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CONVERGENCE THEOREMS FOR (SL)-INTEGRAL IN RIESZ SPACE-CONTEXT

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ABSTRACT. We prove some convergence theorems for the (SL)-integral in Riesz spaces introduced in [A. Boccuto: *Differential and integral calculus in Riesz spaces*, Tatra Mt. Math. Publ. **14** (1998), 293–323], and we prove some results concerning the equivalence between the (SL)- and Kurzweil-Henstock integrals for Riesz space-valued functions.

1. Introduction

There are several generalizations of the Riemann integral both in the classical and abstract theory of integration. A variant of the Kurzweil-Henstock integral coinciding with that for real-valued functions, in general different from that for Riesz space-valued mappings, is a strong Luzin integral ((SL)-integral), that was investigated in the case of real-valued functions by P. Y. Lee and R. Výborný ([15], [16]). The idea concerning with this type of integral is as follows: we introduce a condition, so called *strong Luzin condition* or *property* (SL) which lies strictly between an absolute continuity and a condition N (roughly speaking, a function f *satisfies a condition* N if it maps sets of measure zero onto the sets of measure zero). We say that a function f is (SL)-integrable if it "admits" a "weak primitive" of the class (SL). In the Kurzweil-Henstock integration, the condition (SL) plays a very similar role to that by absolute continuity in the theory of Lebesgue integral. We prove that under suitable hypotheses for our involved Riesz space, an integral function associated with the Kurzweil-Henstock integral is of class (SL). Moreover, we will see that, under the same hypotheses, the (SL)- and the Kurzweil–Henstock integral coincide: this, in general, is true only in some particular Riesz spaces.

Furthermore, some convergence theorems are proved for the (SL)-integral, for Riesz space-valued functions.

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2 Preliminaries

Definition 2.1. Giving an interval $[a,b] \subset \mathbb{R}$, the *division of* $[a,b]$ is called a finite set $T = \{x_0, x_1, \dots, x_n\} \subset [a,b]$, where $x_0 = a$, $x_n = b$ and $x_{i-1} < x_i$ for all $i = 1, \dots, n$. By $\mathcal{T}$ we denote the class of all divisions of $[a,b]$.

By a *partition of* $[a,b]$ we understand a set of type $\{(A_i, \xi_i) \colon i = 1, \dots, n\}$, where $A_i = [x_{i-1}, x_i]$, $\{x_0, x_1, \dots x_n\}$ is a division, and $\xi_i \in A_i$ for all $i = 1, \dots, n$. A partition is said to be *special* if ξ_i is an endpoint of A_i for every $i = 1, \dots, n$. A *mesh* of a partition Π denotes the quantity $|\Pi| \equiv \max_{i=1}^{n}(x_i - x_{i-1})$. Moreover, since no confusion might arise, giving a measurable subset A of the (extended) real line, we denote by $|A|$ its Lebesgue measure (finite or $+\infty$).

Definition 2.2. A *decomposition of* $[a,b]$ is a set of the type

$$\{(A_i, \xi_i) \colon i = 1, \dots, n\}, \tag{1}$$

where $\{A_i\}_{i=1}^{n}$ is a family of pairwise non-overlapping intervals of $[a,b]$ and $\xi_i \in A_i$ for all $i = 1, \dots, n$.

We remark that a decomposition of $[a,b]$ is not necessarily a partition of $[a,b]$.

From now on, let us denote by Δ a set of all positive real-valued functions defined on an interval $[a,b] \subset \mathbb{R}$.

Definition 2.3. Giving a partition or decomposition $\Pi = \{([x_{i-1}, x_i], \xi_i) \colon i = 1, \dots, n\}$ of $[a,b]$, and a function $\delta \in \Delta$, we say that Π is δ*-fine* if $x_i - x_{i-1} \le \delta(\xi_i)$ for all $i = 1, \dots, n$.

Moreover, if $f \colon [a,b] \to \mathbb{R}$ is a mapping, and Π is as above, we denote by $\sum\limits_{\Pi} f$ the quantity

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition 2.4. A function $\delta \colon [a,b] \to \mathbb{R}_0^+$ is called *gage* if the set $Z_\delta \equiv \{x \in [a,b] \colon \delta(x) = 0\}$ has Lebesgue measure zero.

We now endow the set of all gages with the following ordering. Giving two gages δ_1 and δ_2, we say that $\delta_1 \le \delta_2$ if and only if $\delta_1(x) \le \delta_2(x)$ for all $x \in [a,b]$.

Definition 2.5. If δ is a gage, we say that a decomposition $\Pi = \{(A_i, \xi_i) \colon i = 1, \dots, n\}$ is δ*-fine* if $\xi_i \notin Z_\delta$ and $|A_i| \le \delta(\xi_i)$ for all $i = 1, \dots, n$.

Remark 2.6. We remark that there exist some gages δ, without δ-fine partitions. However, for all gages δ there exists a δ-fine decomposition Π (for a proof see [28, Theorem 1, p. 259]).

Definition 2.7. Let $J \subset [a,b] \subset \mathbb{R}$, and $\Delta_J = (\mathbb{R}^+)^J$. We say that a function $P\colon [a,b] \to \mathbb{R}$ is *of class* (SL) or has the *property* (SL) on J if for every set $N \subset [a,b]$ of Lebesgue measure zero and $\forall\,\varepsilon > 0$ there exists a map $\delta\colon J \to \mathbb{R}^+$ such that for any δ-fine decomposition $\Pi = \big\{\big([x_{i-1}, x_i], \xi_i\big) : i = 1, \dots, n\big\}$ of $[a,b]$, with $x_i \in [a,b]$ and $\xi_i \in N \cap J$ $\forall\, i = 1, \dots, n$, we have:

$$\sum_{i=1}^{n} \big|P(x_i) - P(x_{i-1})\big| \le \varepsilon\,. \tag{2}$$

We say that P is *of class* (SL) if it is of class (SL) on $[a,b]$.

Definition 2.8. Let R be any Dedekind complete Riesz space, and $\Delta^{(1)}$ be the set of all gages defined on $[a,b]$. We say that $f\colon [a,b] \to \mathbb{R}$ is (SL)*-integrable* on $[a,b]$ if there exists a function $P\colon [a,b] \to \mathbb{R}$ of class (SL) such that, for every $\varepsilon > 0$, there exists $\delta \in \Delta^{(1)}$ such that

$$\left|\sum_{\Pi} f - \sum_{i=1}^{n} \big[P(x_i) - P(x_{i-1})\big]\right| \le \varepsilon \tag{3}$$

whenever $\Pi = \big\{\big([x_{i-1}, x_i], \xi_i\big) : i = 1, \dots, n\big\}$ is a δ-fine decomposition of $[a,b]$. The function P will be called *weak primitive* of f.

In this case we put (by *definition*) $(\mathrm{SL})\int_a^b f = P(b) - P(a)$.

The (SL) integral is well-defined: indeed we have the following

Proposition 2.9. *Let $N \subset [a,b]$ be a set of Lebesgue measure zero, $f_0\colon [a,b] \to \mathbb{R}$ such that $f_0(x) = 0$ for all $x \notin N$ and $P_0\colon [a,b] \to \mathbb{R}$ be a constant function. Then P_0 is a weak primitive of f_0.*

Proposition 2.10. *Let N and f_0 be as in Proposition 2.9, and P be a weak primitive of f_0. Then P is constant.*

The following fundamental result holds (see [15], [16]).

Theorem 2.11. *A function $f\colon [a,b] \to \mathbb{R}$ is Kurzweil-Henstock integrable on $[a,b]$ if and only if it is (SL)-integrable on $[a,b]$, and in this case we have*

$$(\mathrm{KH})\int_a^b f = (\mathrm{SL})\int_a^b f\,.$$

3. The (SL)-integral in the Riesz-space context

Now, we introduce some preliminary definitions concerning Riesz spaces, and later the (KH)- and (SL)-integrals.

We begin with the following:

Definition 3.1. Let R be any Riesz space. A *unit of* R is an element $u \in R$ such that $u \geq 0$, $u \neq 0$.

Definition 3.2. Let u be a unit of R. We say that the sequence $(r_n)_n$ of elements of R *u-converges* to $r \in R$ if $\forall \varepsilon > 0$, $\exists\, \overline{n} = \overline{n}(\varepsilon) \in \mathbb{N}$ such that $|r_n - r| \leq \varepsilon u$, $\forall\, n \geq \overline{n}$. We say that $(r_n)_n$ *converges relatively* (or (r)-*converges*) to r if it u-converges, for some unit u. In this case, we write $(r)\lim\limits_{n\to+\infty} r_n = r$ or $(r)\lim\limits_n r_n = r$.

Definition 3.3. Let u be a unit of R, and $(\Lambda, \prec)$ be any directed set. We say that the net $(r_\lambda)_{\lambda\in\Lambda}$ *u-converges* to r if $\forall \varepsilon > 0$, $\exists\, \overline{\lambda} = \overline{\lambda}(\varepsilon) \in \Lambda$ such that

$$|r_\lambda - r| \leq \varepsilon u\,, \qquad \forall\, \lambda \in \Lambda\,,\ \lambda \prec \overline{\lambda}\,.$$

We say that $(r_\lambda)_{\lambda\in\Lambda}$ *converges relatively* (or (r)-*converges*) to r if it u-converges, for some unit u. In this case, we write $(r)\lim\limits_{\lambda\in\Lambda} r_\lambda = r$ or $(r)\lim\limits_\lambda r_\lambda = r$.

Definition 3.4. A sequence $(r_n)_n$ of elements of R is said to be *order-convergent* (or (O)-*convergent*) to $r \in R$, if there exists a sequence $(p_n)_n$ in R, such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall\, n \in \mathbb{N}$ (see also [17], [39]). In this case, we write $(\mathrm{O})\lim\limits_n r_n = r$.

Let us note that, in the context of (O)-convergence, the definition of series of the type $\sum\limits_{n=1}^{\infty} a_n$, with $a_n \in R$ $\forall n \in \mathbb{N}$, can be formulated analogously as in classic case.

Definition 3.5. Let $(\Lambda, \prec)$ be any directed set. A net $(r_\lambda)_{\lambda\in\Lambda}$ is said to be *order convergent* (or (O)-*convergent*) to r if

$$\sup_{\rho\in\Lambda} \inf_{\lambda\prec\rho} r_\lambda = \inf_{\rho\in\Lambda} \sup_{\lambda\prec\rho} r_\lambda\,,$$

and in this case we write $(\mathrm{O})\lim\limits_{\lambda\in\Lambda} r_\lambda = r$. We observe that if a net (O)-converges both to r and s, then $r \equiv s$ necessarily.

We remark that if R is Archimedean then (r)-convergence implies (O)-convergence. In general, the converse is not true (for example, in the spaces of type $\mathcal{C}(\Omega) = \{f\colon \Omega \to \mathbb{R}\,,\ f \text{ is continuous}\}$ where Ω is an infinite compact topological space). However, there are many cases when these two concepts coincide, for example in the spaces of type $L^0(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B}, \mu)$ is a measure space with μ-positive, σ-additive and σ-finite measure (see [17], [39]).

Definition 3.6. A Dedekind complete Riesz space R, such that every subset $R_1 \neq \emptyset$ bounded from above containing a countable subset having the same supremum as R_1, is said to be *super Dedekind complete.*

Definition 3.7. Let R be a Riesz space. We say that R has a *property* σ or σ*-property* if for all sequences $(r_n)_n$ of positive elements of R there exists a sequence $(\lambda_n)_n$ of strictly positive real numbers such that the sequence $(\lambda_n r_n)_n$ is bounded (see also [17, Definition 70.1, p. 478]).

Remark 3.8. We observe that if $(X, \mathcal{B}, \mu)$ is a measure space where μ is a σ-finite countably additive $\widetilde{\mathbb{R}}$-valued measure, then the spaces $L^p(X, \mathcal{B}, \mu)$, with $p \in [0, \infty]$, have the property σ (see also [17]). Moreover, we note that there exist some super Dedekind complete Riesz spaces lacking in property σ (for example, we can take the space $R \subset \mathbb{R}^{\mathbb{N}}$ consisting of all sequences with only a finite number of nonzero coordinates: see [17, p. 479]).

Assumptions 3.9. *From now on, we always suppose R to be a Dedekind complete Riesz space. In some suitable cases, we add R an extra element $+\infty$ extending ordering and operations having the same role as the element $+\infty$ with respect to real numbers, in such a way*

$$\begin{cases} +\infty \geq r\,, & \forall\, r \in R\,, \\ 0 \cdot (+\infty) = 0\,, & \\ \lambda \cdot (+\infty) = +\infty\,, & \forall\, \lambda \in \mathbb{R}^+, \\ +\infty + r = +\infty\,, & \forall\, r \in R \cup \{+\infty\}\,. \end{cases}$$

We denote by the symbol $\overline{R}$ the set $R \cup \{+\infty\}$.

Definition 3.10. A net $(p_\delta)_{\delta \in \Delta}$ of elements of $\overline{R}$ is called (O)*-net* if the following conditions are fulfilled:

(3.10.1) $p_{\delta_1} \leq p_{\delta_2}$ whenever $\delta_1, \delta_2 \in \Delta$, $\delta_1 \leq \delta_2$;

(3.10.2) $\inf\{p_\delta \colon \delta \in \Delta\} = 0$.

A sequence $(p_n)_n$ in $\overline{R}$ is called (O)*-sequence* if it is decreasing and $\inf\limits_n p_n = 0$.

Definition 3.11. A map $f \colon [a, b] \to R$ is (KH)*-integrable* (on $[a, b]$) if there exists an element $Y \in R$ such that

$$(\mathrm{O}) \lim_{\delta \in \Delta} \left[\sup \left\{ \left| \sum_{\Pi} f - Y \right| : \Pi \text{ is a } \delta\text{-fine partition of } [a, b] \right\} \right] = 0\,. \tag{4}$$

Remark 3.12. It is easy to see that in Definition 3.11 the element Y is unique, provided that it exists; in this case we write

$$Y = (\mathrm{KH}) \int_a^b f\,.$$

Without loss of generality, in (4) we can take the set of δ-fine special partitions of $[a,b]$ instead of the set of all δ-fine partitions of $[a,b]$: indeed, giving a partition "generated" by a division $T = \{x_0, x_1, \dots, x_n\}$, with $\xi_i \in [x_{i-1}, x_i]$ for all $i = 1, \dots, n$, and such that ξ_i is not an endpoint of $[x_{i-1}, x_i]$ for some i, it will be sufficient to consider the division T' obtained by adding such ξ_i's to the points of T, and to associate the same ξ_i's with T' as in the original partition.

In general, the (HK)- and (SL)-integral are different from each other in a general Riesz space (see also the following example ([3]). However, we will prove that, under suitable hypotheses, these two integrals coincide.

EXAMPLE 3.13. Let R be a Dedekind complete Riesz space without property σ (see Remark 3.8). In R there exists a sequence $(u_n)_n$ such that for all sequences $(\lambda_n)_n$ of positive real numbers, the sequence $(\lambda_n u_n)_n$ is not bounded in R. The function $f\colon [0,1] \to R$, defined by setting

$$f(x) = \begin{cases} u_n & \text{if } x = 1/n\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{5}$$

is not (KH)-integrable on $[0,1]$. This function is (SL)-integrable on $[0,1]$ with (SL)-integral zero, because it coincides with the identically zero function up to the complement of a set of Lebesgue measure 0.

We observe, however, that, if R has the property σ and $f\colon [a,b] \to R$ vanishes up to the complement of a set of Lebesgue measure zero, then f is (SL)-integrable on $[a,b]$ with integral zero (see also [5]).

We also recall a version of the Henstock Lemma for the (KH)-integral (see also [3]).

LEMMA 3.14. *If f is* (KH)*-integrable on $[a,b]$ and $F(I) = (\mathrm{KH})\int_I f$ for every subinterval $I \subset [a,b]$, then*

$$\inf_{\delta\in\Delta}\left\{\sup\sum_{i=1}^{q}\bigl|\,|J_i|f(\xi_i)-F(J_i)\bigr|\colon \Pi \textit{ is a } \delta\textit{-fine decomposition of } [a,b]\right\} = 0 \tag{6}$$

(*where* $\Pi \equiv \{(J_i,\xi_i)\colon i = 1,\dots,q\}$).

We also recall the following result (see [3]):

THEOREM 3.15. *Let R be any Dedekind complete Riesz space, and f_1, f_2, g, $h\colon [a,b] \to R$ be four maps* (KH)*-integrable on $[a,b]$ such that $g \le f_j \le h$ for $j = 1,2$.*

Then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are (KH)*-integrable on $[a,b]$.*

Now, introduce the notion of function of a class (SL) (with respect to (O)-convergence).

Definition 3.16. Let $J \subset [a,b] \subset \mathbb{R}$, and $\Delta_J = (\mathbb{R}^+)^J$. We say that a function $P\colon [a,b] \to R$ is *of class* (SL) or has the *property* (SL) on J if for every set $N \subset [a,b]$ of Lebesgue measure zero there exists an (O)-net $(p_\delta^{(N)})_{\delta \in \Delta_J}$, such that $\forall\, \delta \in \Delta_J$

$$\sup\left\{ \sum_{i=1}^{n} \left|P(x_i) - P(x_{i-1})\right| \colon \left\{\left([x_{i-1}, x_i], \xi_i\right) \colon i = 1, \dots, n\right\} \text{ is a } \delta\text{-fine} \right.$$
$$\left. \text{decomposition of } [a,b],\ x_i \in [a,b] \text{ and } \xi_i \in N \cap J\ \forall\, i \right\} \le p_\delta^{(N)}. \quad (7)$$

We say that P is *of class* (SL) on the assumption that it is of class (SL) on $[a,b]$.

Definition 3.17. Let R be any Dedekind complete Riesz space, and $\Delta^{(1)}$ be the set of all gages defined on $[a,b]$. We say that $f\colon [a,b] \to R$ is (SL)*-integrable* on $[a,b]$ if there exists a function $P\colon [a,b] \to R$ of class (SL) and an (O)-net $(p_\delta)_{\delta \in \Delta^{(1)}}$ such that $\forall \delta \in \Delta^{(1)}$ we have:

$$\sup\left\{ \left| \sum_{\Pi} f - \sum_{i=1}^{n} \left[P(x_i) - P(x_{i-1})\right] \right| \colon \Pi = \left\{\left([x_{i-1}, x_i], \xi_i\right) \colon i = 1, \dots, n\right\} \right.$$
$$\left. \text{is a } \delta\text{-fine decomposition of } [a,b] \right\} \le p_\delta\,.$$

The function P will be called *weak primitive* of f.

In this case we put (according to the *definition*) $(\mathrm{SL})\int_a^b f = P(b) - P(a)$.

In [3] it is proved that the (SL)-integral is well-defined. There are the following propositions there:

Proposition 3.18. *Let R be any Dedekind complete Riesz space, $N \subset [a,b]$ be a set of Lebesgue measure zero, and let $f_0\colon [a,b] \to R$ be such that $f_0(x) = 0$ for all $x \notin N$. Then*

$$(\mathrm{SL})\int_a^b f_0 = 0\,.$$

Proposition 3.19. *Let R be as in Proposition 3.18, and $f, g\colon [a,b] \to R$ be two functions differing only in a set of Lebesgue measure zero. Then f is* (SL)*-integrable on $[a,b]$ if and only if g is either, and in this case,*

$$(\mathrm{SL})\int_a^b f = (\mathrm{SL})\int_a^b g\,.$$

Proposition 3.20. *Let $f\colon [a,b] \to R$ be* (SL)*-integrable on $[a,b]$, and let P_1, P_2 be two weak primitives of f. Then*

$$P_1(\beta) - P_1(\alpha) = P_2(\beta) - P_2(\alpha)$$

for all α, β with $[\alpha, \beta] \subset [a,b]$.

In [3], the following comparison result of (KH)- and (SL)-integrals for *bounded* functions was given considering values in an *arbitrary* Dedekind complete Riesz spaces was given.

Theorem 3.21. *Let R be any Dedekind complete Riesz space and let $f\colon [a,b] \to R$ be bounded. Then the following are equivalent:*

(3.21.1) *f is* (KH)*-integrable on $[a,b]$;*

(3.21.2) *f is* (SL)*-integrable on $[a,b]$.*

In this case we have

$$(\mathrm{KH})\int_a^b f = (\mathrm{SL})\int_a^b f\,.$$

By suitable hypotheses for our Riesz space R, we will prove that the (KH)- and (SL)-integrals coincide even for functions being not necessarily bounded on $[a,b]$. We start with a formulation of hypotheses.

Assumptions 3.22. *From now on, we assume that $R = L^0(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space, with μ-positive, σ-finite, σ-additive and $\widetilde{\mathbb{R}}$-valued.*

First of all, since (r)- and (O)-convergences coincide in our space L^0, we observe that the definitions of functions of class (SL) and of (SL)-integrability can be formulated as follows:

Definition 3.23. Let $J \subset [a,b] \subset \mathbb{R}$, and $\Delta_J = (\mathbb{R}^+)^J$. We say that a function $P\colon [a,b] \to R$ is *of class* (SL) or has the *property* (SL) on J if for every set $N \subset [a,b]$ of Lebesgue measure zero there exists a unit w such that, $\forall\, \varepsilon > 0$, there exists a map $\delta\colon J \to \mathbb{R}^+$ such that, for any δ-fine decomposition $\Pi = \big\{([x_{i-1}, x_i], \xi_i)\colon i = 1, \dots, n\big\}$ of $[a,b]$, with $x_i \in [a,b]$ and $\xi_i \in N \cap J$ $\forall\, i = 1, \dots, n$, we have:

$$\sum_{i=1}^{n} \big|P(x_i) - P(x_{i-1})\big| \le \varepsilon w\,. \tag{8}$$

We say that P is *of class (SL)* provided it is of class (SL) on $[a,b]$.

Definition 3.24. Let $\Delta^{(1)}$ be the set of all gages defined on $[a,b]$. We say that $f\colon [a,b] \to R$ is *(SL)-integrable* on $[a,b]$ if there exists a function $P\colon [a,b] \to R$ of class (SL) and a unit w such that, for every $\varepsilon > 0$, there exists $\delta \in \Delta^{(1)}$ such that

$$\left| \sum_{\Pi} f - \sum_{i=1}^{n} \left[P(x_i) - P(x_{i-1}) \right] \right| \leq \varepsilon w \,, \tag{9}$$

whenever $\Pi = \left\{ \left([x_{i-1}, x_i], \xi_i \right) : i = 1, \dots, n \right\}$ is a δ-fine decomposition of $[a,b]$. The function P will be called *weak primitive* of f.

In this case we put (by *definition*) $(\mathrm{SL})\int_a^b f = P(b) - P(a)$.

Now we prove that if R satisfies Assumptions 3.22 then the (SL)-integral coincides with the (KH)-integral. First of all we prove that, for super Dedekind complete Riesz space-valued maps, "the integral function associated with Kurzweil-Henstock integral in $[a,b]$ is of class (SL) on $[a,b]$".

Proposition 3.25. *Assume that R is a super Dedekind complete Riesz space, and let $f\colon [a,b] \to R$ be* (KH)*-integrable on $[a,b]$. Let F be a* (KH)*-integral function associated with f, that is $F(a) = 0$ and*

$$F(x) = (\mathrm{KH})\int_a^x f \,, \qquad \forall\, x \in (a,b] \,.$$

Then F is of class (SL) *on $[a,b]$.*

Proof. We know that f is integrable on every subinterval $[c,d] \subset [a,b]$. So, the definition of F given above makes sense. Let Ω be such a compact Hausdorff extremely disconnected topological space (unique up to homeomorphisms) existing by virtue of Maeda–Ogasawara–Vulikh representation theorem (see [1]). For all $\omega \in \Omega$, define $f_\omega\colon [a,b] \to \widetilde{\mathbb{R}}$ by setting:

$$f_\omega(x) = f(x)(\omega) \,, \qquad x \in [a,b] \,. \tag{10}$$

Throughout the proof of Lemma 3.4 of [4], we proved that there exists a meager set $N_0 \subset \Omega$ such that $\forall\, \omega \in \Omega \setminus N_0$, $\forall\, x \in [a,b]$,

$$F(x)(\omega) \equiv \left((\mathrm{KH})\int_a^x f \right)(\omega) = (\mathrm{KH})\int_a^x f_\omega \in \mathbb{R} \,, \tag{11}$$

where the f_ω's are as in (10). Let

$$\widetilde{F}_\omega(x) = (\mathrm{KH})\int_a^x f_\omega \,, \qquad \omega \in \Omega \setminus N_0 \,,\ x \in [a,b] \,.$$

For such ω's, the function $\widetilde{F}_\omega$ is of class (SL) on $[a,b]$: that is, in terms of (O)-convergence, for every set $N \subset [a,b]$ of Lebesgue measure zero and $\forall\, \omega \in \Omega \setminus N_0$, we get:

$$\inf_{\delta\in\Delta}\Bigg[\sup\Bigg\{\sum_{i=1}^{n}\big|\widetilde{F}_\omega(x_i)-\widetilde{F}_\omega(x_{i-1})\big|\colon \big\{\big([x_{i-1},x_i],\xi_i\big)\colon i=1,\dots,n\big\} \text{ is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg]=0\,. \tag{12}$$

From (12), taking into account that R is super Dedekind complete and hence all involved suprema and infima can be considered countable, we get, up to the complement of meager subsets of Ω:

$$\begin{aligned}
0 &= \inf_{\delta\in\Delta}\Bigg[\sup\Bigg\{\sum_{i=1}^{n}\big|\widetilde{F}_\omega(x_i)-\widetilde{F}_\omega(x_{i-1})\big|\colon \big\{\big([x_{i-1},x_i]\ \xi_i\big)\colon i=1,\dots,n\big\} \\
&\qquad\qquad \text{is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg] \\
&= \inf_{\delta\in\Delta}\Bigg[\sup\Bigg\{\sum_{i=1}^{n}\big|F(x_i)(\omega)-F(x_{i-1})(\omega)\big|\colon \big\{\big([x_{i-1},x_i],\xi_i\big)\colon i=1,\dots,n\big\} \\
&\qquad\qquad \text{is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg] \\
&= \inf_{\delta\in\Delta}\Bigg[\sup\Bigg\{\Bigg(\sum_{i=1}^{n}\big|F(x_i)-F(x_{i-1})\big|\Bigg)(\omega)\colon \big\{\big([x_{i-1},x_i],\xi_i\big)\colon i=1,\dots,n\big\} \\
&\qquad\qquad \text{is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg] \qquad (13) \\
&= \inf_{\delta\in\Delta}\Bigg\{\Bigg[\sup\Bigg\{\sum_{i=1}^{n}\big|F(x_i)-F(x_{i-1})\big|\colon \big\{\big([x_{i-1},x_i],\xi_i\big)\colon i=1,\dots,n\big\} \\
&\qquad\qquad \text{is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg](\omega)\Bigg\} \\
&= \Bigg\{\inf_{\delta\in\Delta}\Bigg[\sup\Bigg\{\sum_{i=1}^{n}\big|F(x_i)-F(x_{i-1})\big|\colon \big\{\big([x_{i-1},x_i],\xi_i\big)\colon i=1,\dots,n\big\} \\
&\qquad\qquad \text{is a } \delta\text{-fine decomposition of } [a,b]\,,\ x_i\in[a,b] \text{ and } \xi_i\in N\ \forall\, i\Bigg\}\Bigg]\Bigg\}(\omega)\,.
\end{aligned}$$

From this it follows that F is of class (SL) on $[a,b]$. It concludes the proof. □

Now we turn to the theorem on equivalence between two involved integrals.

Theorem 3.26. *Let R satisfy Assumptions 3.22. A function $f\colon [a,b] \to R$ is* (KH)*-integrable on $[a,b]$ if and only if it is* (SL)*-integrable on $[a,b]$, and in this case we have*

$$(\mathrm{KH})\int_a^b f = (\mathrm{SL})\int_a^b f\,.$$

Proof. We begin with "only if" part. Since f is (KH)-integrable on $[a,b]$, then it is (KH)-integrable even on every subinterval of $[a,b]$. Let $F(a) = 0$ and

$$F(x) = (\mathrm{KH})\int_a^x f\,, \qquad \forall\, x \in (a,b]\,.$$

The assertion of "only if" part follows form Henstock Lemma, because (9) holds when P is replaced by F and since F is of class (SL) on $[a,b]$ (thanks to super Dedekind completeness of R, which is a consequence of Assumptions 3.22), by virtue of Proposition 3.25.

We now turn to "if" part. Let $f\colon [a,b] \to R$ be (SL)-integrable, and let $P\colon [a,b] \to R$ be a weak primitive of f. There exists a unit w such that for every $\varepsilon > 0$, there is a gage δ_1 such that

$$\left| \sum_{i=1}^{n} \bigl\{ [P(x_i) - P(x_{i-1})] - (x_i - x_{i-1}) f(\xi_i) \bigr\} \right| \le \varepsilon w$$

for every δ_1-fine decomposition $\Pi_1 = \bigl\{ \bigl([x_{i-1}, x_i], \xi_i\bigr) \colon i = 1, \dots, n \bigr\}$ of $[a,b]$. Let Z_{δ_1} be a δ_1-zeros set and let $f_0 \equiv f\chi_{Z_{\delta_1}}$. We note that f_0 is zero almost everywhere. Then since L^0 has the property σ, f_0 is Kurzweil–Henstock integrable on $[a,b]$ and

$$(\mathrm{KH})\int_a^b f_0 = 0\,.$$

Thus there exists a unit z such that for all $\varepsilon > 0$ there exists a function $\delta_2\colon [a,b] \to \mathbb{R}^+$ such that $\forall\, \delta_2$-fine partition Π_2 of $[a,b]$ and we get

$$\left| \sum_{\Pi_2} f_0 \right| \le \varepsilon z\,.$$

Since P is of class (SL) on $[a,b]$, there exists a unit y such that for all $\varepsilon > 0$ there is a map $\gamma\colon [a,b] \to \mathbb{R}^+$ for each γ-fine decomposition Π_3 of $[a,b]$, $\Pi_3 = \bigl\{ \bigl([\omega_{i-1}, \omega_i], \kappa_i\bigr) \colon i = 1, \dots, n \bigr\}$ with $\kappa_i \in [a,b]$ and $\kappa_i \in Z_{\delta_1}$, $\forall\, i = 1, \dots, n$, and we have:

$$\sum_{i=1}^{n} \bigl| P(\omega_i) - P(\omega_{i-1}) \bigr| \le \varepsilon y\,. \tag{14}$$

Let $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$ if $t \notin Z_{\delta_1}$, and $\delta(t) = \min\{\gamma(t), \delta_2(t)\}$ if $t \in Z_{\delta_1}$. If $\Pi = \{([w_{k-1}, w_k], \zeta_k) : k = 1, \dots, p\}$ is any δ-fine partition of $[a,b]$, then we get:

$$\begin{aligned}
&\left| \sum_{k=1}^{p} (w_k - w_{k-1}) f(\zeta_k) - \big[P(b) - P(a)\big] \right| \\
&\qquad = \left| \sum_{k=1}^{p} (w_k - w_{k-1}) f(\zeta_k) - \sum_{k=1}^{p} \big[P(w_k) - P(w_{k-1})\big] \right| \\
&\qquad \le \left| \sum_{k=1}^{p} (w_k - w_{k-1}) f_0(\zeta_k) \right| + \sum_{\substack{k=1,\dots,p,\\ \zeta_k \in Z_{\delta_1}}} \big|P(w_k) - P(w_{k-1})\big| \\
&\qquad\quad + \left| \sum_{\substack{k=1,\dots,p,\\ \zeta_k \notin Z_{\delta_1}}} (w_k - w_{k-1}) f(\zeta_k) - \big[P(w_k) - P(w_{k-1})\big] \right| \\
&\qquad \le \varepsilon(w + z + y).
\end{aligned}$$

This concludes the proof. □

4. Convergence theorems

In this section we prove some convergence theorems for the (SL)-integral and generalize Theorem 2.11 to suitable Riesz spaces. Let R be as in Assumptions 3.22. We prove that, under these assumptions, in a certain sense, the (SL)-integral contains an improper integral (on $[a,b]$).

THEOREM 4.1. *Let $f\colon [a,b] \to R$ satisfy the following properties:*

(4.1.1) *f is bounded and* (SL)*-integrable in $[a,c]$ for every $c \in (a,b)$;*

(4.1.2) (O) $\lim\limits_{c \to b^-} \left[(\mathrm{SL}) \int_a^c f \right] = l \in R$.

Then f is (SL)*-integrable on $[a,b]$ and*

$$(\mathrm{SL}) \int_a^b f = l\,.$$

Proof. Let us define $P\colon [a,b] \to R$ as follows:

$$P(x) = (\mathrm{SL}) \int_a^x f \ \text{ if } x \in [a,b); \quad P(b) = l\,.$$

Let $(y_n)_{n\in\mathbb{N}\cup\{0\}}$ be a strictly increasing sequence such that $y_0 = a$ and $\lim\limits_{n\to+\infty} y_n = b$. Since f is bounded in $[a,x]$ for each $x \in [a,b)$, then P is of class (SL) on $[y_n, y_{n+1}]$ for every $n \in \mathbb{N}\cup\{0\}$ (see also [3]), and hence f is of class (SL) on $[a,b)$. From this and (4.1.2), by virtue of a "continuity property" of P at b, it follows that P is of class (SL) on $[a,b]$, by properties of functions of class (SL) (see [3]). Since f is (SL)-integrable on *every* interval of the type $[y_n, y_{n+1}]$ and, in our space R, (O)-convergence coincides with (r)-convergence, then, for each $n \in \mathbb{N}$, there exists a unit w_n such that, for all $\varepsilon > 0$, there exists a gage $\delta_n^*\colon [a,b] \to \mathbb{R}_0^+$ such that for every δ_n^*-fine decomposition $\Pi_n = \{([x_{i-1}, x_i], \xi_i)\colon i = 1,2,\dots,k_n\}$ of $[y_n, y_{n+1}]$ we have

$$\left| \sum_{i=1}^{k_n} (x_i - x_{i-1}) f(\xi_i) - \big[P(x_i) - P(x_{i-1})\big] \right| \le \varepsilon w_n\,. \tag{15}$$

In view of the property σ and in correspondence with a sequence $(w_n)_n$ there exist a sequence of positive real numbers $(\lambda_n)_n$ and a unit w, such that

$$\lambda_n w_n \le w, \qquad \forall\, n \in \mathbb{N}\,.$$

Applying the previous argument as in (15) and replacing ε by $\varepsilon\lambda_n/2^n$, we get that for each $n \in \mathbb{N}$ there exists a unit w_n such that for all $\varepsilon > 0$ there exists a gage $\delta_n\colon [a,b] \to \mathbb{R}_0^+$ such that for every δ_n-fine decomposition $\Pi_n = \{([x_{i-1}, x_i], \xi_i)\colon i = 1,2,\dots,k_n\}$ of $[y_n, y_{n+1}]$ and we have:

$$\left| \sum_{i=1}^{k_n} (x_i - x_{i-1})\, f(\xi_i) - \big[P(x_i) - P(x_{i-1})\big] \right| \le \frac{\varepsilon\lambda_n}{2^n}\, w_n \le \frac{\varepsilon}{2^n}\, w\,. \tag{16}$$

Now, let

$$\delta(x) = \begin{cases} 0 & \text{if } x \in \{y_n\colon n \in \mathbb{N}\cup\{0\}\}\,, \\ \min\{x - y_n,\ y_{n+1} - x,\ \delta_n(x)\} & \text{if } x \in (y_n, y_{n+1})\,. \end{cases}$$

It is easy to see that $\delta\colon [a,b] \to \mathbb{R}_0^+$ is a gage. Every δ-fine decomposition must have its intervals contained in some interval of the type $[y_n, y_{n+1}]$. Indeed, if there exists an element $([u,v],\xi)$ of such a decomposition (where $[u,v]$ is a subinterval of $[a,b]$ and $\xi \in [u,v]$) such that $\exists\, n \in \mathbb{N}$ with $u < y_n < v$, then $v - u \le \delta(\xi)$. Since $\delta(y_n) = 0$, then ξ cannot coincide with y_n. If $u \le \xi < y_n$ then

$$v - u \le \delta(\xi) \le y_n - \xi \le y_n - u\,,$$

and hence $v \le y_n$, a contradiction. Moreover, if $y_n < \xi \le v$ then

$$v - u \le \delta(\xi) \le \xi - y_n \le v - y_n\,,$$

and hence $u \geq y_n$, a contradiction.

Now, let Π be any δ-fine decomposition of $[a,b]$: as shown above, Π "induces" a decomposition Π_n of $[y_n, y_{n+1}]$ on every interval $[y_n, y_{n+1}]$; thus

$$\Pi = \bigcup_{n \in T^*} \Pi_n , \tag{17}$$

where T^* is a suitable subset of $\mathbb{N}$. We observe that T^* is a *finite* set: the involved decompositions are formed by a finite number of intervals and $\lim\limits_{n\to+\infty} y_n = b$. Moreover, by δ-fineness, an endpoint of the last interval of an involved decomposition must be strictly smaller than y_{n_0} for some positive integer n_0: in particular, our decomposition Π cannot have *any* interval of type $[t,b]$, and so no δ-fine decomposition can be a partition. Finally we have:

$$\begin{aligned} 0 &\leq \left| \sum_{i \in T^*} (x_i - x_{i-1}) f(\xi_i) - \big[P(x_i) - P(x_{i-1})\big] \right| \\ &\leq \sum_{\Pi_n} \left| \sum_{i=1}^{k_n} (x_i - x_{i-1}) f(\xi_i) - \big[P(x_i) - P(x_{i-1})\big] \right| \\ &\leq \sum_{n=1}^{\infty} \frac{\varepsilon \lambda_n}{2^n} w_n \leq \varepsilon w . \end{aligned} \tag{18}$$

The assertion follows from (18). This concludes the proof. □

The next results are generalizations of theorems proved for real-valued functions by P. Y. Lee and R. Výborný (see [15] and [16, pp. 162–169]). We begin with the following preliminary definitions.

Definition 4.2. Let $J \subset [a,b] \subset \mathbb{R}$, and $\Delta_J = (\mathbb{R}^+)^J$. We say that a sequence of functions $(P_n\colon [a,b] \to R)_n$ is *of class* (USL) or has *property* (USL) (uniform (SL)) on J if for every set $N \subset [a,b]$ of Lebesgue measure zero there exists a unit w such that $\forall\, \varepsilon > 0$, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$, and a map $\delta\colon J \to \mathbb{R}^+$ such that for every $n \geq n_0$ and for any δ-fine decomposition $\Pi = \big\{\big([x_{i-1}, x_i], \xi_i\big) \colon i = 1, \dots, p\big\}$ of $[a,b]$ with $x_i \in [a,b]$ and $\xi_i \in N \cap J$ $\forall\, i = 1, \dots, p$, we have:

$$\sum_{i=1}^{p} \big|P_n(x_i) - P_n(x_{i-1})\big| \leq \varepsilon w . \tag{19}$$

We say that $(P_n)_n$ is *of class* (USL) if it is of class (USL) on $[a,b]$.

Definition 4.3. Let $\Delta^{(1)}$ be a set of all gages defined on $[a,b]$. We say that the sequence $(f_n\colon [a,b] \to R)_n$ is (SL)-*equiintegrable* (or simply *equiintegrable*) on $[a,b]$ if there exists a sequence of functions $(P_n\colon [a,b] \to R)_n$ of class (USL) and a unit w such that for every $\varepsilon > 0$ there exist $\delta \in \Delta^{(1)}$ and $n_0 \in \mathbb{N}$, such that

$$\left| \sum_{\Pi} f_n - \sum_{i=1}^{p} \left[P_n(x_i) - P_n(x_{i-1}) \right] \right| \le \varepsilon w \tag{20}$$

for every $n \ge n_0$, and whenever $\Pi = \{([x_{i-1}, x_i], \xi_i) : i = 1, \dots, p\}$ is a δ-fine decomposition of $[a,b]$.

We now prove the following:

Theorem 4.4. *Let R be* **any** *Dedekind complete Riesz space, let $(f_n\colon [a,b] \to R)_n$ be* (SL)*-equiintegrable, and let F_n be a weak primitive of f_n on $[a,b]$ $\forall\, n \in \mathbb{N}$. Suppose that the sequence $(F_n)_n$ satisfies property* (USL) *on $[a,b]$ and assume that there exists a function $f\colon [a,b] \to R$ such that*

$$(r)\lim_n f_n(x) = f(x)$$

for almost all $x \in [a,b]$ and with respect to the same unit y.

Then there exists a map $F\colon [a,b] \to R$ such that

$$(r)\lim_n \left[F_n(x) - F_n(a) \right] = F(x)\,, \qquad \forall\, x \in [a,b]$$

with respect to a same unit, f is (SL)*-integrable on $[a,b]$ and F is a weak primitive of f on $[a,b]$.*

P r o o f. (See also [15, Theorem 3, pp. 767–768] and [16, Theorem 4.3.3, pp. 163–165].) Let N be such a set (of Lebesgue measure zero) where the sequence $(f_n)_n$ does not converge to f, $\varepsilon > 0$, $z \in R$ be a unit, and δ be a gage chosen with respect to (SL)-equiintegrability of the sequence $(f_n)_n$. Further, let Z_δ be the set of all zeros of δ, $\gamma\colon [a,b] \to \mathbb{R}^+$ associated with ε and $N \cup Z_\delta$ according to the property (USL) of the sequence $(F_n)_n$. Put

$$\delta^*(x) = \begin{cases} \delta(x) & \text{if } x \notin N \cup Z_\delta\,, \\ \gamma(x) & \text{if } x \in N \cup Z_\delta\,. \end{cases}$$

Fix $x \in (a,b]$, and let $\Pi = \{([u_i, v_i], x_i) : i = 1, \dots, p\}$ be a δ^*-fine partition of $[a,x]$. Let Π_0 be such a part of Π for which $x_i \in N \cup Z_\delta$ $\forall\, i = 1, \dots, p$, and

$\Pi_1 = \Pi \setminus \Pi_0$. For every $m, n \in \mathbb{N}$ we have:

$$\begin{aligned}
0 &\le \big|\big[F_m(x) - F_m(a)\big] - \big[F_n(x) - F_n(a)\big]\big| \\
&\le \Big|\sum_{\Pi} \big\{ \big[F_m(v_i) - F_m(u_i)\big] - \big[F_n(v_i) - F_n(u_i)\big] \big\}\Big| \\
&\le \sum_{\Pi_0} \big[F_m(v_i) - F_m(u_i)\big] + \sum_{\Pi_0} \big[F_n(v_i) - F_n(u_i)\big] \\
&\qquad + \Big|\sum_{\Pi_1} \big\{ \big[F_m(v_i) - F_m(u_i)\big] - (v_i - u_i) f_m(x_i) \big\}\Big| \\
&\qquad + \Big|\sum_{\Pi_1} \big\{ \big[F_n(v_i) - F_n(u_i)\big] - (v_i - u_i) f_n(x_i) \big\}\Big| \\
&\qquad + \sum_{\Pi_1} (v_i - u_i) \big| f_m(x_i) - f_n(x_i) \big| \,.
\end{aligned} \tag{21}$$

For m, n larger than a suitable positive integer n^* depending on ε only, the first two terms in the last member of inequality (21) are less than εw, where w is a suitable unit existing by the property (USL), and both the third and fourth term are less than εz, where z is as above. We now estimate the latter term.

By hypothesis, y is a such unit that for every $x \in [a, b] \setminus N$ and for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\forall\, n, m \ge n_0$,

$$\big| f_m(x) - f_n(x) \big| \le \varepsilon y \,.$$

From this it follows that the last term in inequality (21) is less than $\varepsilon(b-a)y$ for sufficiently large n (depending on the involved partition Π). From this it follows that the sequence $\big(F_n(x) - F_n(a)\big)_n$ is Cauchy. Hence in R there exists a limit

$$F(x) \equiv (r) \lim_n \big[F_n(x) - F_n(a)\big] \,,$$

for every $x \in (a, b]$ (or to the same unit as to that one with respect to which the involved sequence is Cauchy), because every Dedekind complete Riesz space is complete with respect to (r)-convergence (see also [17]). Moreover, set $F(a) = 0$. We have that F is of class (SL), since the sequence $(F_n)_n$ is of class (USL). Now change δ, if necessary, by defining it equal to zero on the set N. So, $\forall\, n \ge n^*$ (where n^* is as above) and for every δ-fine decomposition $\Pi = \big\{ \big([s_i, t_i], \xi_i\big) : i = 1, \dots, q \big\}$ of $[a, b]$, we obtain:

$$\Big| \sum_{i=1}^{q} \big\{ \big[F_n(t_i) - F_n(s_i)\big] - (t_i - s_i) f_n(\xi_i) \big\} \Big| \le \varepsilon z \,, \tag{22}$$

where z is as above. Considering that the (r)-limit from (22) as n tends to $+\infty$, we get:

$$\left| \sum_{i=1}^{q} \Big\{ \big[F(t_i) - F(s_i)\big] - (t_i - s_i) f(\xi_i) \Big\} \right| \le \varepsilon z$$

whenever $\Pi = \big\{ \big([s_i, t_i], \xi_i \big) \colon i = 1, \dots, q \big\}$ is a δ-fine decomposition of $[a,b]$. Thus we get (SL)-integrability of f on $[a,b]$ with weak primitive F. This concludes the proof. □

We now turn to the following version of the monotone convergence theorem.

Theorem 4.5. *Let R satisfy Assumptions 3.22, let $f_n \colon [a,b] \to R$, $n \in \mathbb{N}$, be* (SL)*-integrable on $[a,b]$, and suppose that the sequence $\left((\mathrm{SL})\int_a^b f_n \right)_n$ is bounded in R. Moreover, assume that*

$$f_n(x) \le f_{n+1}(x) \quad \big[\text{or } f_n(x) \ge f_{n+1}(x)\big], \qquad \forall\, n \in \mathbb{N}$$

for almost all $x \in [a,b]$, and that there exists a function $f \colon [a,b] \to R$ such that

$$(r) \lim_n f_n(x) = f(x)$$

for almost all $x \in [a,b]$ and with respect to the same unit z.

Then f is (SL)*-integrable on $[a,b]$ and*

$$(\mathrm{SL})\int_a^x f = (r) \lim_n \left[(\mathrm{SL})\int_a^x f_n \right], \qquad \forall x \in [a,b]$$

with respect to a same unit.

P r o o f. According to Theorem 4.4, it is sufficient to prove the equiintegrability of sequence $(f_n)_n$. Considering $-f_n$ and $f_n - f_1$, respectively, instead of f_n, without loss of generality we can suppose that $(f_n)_n$ is increasing and $f_n \ge 0$. Put

$$F_n(x) \equiv (\mathrm{SL})\int_a^x f_n\,, \qquad n \in \mathbb{N},\ x \in [a,b]\,.$$

We have:

$$F_n(x) \le F_n(b)\,, \qquad \forall\, n \in \mathbb{N},\ \forall\, x \in [a,b]\,,$$

the sequence $\big(F_n(b)\big)_n$ is increasing and bounded, and thus in R there the following limit exists:

$$F(x) \equiv (\mathrm{O}) \lim_n F_n(x)\,, \qquad x \in [a,b]\,.$$

So, there is a unit w such that $\forall \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$, such that

$$F(b) - F_n(b) \le \varepsilon w\,, \qquad \forall\, n \ge n_0\,.$$

Let N be a set (of Lebesgue measure zero) where the sequence $(f_n)_n$ does not converge to f. By hypothesis, z is such a unit that $\forall\, \varepsilon > 0$ and $\forall\, x \in [a,b] \setminus N$, there exists $n(x) \ge n_0$ such that

$$f(x) - f_n(x) \le \varepsilon z\,, \qquad \forall\, n \ge n(x)\,.$$

Since f_n is (SL)-integrable $\forall\, n \in \mathbb{N}$, for every $n \in \mathbb{N}$ there exists a unit w_n such that $\forall\, \varepsilon > 0$, there exists a gage δ_n^* such that for every δ_n^*-fine decomposition Π_n of $[a,b]$, $\Pi_n = \big\{\big([u_i,v_i],x_i\big) : i = 1,\dots,q\big\}$, we have:

$$\left| \sum_{i=1}^{q} \big[F_n(v_i) - F_n(u_i) - (v_i - u_i) f_n(x_i)\big] \right| \le \varepsilon w_n\,.$$

Since L^0 has the property σ, there exists a sequence $(\lambda_n)_n$ of positive real numbers and a unit y such that

$$\lambda_n w_n \le y\,, \qquad \forall\, n \in \mathbb{N}.$$

Thus y is such a unit that $\forall\, \varepsilon > 0$, there exists a gage δ_n such that for every δ_n-fine decomposition $\Pi_n = \big\{\big([u_i,v_i],x_i\big) : i = 1,\dots,q\big\}$, we have:

$$\left| \sum_{i=1}^{q} \big[F_n(v_i) - F_n(u_i) - (v_i - u_i) f_n(x_i)\big] \right| \le \frac{\varepsilon \lambda_n}{2^n} w_n \le \frac{\varepsilon}{2^n} y\,.$$

Without loss of generality we can assume

$$\delta_n(x) \ge \delta_{n+1}(x)\,, \qquad \forall\, n \in \mathbb{N},\ \forall\, x \in [a,b]\,.$$

Moreover, let

$$\delta(x) = \begin{cases} \delta_{n(x)}(x) & \text{if } x \notin N, \\ 0 & \text{if } x \in N. \end{cases} \tag{23}$$

We now prove that the sequence $(F_n)_n$ has the property (USL) on $[a,b]$. For every decomposition $\Pi = \big\{\big([u_i,v_i],x_i\big) : i = 1,\dots,q\big\}$ we get:

$$\begin{aligned} \sum_{i=1}^{q} \big|F_n(v_i) - F_n(u_i)\big| &\le \sum_{i=1}^{q} \big|F_{n_0}(v_i) - F_{n_0}(u_i)\big| \\ &\quad + \sum_{i=1}^{q} \big[F_n(v_i) - F_n(u_i) - \big(F_{n_0}(v_i) - F_{n_0}(u_i)\big)\big] \\ &\le \sum_{i=1}^{q} \big|F_{n_0}(v_i) - F_{n_0}(u_i)\big| + \big[F(b) - F_{n_0}(b)\big] \\ &\le \sum_{i=1}^{q} \big|F_{n_0}(v_i) - F_{n_0}(u_i)\big| + \varepsilon w\,, \qquad \forall\, n \ge n_0\,. \end{aligned} \tag{24}$$

There exists a gage γ_{n_0} according to the fact that F_{n_0} is of class (SL) on $[a,b]$. By virtue of (24) γ_{n_0} is the required gage, because sequence $(F_n)_n$ satisfies property (USL) on $[a,b]$.

Finally we prove the (SL)-equiintegrability of the sequence $(f_n)_n$ on $[a,b]$. Let n_0 be as above. Fix $m,n \geq n_0$ arbitrarily. Let δ be as in (23), let $\Pi' = \{([u_i,v_i],x_i)\colon i=1,\dots,q\}$ be a δ-fine decomposition of $[a,b]$. Let Π be such a part of Π' whose x_i's are $m > n(x_i)$ and $\Pi_0 = \Pi' \setminus \Pi$. Since the sequence $(\delta_n)_n$ is decreasing, we have that Π_0 is δ_m-fine, and

$$
\begin{aligned}
&\left|\sum_{\Pi'}\{(v_i-u_i)f_m(x_i) - [F_m(v_i)-F_m(u_i)]\}\right| \\
&\leq \left|\sum_{\Pi}\{(v_i-u_i)f_m(x_i) - [F_m(v_i)-F_m(u_i)]\}\right| \\
&\qquad + \left|\sum_{\Pi_0}\{(v_i-u_i)f_m(x_i) - [F_m(v_i)-F_m(u_i)]\}\right| \\
&\leq \left|\sum_{\Pi}\{(v_i-u_i)f_m(x_i) - [F_m(v_i)-F_m(u_i)]\}\right| + \frac{\varepsilon}{2^m}y\,.
\end{aligned}
$$

Moreover, we get:

$$
\begin{aligned}
&\left|\sum_{\Pi}\{(v_i-u_i)f_m(x_i) - [F_m(v_i)-F_m(u_i)]\}\right| \\
&\leq \sum_{\Pi}(v_i-u_i)\left|f_m(x_i)-f_{n(x_i)}(x_i)\right| \\
&\qquad + \left|\sum_{\Pi}\{[F_{n(x_i)}(v_i)-F_{n(x_i)}(u_i)] - (v_i-u_i)f_{n(x_i)}(x_i)\}\right| \\
&\qquad + \left|\sum_{\Pi}\{[F_{n(x_i)}(v_i)-F_{n(x_i)}(u_i)] - [F_m(v_i)-F_m(u_i)]\}\right|.
\end{aligned}
$$

We have:

$$0 \leq \sum_{\Pi}(v_i-u_i)\left|f_m(x_i)-f_{n(x_i)}(x_i)\right| \leq \varepsilon z\,;$$

$$
\begin{aligned}
0 &\leq \sum_{\Pi}\{[F_m(v_i)-F_m(u_i)] - [F_{n(x_i)}(v_i)-F_{n(x_i)}(u_i)]\} \\
&\leq \sum_{\Pi}\{[F(v_i)-F(u_i)] - [F_{n_0}(v_i)-F_{n_0}(u_i)]\} \\
&\leq F(b)-F_{n_0}(b) \,\leq\, \varepsilon w\,.
\end{aligned}
$$

Furthermore, we get:

$$\begin{aligned}&\left|\sum_{\Pi}\left\{\left[F_{n(x_i)}(v_i)-F_{n(x_i)}(u_i)\right]-(v_i-u_i)f_m(x_i)\right\}\right|\\&\le\sum_{\kappa=1}^{\infty}\left|\sum_{n(x_i)=\kappa}\left[F_\kappa(v_i)-F_\kappa(u_i)\right]-\left[(v_i-u_i)f_\kappa(x_i)\right]\right|\le\sum_{\kappa=1}^{\infty}\frac{\varepsilon}{2^\kappa}y=\varepsilon y\,.\end{aligned}$$

Finally we obtain:

$$\left|\sum_{\Pi'}\left\{(v_i-u_i)\,f_m(x_i)\;-\left[F_m(v_i)-F_m(u_i)\right]\right\}\right|\le\varepsilon(z+w+2y)\,.$$

This concludes the proof. □

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