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ON WEAKLY COMPACT OPERATORS

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Summary. The paper deals with some questions of weakly compact mappings and brings an example disproving the assertion that every finitely additive vector Borel measure with bounded semivariation represents a weakly compact operator.

In [P] Th. 5 and Cor. 1 P e l c z y n s k i proved the following.

Theorem. *Let S be a compact Hausdorff topological space and X a Banach space such that no subspace of X is isomorphic to c_0. Then every continuous linear operator L from $C(S)$ into X is weakly compact.*

Recall that c_0 denotes the Banach space of all to zero converging sequences of numbers with the sup norm; $C(S)$ is the Banach space of all continuous real or complex-valued functions on S with the sup norm. That $L\colon C(S) \to X$ is weakly compact means that for every bounded sequence (f_n) of functions from $C(S)$ the sequence (Lf_n) contains a subsequence (Lg_k) weakly convergent to an element x in X.

Corollary 1. *Let X be a Banach space. Then each continuous linear operator from $C(S)$, where S is an infinite compact metric space, into X is weakly compact if and only if X contains no subspace isomorphic to c_0.*

The assumption of metrizability of S is essential. It follows from the fact that each continuous linear operator from l^∞ [$l^\infty = m$ denotes the Banach space of all bounded sequences of numbers with the sup norm] into c_0 is weakly compact [Go], p. 168, G o o d n e r, or [Gr], p. 229, G r o t h., into every separable Banach space. More over into a quasicomplete Hausdorff separable locally convex topological vector space, if S is a Stonian space, i.e., the closure of any open set is again open.

2000 M a t h e m a t i c s S u b j e c t C l a s s i f i c a t i o n: 28E10, 81P10.
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An arbitrary continuous linear mapping from $C(S)$ into X, with S locally compact, is weakly compact if and only if X is weakly Σ-complete, i.e., satisfies the following condition: For every sequence $(x_n)_{n\in N}$ of points of X, such that

$$\Sigma_n \big| < x_n, x' > \big| < \infty$$

for all $x' \in X'$, there exists $x \in X$ such that

$$< x, x' >= \Sigma_n < x_n, x' >,$$

(cf. [P], [BeP], [Th], Thomas). This condition is satisfied when X is semi-reflexive or weakly sequentially complete. In the paper [P] Pelczynski describes weakly Σ-complete Banach space as a space not containing a subspace isomorphic to c_0 on the basis of a result of [Be P], p. 160, according to which a Banach space is weakly Σ-complete if (and only if) there is no subspace isomorphic to c_0 in X. It is not known if this equivalence holds for arbitrary quasi-complete spaces [Th], p. 20. On the other hand there exist an infinite compact metric space S and a continuous linear operator L on $C(S)$ into l^∞ which is not weakly compact. As known (cf [P], p. 221, Pelczynski A.) an operator having this property may be given by the formula

$$L(f) = \big(f(s_n) - f(a)\big), \; f \in C(S), \quad n = 1, 2, \dots$$

where (s_n) is a convergent sequence of different points in S and $a = \lim\limits_{n\to\infty} s_n$. Note that in this case $L(f)$ even takes its values in c_0.

Let ε_s be the Dirac measure concentrated in $s \in S$. Define, for any positive integer n, the real-valued Borel measure l_n by $l_n(E) = (\varepsilon_{s_n} - \varepsilon_a)(E)$, E a Borel subset of S. Define

$$\boldsymbol{m}(E) = \big(l_n(E)\big)_{n=1}^{\infty}.$$

Then $\boldsymbol{m}$ is a finitely additive set function on Borel sets $\boldsymbol{B}(S)$ with values in l^∞. The set function $\boldsymbol{m}$ has the bounded semivariation, i.e., the quantity

$$\sup \left\| \sum_{i=1}^{k} d_i \, \boldsymbol{m}(E_i) \right\|$$

is finite, where the supremum is taken over all disjoint sets E_i in $\boldsymbol{B}(S)$ with the union S and all complex numbers d_i with the absolute value equal or less than one.

COUNTEREXAMPLE. We will give a promised counterexample. By the assigning

$$f \to \int f d\boldsymbol{m} = \left(\int f(s)\, dl_n(s)\right)_{n=1}^{\infty}$$

a representation of the continuous linear operator $L\colon C(S)$ into l^∞ is given by means of the finitely additive set function $\boldsymbol{m}\colon \boldsymbol{B}(S) \to l^\infty$. We have, of course,

$$\int f\,d\boldsymbol{m} = \left(\int f(s)\,dl_n(s)\right)_{n=1}^{\infty} = \left(\int f(s)\,d\varepsilon_{s_n}(s) - \int f(s)\,d\varepsilon_a\right)$$
$$= \big(f(s_n) - f(a)\big)_{n=1}^{\infty} \in l^\infty(\text{even} \in c_0)\,.$$

Take $S = [0,1]$ and put

$$f_k(s) = \frac{ks}{1+k^2s^2}, \quad k = 1,2,\dots$$

for s in S and $s_n = 1/n$, $n = 1,2,\dots$, so $a = 0$. For all s and $k = 1,2\dots$, clearly, $0 \leqslant f_k(s) \leqslant 1/2, f_k(s_k) = 1/2$, $\sup\limits_s f_k(s) = \frac{1}{2}$. We $\frac{1}{2} = \|f_k\| \leqslant \frac{1}{2}$, $k = 1,2,\dots$ and $f_k(s) \to 0$ for $k \to \infty$ for every s and

$$\left\|\int f_k d\boldsymbol{m}\right\| = \sup_n \left|f_k(s_n)\right| = \frac{1}{2}, k = 1,2,\dots$$

Thus (f_k) is a bounded sequence of functions from $C(S)$ converging to zero at every point s in S, i.e., (f_k) converges weakly to 0 in $C(S)$. Hence the assigning $f \to L(f) = \int f d\boldsymbol{m}$ is not weakly compact operator on $C(S)$ into l^∞ because $L(f_k)$ does not converge to zero in l^∞ as it should be for a weakly compact operator from $C(S)$ into l^∞. [DS] IV.7.4. Dunford–Schwartz.

The set function $\boldsymbol{m}$ considered here is a finitely additive on $\boldsymbol{B}(S)$ with bounded semivariation which is not countably additive (and hence) nor regular, takes its values in the Banach space $X = l^\infty$; $\boldsymbol{m}$ does not, however, represent a weakly compact operator on $C(S)$ into X. This example shows that it is not true that every **finitely** additive set function on Borel sets in S, with values in Banach space X with bounded semivariation defines a weakly compact operator on $C(S)$ into X as asserted in [Wh] Theorem 1.4.

It is easy to prove that if m has bounded semi-variation then the set

$$V_{\boldsymbol{m}} = \left\{\sum_{j=1}^{k} \boldsymbol{m}(E_j)\alpha_j \mid k \in N^*, |\alpha_j| \leqslant 1\right\}$$

is a bounded set in X.

We will say that $\boldsymbol{m}$ has weakly compact semi-variation if $\boldsymbol{m}$ has bounded semi-variation and if the set $V_{\boldsymbol{m}}$ is included in a weakly compact set W of X.

It is clear that, when $\boldsymbol{m}$ has weakly compact semi-variation, for any continuous function $f\colon [a,b] \to R$, we have $\int f\ d\boldsymbol{m} \in \|f\|W$ if $V_{\boldsymbol{m}} \subseteq W$.

If $V_{\boldsymbol{m}} \subseteq W$, then $\boldsymbol{m}(S) \subseteq W$.

On the other hand our set function $\boldsymbol{m}$ representing a continuous linear operator on $C(S)$ into c_0 takes its values in the second dual $c_0'' = l^\infty$ and for every $u \in l^1$ the set function $u \circ \boldsymbol{m}(E) = \boldsymbol{m}(E)u$ is regular and countably additive. But this is quite in the accordance with the representation theorem of continuous linear operators on $C(S)$ into X, cf. [DS] Th. VI.7.3.

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