Tatra Mt. Math. Publ. **38** (2007), 41–56

**Tatra Mountains**
*Mathematical Publications*

# SCALAR NONLINEAR INTEGRAL EQUATIONS

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ABSTRACT. In a series of papers [T. A. Burton: *Integral equations, Volterra equations, and the remarkable resolvent: contractions*, Electron. J. Qual. Theory Differ. Equ. **2** (2006), 1–17], [T. A. Burton: *Integral equations, $L^p$-forcing, remarkable resolvent: Liapunov functionals*, Nonlinear Anal. **68** (2008), 35–46], [T. A. Burton: *Integral equations, large and small forcing functions: periodicity*, Math. Comput. Modelling **45** (2007), 1363–1375.] we have studied scalar linear integral equations and the role of the resolvent in duplicating the forcing function so that large forcing functions frequently have little effect on the solution, while small forcing functions can exert enormous control over the behavior of solutions. Much of that work depends entirely on linearity. The purpose of this paper is to see what can be proved in the nonlinear case and just how much techniques must be changed. The following question is forever before us. What is the qualitative difference between the solutions of

$$x(t) = (t+1)^{\frac{1}{2}} \sin(t+1)^{\frac{1}{3}} + \sin t - \int_0^t C(t,s)x(s)\,\mathrm{d}s$$

and

$$x(t) = \sin t - \int_0^t C(t,s)x(s)\,\mathrm{d}s$$

under "reasonable" conditions on $C(t,s)$? There would be no story to tell unless it were true that there is essentially no difference at all. The large function $(t+1)^{\frac{1}{2}} \sin(t+1)^{\frac{1}{3}}$ is simply swallowed up, while the tiny function $\sin t$ exerts enormous control over everything. The example presents us with three problems. Replace $x(s)$ in the integral by $g\big(x(s)\big)$ where $g(x)$ has the sign of $x$. First, determine exactly which properties on $C(t,s)$ constitute "reasonable" conditions. Then determine a vector space of big functions, like $a(t) = t$, which have little effect on solutions. Finally, determine a vector space of small functions, $a(t)$, which "rule the solution with an iron hand."

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2000 Mathematics Subject Classification: 47H10, 45A05, 45J05, 45M10.
Keywords: integral equations, contractions, Liapunov functions.

## 1. Introduction

In a series of papers [2], [4], [5] we have studied a scalar linear integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)\,\mathrm{d}s\,, \tag{1}$$

together with the resolvent equation

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)\,\mathrm{d}u\,, \tag{2}$$

and variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)\,\mathrm{d}s\,. \tag{3}$$

Those formulas are found in Burton [1; Chapter 7] and Miller [9; p. 190], for example.

All of the work is motivated by the following question. Under "reasonable" conditions on the kernel $C(t,s)$, what is the qualitative difference between the solutions of

$$x(t) = (t+1)^{\frac{1}{2}}\sin(t+1)^{\frac{1}{3}} + \sin t - \int_0^t C(t,s)x(s)\,\mathrm{d}s \tag{4a}$$

and

$$x(t) = \sin t - \int_0^t C(t,s)x(s)\,\mathrm{d}s\,? \tag{4b}$$

In fact, the solutions are almost indistinguishable. Moreover, there are large vector spaces containing unbounded functions for which $\int_0^t R(t,s)a(s)\,\mathrm{d}s$ is almost an exact copy of $a(t)$, while there are also large vector spaces of small functions which that integral is totally unable to copy. Those results depend strongly on the linearity.

The application to physical problems is obvious. There will be large disturbances which will have little effect on the problem, while some small disturbances will have continuing effect. Ignoring small perturbations can lead to disaster, while worrying over large perturbations may be totally unnecessary.

## 2. An effective Liapunov functional

Here, we begin the study of similar properties in a nonlinear case, trying to see exactly how the techniques must be changed to accommodate the nonlinearities. Our work will center on the equation

$$x(t) = a(t) - \int_0^t C(t,s)g\big(x(s)\big)\,\mathrm{d}s \tag{6}$$

where $a\colon [0,\infty) \to \Re$, $g\colon \Re \to \Re$, and $C\colon \Re \times \Re \to \Re$ are at least continuous and

$$xg(x) > 0 \qquad \text{for} \qquad x \neq 0\,. \tag{7}$$

These conditions are sufficient to ensure that (6) has at least one solution which, if it remains bounded, can be continued to $[0,\infty)$. One may consult Burton [1; Chapter 7], Miller [9], Corduneanu [6], or Grippenburg-Londen-Staffans [7] for details. We will now offer two results to the general effect that if $a' \in L^p$ then any solution of (6) satisfies $g\big(x(t)\big) \in L^p$. Here, $C_1$ or $C_t$ means the partial derivative of $C(t,s)$ with respect to $t$. If a derivative is indicated, we assume it to be continuous. We use $C_1$ and $C_t$ interchangeably to avoid confusion with dummy variables. Our first result is exactly the same as the linear case, but notice how (8) integrates only the first coordinate and how well $a(t)$ and $x(t)$ separate in $V'$; that will be a major concern here.

**Theorem 2.1.** *Suppose that $a' \in L^1[0,\infty)$ and there is an $\alpha > 0$ with*

$$-C(t,t) + \int_0^\infty |C_1(u+t,t)|\,\mathrm{d}u \le -\alpha\,. \tag{8}$$

*If $x$ is a solution of (6) then $g(x) \in L^1[0,\infty)$ and $x$ is bounded so $g(x) \in L^p$ for $p \ge 1$.*

Proof. If $x(t)$ solves (6) then it also solves

$$x'(t) = a'(t) - C(t,t)g\big(x(t)\big) - \int_0^t C_1(t,s)g\big(x(s)\big)\,\mathrm{d}s \tag{9}$$

and we define a Liapunov functional by

$$V\big(t,x(\cdot)\big) = |x(t)| + \int_0^t \int_{t-s}^\infty |C_1(u+s,s)|\,\mathrm{d}s\,\big|g\big(x(s)\big)\big|\,\mathrm{d}s\,. \tag{10}$$

We let $V(t) := V\big(t, x(\cdot)\big)$ and find that

$$\begin{aligned} V'(t) &\le |a'(t)| - C(t,t)\big|g\big(x(t)\big)\big| + \int_0^t \big|C_1(t,s)g\big(x(s)\big)\big|\,\mathrm{d}s \\ &\quad + \big|g\big(x(t)\big)\big| \int_0^\infty |C_1(u+t,t)|\,\mathrm{d}u - \int_0^t \big|C_1\big(t,s)\big)g\big(x(s)\big)\big|\,\mathrm{d}s \\ &\le |a'(t)| - \alpha\big|g\big(x(t)\big)\big|. \end{aligned}$$

Thus, so long as the solution is defined then

$$|x(t)| \le V(t) \le V(0) + \int_0^t |a'(s)|\,\mathrm{d}s - \alpha \int_0^t \big|g\big(x(s)\big)\big|\,\mathrm{d}s\,.$$

Hence, $|x(t)| \le V(0) + \int_0^\infty |a'(s)|\,\mathrm{d}s$ so $x$ can be continued on $[0,\infty)$. Moreover, $g(x) \in L^1[0,\infty)$. □

One may note that we do not need $a'$ integrable on the half-line to parlay the above proof into existence of the solution for $t > 0$ using only (8). It is also a curious fact that we did not need $\int_0^x g(s)\,\mathrm{d}s$ to diverge with $x$ in order to obtain boundedness. This will change in the next result.

In this result $a'$ and $g(x)$ separate completely in $V'$ so that we can integrate them separately and achieve the conclusion. In the next result our main problem is to contrive a Liapunov functional so that $a'$ and $g(x)$ will separate in $V'$ in such a way that no assumption on the nonlinearity need be made. That is a significant challenge, but it can be conquered exactly as desired. In a later result we will be forced to use Young's inequality to achieve a separation that will require monotonicity of $g$.

The reader should note that it is $a' \in L^{2n}[0,\infty)$, resulting in $g(x) \in L^{2n}[0,\infty)$. Notice also how both coordinates of $C$ are integrated, but the burden falls on the second coordinate as $n$ becomes large in marked contrast to the last result. These properties are so important to help us understand the richness of the nonconvolution case.

**Theorem 2.2.** *Suppose there is a positive integer $n$ with $a'(t) \in L^{2n}[0,\infty)$, a constant $\alpha > 0$, and a constant $N > 0$ with*

$$\frac{2n-1}{2nN^{\frac{2n}{2n-1}}} - C(t,t) + \frac{2n-1}{2n}\int_0^t |C_1(t,s)|\,\mathrm{d}s + \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|\,\mathrm{d}u \le -\alpha\,. \tag{11}$$

*If $x$ is a solution on $[0,\infty)$, then $g(x) \in L^{2n}[0,\infty)$. If $\int_0^x g^{2n-1}(s)\,\mathrm{d}s \to \infty$ as $|x| \to \infty$ then any solution $x$ of* (6) *is bounded.*

Proof. Define

$$V\big(t,x(\cdot)\big) = \int_0^x g^{2n-1}(s)\,\mathrm{d}s + \frac{1}{2n}\int_0^t \int_{t-s}^\infty |C_t(u+s,s)|\,\mathrm{d}u\, g^{2n}\big(x(s)\big)\,\mathrm{d}s\,.$$

Then we have

$$\begin{aligned}
V'(t) &= a'(t)g^{2n-1}(x) - C(t,t)g^{2n}(x) - g^{2n-1}(x)\int_0^t C_1(t,s)g\big(x(s)\big)\,\mathrm{d}s \\
&\quad + \frac{1}{2n}\int_0^\infty |C_1(u+t,t)|\,\mathrm{d}u\, g^{2n}\big(x(t)\big) - \frac{1}{2n}\int_0^t |C_1(t,s)|g^{2n}\big(x(s)\big)\,\mathrm{d}s \\
&\le \frac{(2n-1)g^{2n}(x)}{2nN^{\frac{2n}{2n-1}}} + \frac{\big(Na'(t)\big)^{2n}}{2n} - C(t,t)g^{2n}(x) \\
&\quad + \int_0^t |C_t(t,s)|\left[\frac{(2n-1)g^{2n}\big(x(t)\big)}{2n} + \frac{g^{2n}\big(x(s)\big)}{2n}\right]\mathrm{d}s \\
&\quad + g^{2n}\big(x(t)\big)\frac{1}{2n}\int_0^\infty |C_1(u+t,t)|\,\mathrm{d}u - \frac{1}{2n}\int_0^t |C_1(t,s)|g^{2n}\big(x(s)\big)\,\mathrm{d}s \\
&\le \frac{\big(Na'(t)\big)^{2n}}{2n} - \alpha g^{2n}\big(x(t)\big).
\end{aligned}$$

Thus,

$$\int_0^x g^{2n-1}(s)\,\mathrm{d}s \le V(t) \le V(0) + \int_0^t \frac{\big(Na'(s)\big)^{2n}}{2n}\,\mathrm{d}s - \alpha\int_0^t g^{2n}\big(x(s)\big)\,\mathrm{d}s\,.$$

When the solution can be defined for all future time, $g(x) \in L^{2n}[0,\infty)$. If the integral on the left diverges with $|x|$, then $x$ is bounded. $\square$

This result shows that when (11) holds and $\int_0^x g(s)\,\mathrm{d}s \to \infty$ as $|x| \to \infty$ then for $0 < \beta < 1$ and

$$a(t) = (t+1)^{\beta} + \sin(t+1)^{\beta} + (t+1)^{\frac{1}{2}} \sin(t+1)^{\frac{1}{3}} \tag{12}$$

then we have $g(x) \in L^{2n}[0,\infty)$ for some $n > 0$.

The fact that no condition on divergence of the integral of $g$ is needed for boundedness in Theorem 2.1, but divergence is needed in Theorem 2.2 is a common problem usually traceable to method of proof rather than a property of the equation. Even if $\int_0^x g(s)\,\mathrm{d}s$ fails to diverge, one may use $g \in L^{2n}$ in several ways to obtain boundedness of the solution. For example, by squaring (6) we get $(\frac{1}{2})x^2(t) \le a^2(t) + \int_0^t C^2(t,s)\,\mathrm{d}s \int_0^t g^2\big(x(s)\big)\,\mathrm{d}s$. Moreover, if $|g(x)| \le |x|$ and $C(t,s)$ is continuous, then the solution of (6) is continuable for every continuous $a(t)$; thus, failure of $\int_0^x g(s)\,\mathrm{d}s$ to diverge in both directions and the solution being noncontinuable would require an exceedingly chaotic $g(x)$.

## 3. Asymptotic periodicity

Let $\mathcal{P}_T$ be the set of continuous $T$-periodic scalar functions and let $Q$ be the set of continuous functions $q\colon [0,\infty) \to R$ such that $q(t) \to 0$ as $t \to \infty$. Denote by $(Y, \|\cdot\|)$ the Banach space of functions $\phi\colon [0,\infty) \to R$ where $\phi \in Y$ implies that $\phi = p + q$ with $p \in \mathcal{P}_T$ and $q \in Q$.

In our work here, we will use a contraction mapping and that will require that $g$ have a bounded derivative, as required in (15) below. But that is a requirement based only on the method of proof and it can likely be removed using a different kind of proof, possibly by the method of *a priori* bounds.

Suppose that

$$C(t+T, s+T) = C(t,s), \tag{13}$$

that

$$\int_{-\infty}^{t} C(t,s)\,\mathrm{d}s \qquad \text{is bounded and continuous,} \tag{14}$$

and that $g$ has a continuous derivative, denoted by $g^*$, with

$$|g^*(x)| \le 1. \tag{15}$$

For a fixed $\phi = p + q \in Y$ by the mean value theorem for derivatives we have

$$g\big(\phi(t)\big) = g\big(p(t)\big) + g\big(p(t) + q(t)\big) - g\big(p(t)\big) = g\big(p(t)\big) + g^*\big(\xi(t)\big)q(t)$$

where $\xi(t)$ is between $p(t) + q(t)$ and $q(t)$. This means that $g(\phi) \in Y$. Note that this did not require (15) since $g^*$ would be bounded along the fixed function, $\phi$. Thus, the ideas seem fully nonlinear.

We will also require

$$\int_{-\infty}^{0} |C(t,s)| \, \mathrm{d}s \to 0 \qquad \text{as} \quad t \to \infty \tag{16}$$

and for $q \in Q$ then

$$\int_{0}^{t} C(t,s)q(s) \, \mathrm{d}s \to 0 \qquad \text{as} \quad t \to \infty. \tag{17}$$

In order to have a contraction we ask that there exists an $\alpha < 1$ with

$$\int_{0}^{t} |C(t,s)| \, \mathrm{d}s \le \alpha. \tag{18}$$

To prove that (6) has an asymptotically periodic solution, we begin by defining a mapping from (6) by $\phi = p + q \in Y$ implies that

$$(P\phi)(t) = a(t) - \int_{0}^{t} C(t,s) g\big(\phi(s)\big) \, \mathrm{d}s \,. \tag{19}$$

The next result is a nonlinear version of Theorem 2.3 in Burton [5].

**Theorem 3.1.** *If* (13–18) *hold and if* $a \in Y$, *so is the unique solution of* (6).

Proof. Clearly, (19) is a contraction, but we must show that $P\colon Y \to Y$. Write $a = p^* + q^* \in Y$ and for $\phi = p + q \in Y$ and

$$(P\phi)(t) = p(t) - \int_{-\infty}^{t} C(t,s)g\bigl(p(s)\bigr)\,\mathrm{d}s - \int_{0}^{t} C(t,s)g^*\bigl(\xi(s)\bigr)q(s)\,\mathrm{d}s + \int_{-\infty}^{0} C(t,s)g\bigl(p(s)\bigr)\,\mathrm{d}s\,.$$

Clearly, the first two terms on the right are periodic, while the remainder is in $Q$. Thus, $P\colon Y \to Y$ and there is a fixed point. $\square$

As shown in [5] for the linear case, under mild conditions the solution approaches a non-constant periodic function.

## 4. Another Liapunov function

We come now to a Liapunov functional which will require much more about the behavior of $C$, but it is a naturally nonlinear functional. It is closely adapted from work of Levin [8] and we have used it several times in our linear work in Burton [2]–[5], for example.

**Theorem 4.1.** *Let $H(t,s) := C_t(t,s)$, and suppose there is an $\alpha > 0$ with $C(t,t) \geq \alpha$ and*

$$H(t,s) \geq 0,\ H_t(t,s) \leq 0,\ H_s(t,s) \geq 0,\ H_{st}(t,s) \leq 0. \tag{23}$$

(i) *If $a' \in L^2[0,\infty)$, then any solution of* (6) *or* (9) *on $[0,\infty)$ satisfies $g(x) \in L^2[0,\infty)$.*

(ii) *If $a'$ is bounded, if $g^2(x) \to \infty$ as $|x| \to \infty$, and if there is an $M > 0$ with*

$$\int_0^t H_s(t,s)(t-s)\int_s^t |a'(u)|^2\,\mathrm{d}u\,\mathrm{d}s + H(t,0)t\int_0^t |a'(u)|^2\,\mathrm{d}u \leq M, \tag{24}$$

*then any solution of* (6) *or* (9) *has $\int_0^x g(s)\,\mathrm{d}s$ bounded.*

Proof. Define

$$
\begin{aligned}
V(t) = {} & \int_0^x g(s)\,\mathrm{d}s + \left(\frac{1}{2}\right) \int_0^t H_s(t,s) \left( \int_s^t g\big(x(u)\big)\,\mathrm{d}u \right)^2 \mathrm{d}s \\
& + \left(\frac{1}{2}\right) H(t,0) \left( \int_0^t g\big(x(s)\big)\,\mathrm{d}s \right)^2 \\
= {} & \int_0^x g(s)\,\mathrm{d}s + \left(\frac{1}{2}\right) \int_0^t C_{st}(t,s) \left( \int_s^t g\big(x(u)\big)\,\mathrm{d}u \right)^2 \mathrm{d}s \\
& + \left(\frac{1}{2}\right) C_t(t,0) \left( \int_0^t g\big(x(s)\big)\,\mathrm{d}s \right)^2
\end{aligned}
\tag{25}
$$

so that if $V(t)$ is bounded, so is $\int_0^x g(s)\,\mathrm{d}s$. Next, write

$$
x' = a'(t) - C(t,t)g(x) - \int_0^t H(t,s) g\big(x(s)\big)\,\mathrm{d}s\,.
$$

Then the derivative of $V$ along a solution is

$$
\begin{aligned}
V'(t) = {} & a'(t)g(x) - C(t,t)g^2(x) - g(x) \int_0^t C_t(t,s) g\big(x(s)\big)\,\mathrm{d}s \\
& + (1/2) \int_0^t H_{st}(t,s) \left( \int_s^t g\big(x(u)\big)\,\mathrm{d}u \right)^2 \mathrm{d}s \\
& + g\big(x(t)\big) \int_0^t H_s(t,s) \int_s^t g\big(x(u)\big)\,\mathrm{d}u\,\mathrm{d}s \\
& + (1/2) H_t(t,0) \left( \int_0^t g\big(x(s)\big)\,\mathrm{d}s \right)^2 + H(t,0) g\big(x(t)\big) \int_0^t g\big(x(s)\big)\,\mathrm{d}s\,.
\end{aligned}
$$

We integrate the fifth term on the right by parts and obtain

$$
\begin{aligned}
g\big(x(t)\big)\Bigg[H(t,s)\int_s^t g\big(x(u)\big)\,\mathrm{d}u\Big|_0^t + \int_0^t H(t,s)g\big(x(s)\big)\,\mathrm{d}s\Bigg]& \\
= -g\big(x(t)\big)H(t,0)\int_0^t g\big(x(u)\big)\,\mathrm{d}u\, + g\big(x(t)\big)\int_0^t H(t,s)g\big(x(s)\big)\,\mathrm{d}s\,.&
\end{aligned}
$$

Canceling terms and taking into account sign conditions yields

$$
\begin{aligned}
V'(t)\, &\le a'(t)g(x) - C(t,t)g(x)^2 \\
&\le \left(\frac{1}{2}\alpha\right)|a'(t)|^2 + \left(\frac{\alpha}{2}\right)g^2(x) - \alpha g^2(x) \\
&\le \left(\frac{1}{2}\right)\frac{(|a'(t)|^2)}{\alpha} - \left(\frac{\alpha}{2}\right)g^2(x).
\end{aligned}
$$

Hence,

$$
2\int_0^x g(s)\,\mathrm{d}s\, \le 2V(t) \le 2V(0) + \left(\frac{1}{\alpha}\right)\int_0^t |a'(s)|^2\,\mathrm{d}s\, - \alpha\int_0^t g^2\big(x(s)\big)\,\mathrm{d}s
$$

so (i) follows. Note that $\int_0^x g(s)\,\mathrm{d}s$ is bounded if $V$ is bounded.

Now, assume $a'(t)$ bounded and let (24) hold; we will bound $V$ and, hence, $\int_0^x g(s)\,\mathrm{d}s$. From $V'$ we see that there is a $\mu > 0$ such that if $V'(t) > 0$ then $|x(t)| < \mu$. Suppose, by way of contradiction, that $V$ is not bounded. Then there is a sequence $\{t_n\} \uparrow \infty$ with $V'(t_n) \ge 0$ and $V(t_n) \ge V(s)$ for $0 \le s \le t_n$; thus, $|x(t_n)| \le \mu$. If $0 \le s \le t_n$ then

$$
0 \le 2V(t_n) - 2V(s) \le -\alpha\int_s^{t_n} g^2\big(x(u)\big)\,\mathrm{d}u\, + \left(\frac{1}{\alpha}\right)\int_s^{t_n} |a'(u)|^2\,\mathrm{d}u\,.
$$

Using these values in the formula for $V$, taking $|x(t_n)| \le \mu$, $t = t_n$, and applying the Schwarz inequality yields

$$\begin{aligned}V(t) \le &\int_0^{\pm\mu} g(s)\,\mathrm{d}s\\ &+ \int_0^t H_s(t,s)(t-s)\int_s^t \left(\frac{1}{\alpha^2}\right)|a'(u)|^2\,\mathrm{d}u\,\mathrm{d}s\\ &+ H(t,0)t\left(\frac{1}{\alpha^2}\right)\int_0^t |a'(u)|^2\,\mathrm{d}u\\ \le &\int_0^{\pm\mu} g(s)\,\mathrm{d}s + \left(\frac{1}{\alpha^2}\right)M\end{aligned}$$

Thus, $V(t)$ and $\int_0^x g(s)\,\mathrm{d}s$ are bounded. □

## 5. A functional for the integral equation

In working with our next Liapunov functional we find

$$V'(t) = 2a(t)g(x) - 2xg(x) \tag{26}$$

and we need to separate $a(t)g(x)$. There are many *ad hoc* ways of doing that but under certain conditions there is a very exact result.

**Lemma 5.1.** *Let $g(x) = -g(-x)$, $g$ be strictly increasing, for $x \ge 0$ let $\phi(x) := \frac{\mathrm{d}}{\mathrm{d}x} xg^{-1}(x)$ be monotone increasing to infinity with $\phi(0) = 0$. Then*

$$2|a(t)g(x)| \le xg(x) + \int_0^{2|a(t)|} \phi^{-1}(s)\,\mathrm{d}s\,. \tag{27}$$

Proof. Young's inequality states that if $\phi\colon [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, satisfies $\phi(0) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$, and if $\psi = \phi^{-1}$ then for $\Phi(x) = \int_0^x \phi(u)\,\mathrm{d}u$ and $\Psi(x) = \int_0^x \psi(u)\,\mathrm{d}u$ we have

$$2|a(t)g(x)| \le \Phi\big(g(x)\big) + \Psi\big(2|a(t)|\big). \tag{28}$$

But

$$\Phi\big(g(x)\big) = \int_0^{g(x)} \frac{\mathrm{d}}{\mathrm{d}s} s g^{-1}(s)\,\mathrm{d}s = g(x)g^{-1}\big(g(x)\big) = xg(x),$$

as required. □

Our next result has its roots in Volterra [10] in 1928, in Levin [8] in 1963, and in Burton [3] in 1993. A much expanded linear version appeared in Burton [4]. Part (ii) is new.

**Theorem 5.2.** *If $a\colon [0,\infty) \to R$ is continuous, while*

$$C(t,s) \geq 0, \quad C_s(t,s) \geq 0, \quad C_t(t,s) \leq 0, \quad C_{st}(t,s) \leq 0 \tag{29}$$

*then along the solution of* (6) *the functional*

$$V(t) = \int_0^t C_s(t,s)\left(\int_s^t g\big(x(u)\big)\,\mathrm{d}u\right)^2 \mathrm{d}s + C(t,0)\left(\int_0^t g\big(x(s)\big)\,\mathrm{d}s\right)^2 \tag{30}$$

*satisfies*

$$V'(t) \leq 2a(t)g(x) - 2xg(x).$$

(i) *If there are constants $B$ and $K$ with*

$$\sup_{t\geq 0} \int_0^t C_s(t,s)\,\mathrm{d}s = B < \infty \quad \text{and} \quad \sup_{t\geq 0} C(t,0) = K < \infty \tag{31}$$

*then along the solution of* (6) *we have*

$$\big(a(t) - x(t)\big)^2 \leq 2(B+K)V(t) \tag{32}$$

(ii) *If the conditions of Lemma 5.1 hold then along a solution of* (6) *we have*

$$V'(t) \leq -x(t)g\big(x(t)\big) + \int_0^{2|a(t)|} \phi^{-1}(s)\,\mathrm{d}s\,.$$

*Hence, if the last term is in $L^1[0,\infty)$ then so is $x(t)g\big(x(t)\big)$. Moreover, $V$ is then bounded so if* (31) *holds then $|a(t)-x(t)|$ is bounded.*

Proof. We have

$$V(t) = \int_0^t C_s(t,s)\left(\int_s^t g\big(x(u)\big)\,\mathrm{d}u\right)^2 \mathrm{d}s + C(t,0)\left(\int_0^t g\big(x(s)\big)\,\mathrm{d}s\right)^2$$

and differentiate along a solution of (6) to obtain

$$\begin{aligned} V'(t) = &\int_0^t C_{st}(t,s)\biggl(\int_s^t g\bigl(x(u)\bigr)\,\mathrm{d}u\biggr)^2 \mathrm{d}s \\ &+ 2g(x)\int_0^t C_s(t,s)\int_s^t g\bigl(x(u)\bigr)\,\mathrm{d}u\,\mathrm{d}s \\ &+ C_t(t,0)\biggl(\int_0^t g\bigl(x(s)\bigr)\,\mathrm{d}s\biggr)^2 \\ &+ 2g(x)C(t,0)\int_0^t g\bigl(x(s)\bigr)\,\mathrm{d}s\,. \end{aligned}$$

We now integrate the third-to-last term by parts to obtain

$$\begin{aligned} &2g(x)\biggl[C(t,s)\int_s^t g\bigl(x(u)\bigr)\,\mathrm{d}u\,\bigg|_0^t + \int_0^t C(t,s)g\bigl(x(s)\bigr)\,\mathrm{d}s\biggr] \\ &\qquad = 2g(x)\biggl[-C(t,0)\int_0^t g\bigl(x(u)\bigr)\,\mathrm{d}u + \int_0^t C(t,s)g\bigl(x(s)\bigr)\,\mathrm{d}s\biggr]. \qquad (1) \end{aligned}$$

Cancel terms, use the sign conditions, and use (6) in the second line below to unite the Liapunov functional and the equation obtaining

$$\begin{aligned} V'(t) = &\int_0^t C_{st}(t,s)\biggl(\int_s^t g\bigl(x(u)\bigr)\,\mathrm{d}u\biggr)^2 \mathrm{d}s + C_t(t,0)\biggl(\int_0^t g\bigl(x(s)\bigr)\,\mathrm{d}s\biggr)^2 \\ &+ 2g(x)\bigl[a(t) - x(t)\bigr] \\ &\le 2g(x)a(t) - 2xg(x) \le -xg(x) + \int_0^{2|a(t)|} \phi^{-1}(s)\,\mathrm{d}s\,. \end{aligned}$$

The lower bound given in (32) may be derived as in B u r t o n [3]. The final conclusion is now immediate. □

## 6. Continuous dependence

The applied mathematician correctly claims that our conditions of $a'$ bounded or in $L^p$ may be difficult to establish because of uncertainties and even stochastic forces. There is a simple way around that if $\frac{\mathrm{d}}{\mathrm{d}x}g(x) =: g^*(x)$ is bounded. For a given function $b(t)$ seek a function $a(t)$ which satisfies one of our boundedness theorems with $|a(t)-b(t)|$ bounded. Here is a sample theorem. We take a simple condition known to imply that $a \in BC$ implies that the solution of (6) is in $BC$ when $g(x) = x$. Many other conditions are known such as (29) and the argument in the proof of Theorem 4.1.

**Proposition 6.1.** *Suppose that $|g^*(x)| \le 1$ and that there is an $\alpha < 1$ with $\int_0^t |C(t,s)|\,\mathrm{d}s \le \alpha$.*

*If*

$$x(t) = a(t) - \int_0^t C(t,s)g\big(x(s)\big)\,\mathrm{d}s$$

*and*

$$y(t) = b(t) - \int_0^t C(t,s)g\big(y(s)\big)\,\mathrm{d}s$$

*with $a - b \in BC$, so is $x - y$.*

P r o o f. Note that for fixed solutions $x$ and $y$ we have

$$\begin{aligned} x(t) - y(t) &= a(t) - b(t) - \int_0^t C(t,s)\Big[g\big(x(s)\big) - g\big(y(s)\big)\Big]\,\mathrm{d}s \\ &= a(t) - b(t) - \int_0^t C(t,s)g^*\big(\xi(s)\big)\big[x(s) - y(s)\big]\,\mathrm{d}s \end{aligned}$$

by the mean value theorem for derivatives where $\xi(s)$ is between $x(s)$ and $y(s)$. The resulting integral equation has a bounded solution. □

## 7. Unsolved problems

Note the transition from Theorem 2.1 to Theorem 2.2. By contriving a Liapunov functional with higher powers of $g(x)$ we are able to pass from the requirement of $a' \in L^1$ (which allows only bounded $a(t)$) to $a' \in L^{2n}$ which allows $a(t) = (t+1)^{\beta}$ for $0 < \beta < 1$. Now look at Theorem 4.1 in which we allow $a' \in L^2$. This allows $a(t) = \ln(t+1)$ which is surprisingly large in view of classical results. But it would be a real *coup* to introduce a higher power of $g(x)$ in the Liapunov functional and allow $a' \in L^p$ for $0 < p < \infty$. As with all Liapunov theory, patience and imagination should achieve the result.

In [5] for the linear case we obtain the counterpart of Theorem 3.1. But we also differentiate (6) and obtain the same result for $a' \in Y$; this includes $a(t) = t + \sin t$. It is a great surprise that the solution is in $Y$ since this means that the function $t$ is completely absorbed, yielding virtually no effect on the long-term behavior of the solution, while $\sin t$ exerts continued influence. Again, it would be a real *coup* to prove this for (9). The problem is that when $g(x) = x$ then (9) has $x' = -C(t,t)x + f\big(t, x(\cdot)\big)$ which can be written as an integral equation, an effective mapping of $Y \to Y$. A clever map must be constructed for the nonlinear case.

Finally, we have handled the terms of $a(t)$ in (4a) and (4b) separately, but we have not put them together in (6).

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Received November 7, 2006

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