forall_t |(X+t) \cap L| \leqslant \aleph_0$. Also we know that if L is a Luzin set and X is meager then $\exists_t (X+t) \cap L = \emptyset$ (see [4]). The same holds if L is a Sierpiński set and X is a negligible set (see [7]).

Therefore it is natural to ask whether for any countable $C \subseteq L$ one can find t such that $(X+t) \cap L = C$. Since we will frequently consider such notion of sets, for brevity let us define the following notation:

Definition 3.1. For any sets $X, Y \subseteq \mathbb{R}(2^\omega)$ we write $X \triangleright Y$ iff for each countable $C \subseteq Y$ there exists t such that $(X+t) \cap Y = C$.

We have:

Theorem 3.2. *Assume CH. If an invariant Borel generated σ-ideal $\mathcal{I}$ contains an $\aleph_0$-covering set, then there exists an $\mathcal{I}$-Luzin set L and $Z \in \mathcal{I}$ such that $Z \triangleright L$.*

Proof. By virtue of Observation (2.2) we conclude that there exists a set $Z \in \mathcal{I}$ such that for all countable disjoint sets C, D we have $\exists_t C \subseteq Z + t \wedge D \cap (Z+t) = \emptyset$.

Let $(C_\mu)_{\mu<2^\omega}$ be an enumeration of all countable subsets of $\mathbb{R}$.

Let $(B_\mu)_{\mu<2^\omega}$ be an enumeration of all Borel subsets of $\mathbb{R}$ from $\mathcal{I}$.

Next, by induction we construct a sequence $(x_\mu)_{\mu<2^\omega}$ and a sequence $(t_\mu)_{\mu<2^\omega}$.

Suppose that we are at the stage $\mu < 2^\omega$ and we assume that $\{t_\alpha, x_\alpha\}_{\alpha<\mu}$ are chosen.

To define t_μ we consider two cases:

Case 1: $C_\mu \nsubseteq \{x_\alpha\}_{\alpha<\mu}$. Then put $t_\mu = 0$.

Case 2: $C_\mu \subseteq \{x_\alpha\}_{\alpha<\mu}$. Then we choose t_μ to satisfy

$$[\{x_\alpha\}_{\alpha<\mu} \cap (Z + t_\mu)] = C_\mu .$$

Next, we choose x_μ to satisfy

$$x_\mu \notin (Z + t_\mu) \cup \bigcup_{\alpha<\mu} \{x_\alpha\} \cup B_\alpha \cup (Z + t_\alpha).$$

Define $L = \{x_\alpha\}_{\alpha<2^\omega}$.

It is obvious that L is a $\mathcal{I}$-Luzin set (generalized $\mathcal{I}$-Luzin set, respectively).

Suppose that $C \subseteq L$ is a countable set. Let $\mu < 2^\omega$ be such that $C \subseteq \{x_\alpha\}_{\alpha<\mu}$ and $C_\mu = C$. By construction of L we get $(Z + t_\mu) \cap L = C_\mu = C$ which completes the proof. □

In particular we have

Corollary 3.3. *Assume CH. There exists a Luzin (Sierpiński, respectively) set L and a meager (negligible, respectively) set X such that $X \triangleright L$.*

On the other hand, we have the following "dual" theorem:

Theorem 3.4. *Assume that $\mathcal{I}$ is an invariant Borel generated σ-ideal such that $cov(\mathcal{I}) = 2^\omega$. If $X \in \mathcal{I}$ is such that $-X \in \mathcal{I}$, then there exists a generalized $\mathcal{I}$-Luzin set L such that $L \triangleright X$.*

Proof. Let $(C_\mu)_{\mu<2^\omega}$ be an enumeration of all countable subsets of X (possibly with repetitions, however such that the empty set appears only once) and let $(B_\mu)_{\mu<2^\omega}$ be an enumeration of a Borel base of the σ-ideal $\mathcal{I}$. By induction we construct a sequence $(t_\mu)_{\mu<2^\omega}$.

At stage $\mu < 2^\omega$ we assume that $(t_\alpha)_{\alpha<\mu}$ are chosen. We pick $t_\mu \in \mathbb{R}$ such that

1. $(C_\mu + t_\mu) \cap \left[\bigcup_{\alpha<\mu} B_\alpha \cup (X + t_\alpha) \right] = \emptyset$;
2. $(X + t_\mu) \cap \bigcup_{\alpha<\mu} C_\alpha + t_\alpha = \emptyset$.

Finally, we put $L = \bigcup_{\mu<2^\omega} (C_\mu + t_\mu)$. First, let us observe that if $\alpha < \beta < 2^\omega$ then $(C_\beta + t_\beta) \cap (X + t_\alpha) = \emptyset$, therefore $\forall_{\alpha<\beta<2^\omega} (C_\alpha + t_\alpha) \cap (C_\beta + t_\beta) = \emptyset$. From this we conclude that $|L| = 2^\omega$.

Let $\mu < 2^\omega$. We have $L \cap (X + t_\mu) = \left[\bigcup_{\alpha<\mu} C_\alpha + t_\alpha \right] \cap (X + t_\mu) \cup (C_\mu + t_\mu) \cup \left[\bigcup_{\alpha>\mu} (C_\alpha + t_\alpha) \cap (X + t_\mu) \right] = \emptyset \cup (C_\mu + t_\mu) \cup \emptyset = C_\mu + t_\mu$, therefore $(L - t_\mu) \cap X = C_\mu$. □

In particular we have:

Corollary 3.5. *Assume CH. For every meager (negligible, respectively) set X there exists a Luzin (Sierpiński, resp.) set L such that $L \triangleright X$.*

4. Homeomorphic images

Among many families of small sets considered in set theory we can find σ-ideals which has the property that continuous image of an element from this ideal is also in this ideal. However, we can also find many such families, which do not have this property (eventually in some models). For example, the collection of strong measure zero set does not have this property (under CH). But we may consider weakening of this notion, namely many families of small sets have the property that if X is an element of this family, $h \colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism, then $h[X]$ is again an element of this family. In this case the σ-ideal of strong measure sets has this property. Unfortunately, it is still unknown whether the "dual" collection of strong first category sets has this property or does not.

In the next theorem we show that the family of $\aleph_0$-covering is not preserved under homeomorphism of the real line. In fact, we show that this notion is not preserved even in the case of simply polynomial homeomorphism, namely $h(x) = x^3$.

Let us introduce some temporary notation. By h denote a homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3$.

If $G \leqslant \mathbb{R}$ and $|G| \leqslant \aleph_0$ then we let $\sim_G$ denote the relation $x \sim_G y \iff x - y \in G$.

Let $\approx$ denote the relation $x \approx y \iff h(x) - h(y) \in \mathbb{Q}$.

Let $\mathcal{A}_G$ denote the partition generated by $\sim_G$ and let $\mathcal{B}$ denote the partition generated by $\approx$.

THEOREM 4.1. *There exists a set $Z \subseteq \mathbb{R}$ such that Z is $\aleph_0$-covering but the image $h[Z]$ is not.*

Proof. Let us start with the following lemma:

LEMMA 4.2. *For all but countably many pairs (A, B) with $A \in \mathcal{A}_G$ and $B \in \mathcal{B}$ we have $|A \cap B| \leqslant 1$.*

Proof. Let $S = \{a \in \mathbb{R} \colon \exists_{q \in G \setminus \{0\}} a \approx a + q\}$. If $A \in \mathcal{A}_G$ and $B \in \mathcal{B}$ are such that $|A \cap B| \geqslant 2$, then $A \cap B \cap S \neq \emptyset$. As $S = \{a \in \mathbb{R} \colon \exists_{q \in G \setminus \{0\}} \exists_{r \in \mathbb{Q}} 3a^2 q + 3aq^2 + (q^3 - r) = 0\}$, S is a countable set and the proof is finished. □

We will construct a set $X \subseteq \mathbb{R}$ with the following properties:

- X^c is an $\aleph_0$-covering;
- X is a selector of the partition $\mathcal{B}$, therefore there will be no $t \in \mathbb{R}$ such that $t + \mathbb{Q} \subseteq h[X^c]$.

The complement of such constructed set will have the required properties.

Let $\langle G_\alpha \colon \alpha < 2^\omega \rangle$ be an enumeration of all countable subgroups of $\mathbb{R}$.

We proceed by induction on $\mu < 2^\omega$. Assume that we are at stage $\mu < 2^\omega$ and we have constructed a sequence $\langle S_\alpha \colon \alpha < \mu \rangle$ such that

1. $\bigcup\limits_{\alpha < \mu} S_\alpha$ is a partial selector of the partition $\mathcal{B}$.
2. $\forall_{\alpha \neq \beta} S_\alpha \cap S_\beta = \emptyset$ and
3. $\forall_{\alpha < \mu} |S_\alpha| \leqslant \aleph_0$.

Since $\left| \bigcup\limits_{\alpha < \mu} S_\alpha \right| < 2^\omega$, by Lemma 4.2 there exists $A \in \mathcal{A}_{G_\mu}$ such that $A \cap \bigcup\limits_{\alpha < \mu} S_\alpha = \emptyset$ and $\forall_{B \in \mathcal{B}} |A \cap B| \leqslant 1$.

Let $\mathcal{B}' = \{B \in \mathcal{B} \colon A \cap B \neq \emptyset\}$ and for each $B \in \mathcal{B}'$ let us choose $s_B \in B \setminus A$ arbitrarily. Next let us put $S_\mu = \{s_B \colon B \in \mathcal{B}'\}$.

Having constructed the sequence $\langle S_\mu \colon \mu < 2^\omega \rangle$ let X be any extension of the partial selector $\bigcup\limits_{\alpha < 2^\omega} S_\alpha$ to a complete selector of the partition $\mathcal{B}$.

Such constructed set X satisfies the required conditions.

Indeed, if $C \subseteq \mathbb{R}$ is countable, then there exists $\mu < 2^\omega$ such that $\langle C \rangle = G_\mu$. Then there exists $A \in \mathcal{A}_{G_\mu}$ such that $X \cap A = \emptyset$, i.e., $\exists_{t \in \mathbb{R}} C + t \subseteq X^c$. □

5. Problems

In this section we formulate some unsolved problems. Namely:

We know (see [3] for a weaker version of this result) that there exists a function $\rho\colon F \to 2^\omega$ (where F is an F_σ meager set) such that

1. $\forall_E\ E$ is closed $\Rightarrow\ \rho^{-1}[E]$ is an F_σ set.
2. $\forall_{y\in 2^\omega}\, \rho^{-1}(\{y\})$ is an $\aleph_0$-covering.

Question 5.1. Does there exist a function $\rho\colon 2^\omega \to 2^\omega$ such that $\forall_E\ E$ is closed $\Rightarrow\ \rho^{-1}[E] \in F_\sigma$ and the partition $\{\rho^{-1}(\{y\})\colon y \in 2^\omega\}$ has the "color" property from Observation (2.2)?

Suppose that $\mathcal{I}$ is one of the ideals $\{\mathcal{M}, \mathcal{N}, \mathcal{E}\}$.

Question 5.2. Is it true that for every $\aleph_0$-covering Borel set X there exists a set $Y \subseteq X$ from $\mathcal{I}$ such that it is $\aleph_0$-covering, too? As the ideal $\mathcal{I}$ has a Borel base we can require Y to be Borel.

Note that for non–Borel sets X this conjecture is not true: for example put $\mathcal{I} = \mathcal{M}$ and as X take any $\aleph_0$-covering Luzin set.

Question 5.3. Does Theorem 2.3 hold also in the case of the real line instead 2^ω?

6. Generalizations

Definition 6.1. We say that $X \subseteq \mathbb{R}$ is an $\aleph_0$-separating set if for any two countable sets of reals $C \subseteq D$ there exists $t \in \mathbb{R}$ such that $(X+t)\cap D = C$.

Notice that if $\mathcal{F}$ is a disjoint system with the color property and $|\mathcal{F}| > 1$, then any set $X \subseteq \mathbb{R}$ for which there are $A, B \in \mathcal{F}$ such that $A \subseteq X \subseteq \mathbb{R}\setminus B$ is $\aleph_0$-separating. Hence, Observation 2.2 says that there is a lot of $\aleph_0$-separating subsets of an $\aleph_0$-covering set. By Theorem 2.3 there is a partition of 2^ω into two $\aleph_0$-separating sets, one of them is an F_σ meager set and the other is a G_δ measure zero set.

Observation 6.2. *Let X be an $\aleph_0$-covering set. There exists a partition $\{X_\alpha\colon \alpha < 2^\omega\}$ of X with the color property, i.e., for every disjoint system $\{D_\alpha\colon \alpha < 2^\omega\}$ with the countable union $\bigcup_{\alpha<2^\omega} D_\alpha$ there exists $t \in \mathbb{R}$ such that $D_\alpha + t \subseteq X_\alpha$ for all $\alpha < 2^\omega$. Moreover, we can find X_α so that they are Bernstein sets relatively in X.*

Proof. Let $\{D^\xi_\alpha : \alpha < 2^\omega\}$, $\xi < 2^\omega$, be an enumeration of all disjoint systems $\{D_\alpha : \alpha < 2^\omega\}$ with $\left|\bigcup_{\alpha<2^\omega} D_\alpha\right| \leqslant \omega$. Let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of relatively Borel subsets of X of size 2^ω with 2^ω repetitions. By induction on $\xi < 2^\omega$ we define $Y_\xi, X_{\alpha,\xi} \subseteq X$ for $\alpha < 2^\omega$, $t_\xi \in \mathbb{R}$, and $b_{\alpha,\xi} \in B_\xi$ for $\alpha < \xi$ such that

1. $Y_\xi = \bigcup\{X_{\alpha,\eta} : \alpha < 2^\omega \text{ and } \eta < \xi\}$,
2. $X_{\alpha,\xi} = D^\xi_\alpha + t_\xi \subseteq X \setminus \big(Y_\xi \cup \{b_{\beta,\eta} : \beta < \eta < \xi\}\big)$, and
3. $b_{\alpha,\xi} \in B_\xi \setminus \Big(Y_{\xi+1} \cup \big\{b_{\beta,\eta} : \beta < \eta \text{ and } \big(\eta < \xi \text{ or } (\eta = \xi \text{ and } \beta < \alpha)\big)\big\}\Big)$.

The choice of t_ξ is possible because the set $D = \bigcup_{\alpha<2^\omega} D^\xi_\alpha$ is countable, $|Y_\xi| < 2^\omega$, and $\big|\{t : D + t \subseteq X\}\big| = 2^\omega$.

Set $X_\alpha = \bigcup_{\xi<2^\omega} X_{\alpha,\xi} \cup \{b_{\alpha,\xi} : \xi > \alpha\}$. By the choice of $b_{\alpha,\xi}$, X_α is a Bernstein set for each α. Let $\{D_\alpha : \alpha < 2^\omega\}$ be a disjoint system with $\left|\bigcup_{\alpha<2^\omega} D_\alpha\right| \leqslant \omega$. There is ξ such that $D_\alpha = D^\xi_\alpha$, and then by (2), $D_\alpha + t_\xi = X_{\alpha,\xi} \subseteq X_\alpha$. □

Theorem 6.3. *Assume CH. Let $\mathcal{I}$ be an invariant σ-ideal with base of size $\leqslant 2^\omega$ and let $\mathcal{F} \subseteq \mathcal{I}$ be a family of $\aleph_0$-separating sets with $|\mathcal{F}| \leqslant 2^\omega$. For every $A \in \mathcal{I}^+$ there exists an $\mathcal{I}$-Luzin set $L \subseteq A$ such that $\forall_{X\in\mathcal{F}} X \triangleright L$.*

Proof. Let $\mathcal{F} = \{X_\alpha : \alpha < 2^\omega\}$ and $\{B_\alpha : \alpha < 2^\omega\}$ be a base of $\mathcal{I}$, both enumerations with repetitions if necessary, and let $\{C_\mu : \mu < 2^\omega\}$ be an enumeration of $[\mathbb{R}]^{\leqslant\omega}$ with 2^ω repetitions. We define $L = \{x_\mu : \mu < 2^\omega\}$ and $\{t_{\mu,\alpha} : \alpha < \mu < 2^\omega\}$ by induction on $\mu < 2^\omega$.

As μ is countable, for each β there is $t_{\mu,\beta} \in \mathbb{R}$ such that

$$(X_\beta + t_{\mu,\beta}) \cap \{x_\alpha : \alpha < \mu\} = C_\mu \cap \{x_\alpha : \alpha < \mu\}.$$

We choose x_μ so that

$$x_\mu \in A \setminus \left[\{x_\alpha : \alpha < \mu\} \cup \bigcup_{\alpha<\mu} B_\alpha \cup \bigcup_{\beta<\alpha<\mu} (X_\beta + t_{\alpha,\beta})\right].$$

Obviously, L is $\mathcal{I}$-Luzin and if $C \in [L]^{\leqslant\omega}$ and $\beta < 2^\omega$, then there is $\mu > \beta$ such that $C = C_\mu \subseteq \{x_\alpha : \alpha < \mu\}$. Then $(X_\beta + t_{\mu,\beta}) \cap L = C_\mu = C$. □

Theorem 6.4. *Let $\mathcal{I}$ be an invariant σ-ideal with base of size $\leqslant 2^\omega$ such that $cov(\mathcal{I}) = 2^\omega$. Let $\mathcal{F}$ be a family of size $\leqslant 2^\omega$ consisting of sets X such that both $X \in \mathcal{I}$ and $-X \in \mathcal{I}$. Then there exists a generalized $\mathcal{I}$-Luzin set L such that L is $\aleph_0$-covering and $\forall_{X\in\mathcal{F}} L \triangleright X$.*

Proof. Let $\mathcal{F} = \{X_\alpha : \alpha < 2^\omega\}$ and $\{B_\alpha : \alpha < 2^\omega\}$ be a base of $\mathcal{I}$, both enumerations with repetitions if necessary, and let $\{C_\mu : \mu < 2^\omega\}$ be an enumeration of $[\mathbb{R}]^{\leqslant\omega}$ with 2^ω repetitions. By induction on $\mu < 2^\omega$ we define $t_\mu \in \mathbb{R}$ such that

$$(C_\mu + t_\mu) \cap \left(\bigcup_{\alpha<\mu} B_\alpha \cup \bigcup_{\alpha<\beta<\mu} (X_\alpha + t_\beta) \right) = \emptyset\,, \tag{3}$$

$$\forall_{\alpha<\mu} (X_\alpha + t_\mu) \cap \bigcup_{\beta<\mu} (C_\beta + t_\beta) = \emptyset\,. \tag{4}$$

Set $L = \bigcup_{\mu<2^\omega} (C_\mu + t_\mu)$. Clearly L is $\aleph_0$-covering, and by (3) L is a generalized $\mathcal{I}$-Luzin set.

Let α be given and we prove that $L \triangleright X_\alpha$. Let $C \subseteq X_\alpha$ be countable. There is $\mu > \alpha$ such that $C = C_\mu$. The set $L \cap (X_\alpha + t_\mu)$ is the union of three sets

$$\bigcup_{\beta<\mu} [(C_\beta + t_\beta) \cap (X_\alpha + t_\mu)] \cup (C_\mu + t_\mu) \cup \bigcup_{\beta>\mu} [(C_\beta + t_\beta) \cap (X_\alpha + t_\mu)]$$

where the first set in the union is empty by (4) and the third set is empty by (3). Therefore $(L - t_\mu) \cap X_\alpha = C_\mu = C$. □

We have these properties of the relation $\triangleright$:

1. If $X \triangleright L$ and $L' \subseteq L$, then $X \triangleright L'$.
2. If $X \triangleright L$ and L is $\aleph_0$-covering, then so is X.

Corollary 6.5. *Assume CH.*

1. *For every meager (negligible) set X there exists a Luzin (Sierpiński) set L such that $L \triangleright X$.*
2. *Let $M \cup N$ be the partition of 2^ω from Theorem 2.3. There exist Luzin sets $L_2 \subseteq L_1$ and Sierpiński sets $S_2 \subseteq S_1$ such that $L_1 \triangleright M \triangleright L_2$ and $S_1 \triangleright N \triangleright S_2$.*

P r o o f . To prove 2. first apply Theorem 3.4 to the sets M and N and the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ so that we find L_1 and S_1, respectively, and then apply Theorem 3.2 to find $L_2 \subseteq L_1$ and $S_2 \subseteq S_1$. □

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