Tatra Mt. Math. Publ. **34** (2006), 289–306

Tatra Mountains
Mathematical Publications

REGULARITY PROBLEMS AND RESULTS CONCERNING COMPOSITE FUNCTIONAL EQUATIONS

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ABSTRACT. In recent years, the investigation of functional equations containing the iterates of unknown functions have resulted in the development of a number of new methods and a new chapter in the regularity theory of functional equations. It is well-known that the solutions of the classical Cauchy equation are either very regular (infinitely many times differentiable) or very irregular (i.e., their graphs are everywhere dense on the plane). Thus, very weak regularity assumptions of a solution of the Cauchy equation force its very strong regularity properties. A situation for composite functional equations (that is for equations containing the iterates of unknown functions) is not so extreme. However, in a great deal of situations, starting with monotonicity and continuity properties of the solutions, their local Lipschitz continuity, differentiability everywhere but at countably many points, or further regularity properties can be proved.

The proofs require the Lebesgue's theorem about the almost everywhere differentiability of monotonic functions, the Bernstein–Doetsch theorem on convex functions, the theorems of Steinhaus and Piccard concerning the algebraic sum of two sets of positive Lebesgue measure or of the second Baire category. Having proved the sufficient differentiability properties of the unknown functions, a functional equation can be differentiated with respect to the variables, and the composite terms can be eliminated. Thus, for derivatives of unknown functions, we obtain a new functional equation that does not involve the iterates of unknown functions, and, as a result, it can be solved using more standard methods of the theory of functional equations.

1. Introduction

A regularity theory of functional equations started probably with observations that the continuous or monotonic or measurable solutions $f\colon \mathbb{R} \to \mathbb{R}$ of

2000 Mathematics Subject Classification: Primary 39Bxx, 39B05, 39B62.
Keywords: functional equations, regularity theorems, regularity problems.

The preparation of this paper was supported by the research grants OTKA T-030082, T-038072.

the Cauchy functional equation

$$f(x+y)=f(x)+f(y), \qquad x,y\in\mathbb{R}, \tag{1}$$

are always of the form $f(x)=cx$, where c is an arbitrary real constant (cf. Aczél [Acz66], Kuczma [Kuc85]) and thus they are analytic. Another well-known result (see ibid.) is that there are also non-regular solutions of this equation. This is due to the fact that a general solution of (1) can be prescribed arbitrarily on a fixed Hamel base and can be extended to $\mathbb{R}$ in a unique way. Therefore, we have the following alternative: The solutions of the functional equation (1) are either very regular (in this case, analytic) or very irregular (in this case, nowhere continuous, non-monotonic on any proper interval, non-measurable, etc.).

As it turned out in last decades, a similar alternative can be set for most of the non-composite functional equations in several variables. For instance, all the following equations

$$\begin{gathered}
f(x+y)=g(x)+h(y) \qquad \text{(Pexider equation)},\\
f\Big(\frac{x+y}{2}\Big)=\frac{f(x)+f(y)}{2} \qquad \text{(Jensen equation)},\\
f(x+y-xy)+f(xy)=f(x)+f(y) \qquad \text{(Hosszú equation)},\\
f(xy)+g\big(x(1-y)\big)+h\big((1-x)y\big)+k\big((1-x)(1-y)\big)=0,\\
\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+ky)=0 \qquad \text{(Fréchet equation)},\\
\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+ky)=n!f(y) \qquad \text{(monomial equation)}
\end{gathered}$$

can be treated with the same methods from the point of the regularity theory. The key idea followed from these investigations is to assume *the weak regularity properties of the unknown functions* (such as measurability, Baire property, monotonicity, continuity, etc.) and to derive *higher-order regularity properties*. Results of this kind are called *regularity theorems for functional equations*. The most general and powerful results have been obtained by Antal Járai (cf. Daróczy–Járai [DJ79], Járai [Jár79], [Jár82], [Jár84b], [Jár86], [Jár91], [Jár92a], [Jár93], [Jár94a], [Jár94b], [Jár94c], [Jár95], [Jár98b], Járai–Maksa [JM95], Járai–Sander [JS97]). Contributions were also made by several other authors as well: Aczél and Chung [AC82], Baker [Bak71b], Itzkowitz [Itz71], Daróczy [Dar71b], Grosse-Erdmann [Gro89], Paganoni [Pag74a], Páles [Pál92d], Sander [San78a], Świątak [Swi76], Tsutsumi and Haruki [TH82], [TH84], [TH91]. A summary of all these results can be found in Járai–Székelyhidi [JSz96], and in the monographs [Jár96] and [Jár04]. The open problems are collected in [Pál02a].

These results are useful for many reasons. For instance, if a sufficiently high-order differentiability of unknown functions is derived then, after the differentiation with respect to variables, the functional equation can often be reduced to a differential equation (see a general method in [Pál92d] and [Ház04]) for each of the unknown functions, and hence a well-developed and efficient theory of ordinary differential equations can be applied to determine the explicit form of the solutions.

In this paper we treat and describe the developments concerning functional equations where the unknown functions appear in argument of other unknown functions. Such functional equations are termed *composite equations* in the sequel. Many of these equations are derived from various characterizations in the axiomatization of utility theory (see a recent book of Luce [Luc00], the survey paper of Aczél, Falmagne and Luce [AFL00]).

Referring to an open problem of Aczél, Luce and Maksa [ALM96], Aczél and Maksa [AM97P], and also Aczél [Acz98PB] arose the problem of solving the following functional equation

$$F^{-1}\big(F(x)F(y)\big)z = F^{-1}\big(F(xz)F\big(yP(x,z)\big)\big), \qquad x,y \in [0,1[\,,\ z \in [0,1]\,,$$

where $P\colon\ [0,1[\times[0,1]\to[0,1]$ and $F\colon\ [0,1[\ \to\ ]0,1]$ is a strictly decreasing bijection. In [ALM96] the differentiability assumptions were necessary to determine F and P. Several efforts to eliminate or reduce the differentiability conditions have been made (see Brillouët–Belluot [BB97], [Acz98a], and [BB98]). Arczél's idea [Acz98b] moved forward with the solution. He eliminated the two-variable function P and deduced the following equation

$$\frac{F(x)}{F(xy)} = F\left(\frac{G(y)}{G(xy)}\right), \qquad x,y \in\ ]0,1[\ , \tag{2}$$

which, by introducing the new unknown functions, is a special case of the following "additivized" equation

$$f(x+y)-f(x) = \phi\big(g(x+y)-g(y)\big)\,, \qquad x,y>0\,,\ x+y<\alpha\,. \tag{3}$$

The problem was cracked in [Pál98PA], [Pál98PB] where (2) and the related equation (3) were solved. The result with its application was published in [AMP99].

The most important steps in solutions of equation (3) and also of more general composite functional equations can be summarized in a following scheme (cf. also [Pál99b] for further details):

(I) Assuming the monotonicity and functional equation, the Jensen convexity of internal unknown functions is proved.

(II) Using the Bernstein–Doetsch theorem we show that the internal functions are differentiable everywhere but at countably many points.

(III) Using the Lebesgue's differentiation theorem, we show the differentiability of external unknown functions.

(IV) The functional equation is differentiated with respect to its variables and composite terms are eliminated.

(V) The resulting equation is a non-composite functional equation with respect to derivatives of internal functions which can be solved by means of standard techniques.

To illustrate the methods, we consider the following functional equation

$$\begin{aligned} f(x+y)-f(x)+\phi\big(g(y+z)-g(y)\big) \\ =\psi\big(g(x+y+z)-g(y+z)-g(x+y)+g(y)\big) \\ (x,y,z>0\,,\ x+y+z<\alpha)\,. \end{aligned} \tag{4}$$

Throughout this paper in this paper $0<\alpha\leqslant\infty$ and $f\colon I\to\mathbb{R}$, $g\colon I\to\mathbb{R}$, $\phi\colon J\to\mathbb{R}$, and $\psi\colon H\to\mathbb{R}$ are strictly monotone functions defined on the sets

$$I:=\,]0,\alpha[\,,\qquad J:=\big\{g(y+z)-g(y)\colon y,z>0,\ y+z<\alpha\big\}\,, \tag{5}$$

$$H:=\big\{g(x+y+z)-g(y+z)-g(x+y)+g(y)\colon x,y,z>0,\ x+y+z<\alpha\big\}\,.$$

For the origin of this equation, cf. [MMP00] and [MP04].

2. Regularity results and solutions of (4)

In our first result, we deduce the stronger regularity properties of internal functions f and g as well as external functions ϕ and ψ.

Theorem 1. *Assume that $f,g\colon I\to\mathbb{R}$, $\phi\colon J\to\mathbb{R}$ and $\psi\colon H\to\mathbb{R}$ are strictly monotone functions satisfying the functional equation (4). Then f and g are strictly convex or concave continuously differentiable functions with non-vanishing derivatives. Furthermore, J and H are open intervals and ϕ, ψ are differentiable on J and H, respectively.*

Proof. Observe that the left hand side of (4) is a strictly monotone function of z on interval $]0,\alpha-x-y[$ for all fixed $x,y>0$. Thus the same is true for the left hand side of (4), i.e.,

$$z\mapsto g(x+y+z)-g(y+z)\qquad\big(z\in\,]0,\alpha-x-y[\big)$$

is strictly monotone if $x,y>0$, $x+y<\alpha$. Being the variable y arbitrarily small, we get that

$$z\mapsto g(x+z)-g(z)\qquad\big(z\in\,]0,\alpha-x[\big) \tag{6}$$

is strictly monotone if $0 < x < \alpha$.

Assuming that (6) strictly increases, for arbitrary $0 < u < v < \alpha$, we get

$$\begin{aligned} g\Big(\frac{u+v}{2}\Big) - g(u) &= g\Big(u + \frac{v-u}{2}\Big) - g(u) \\ &< g\Big(u + \frac{v-u}{2} + \frac{v-u}{2}\Big) - g\Big(u + \frac{v-u}{2}\Big) \\ &= g(v) - g\Big(\frac{u+v}{2}\Big), \end{aligned}$$

therefore g is strictly Jensen-convex.

Due to its monotonicity, g is also locally bounded, hence, in view of the celebrated Bernstein–Doetsch theorem (cf. Bernstein–Doetsch [BD15], Kuczma [Kuc85], Roberts–Varberg [RV73]), g is strictly convex. If (6) strictly decreases, then we can similarly see that g is strictly concave.

By the strict convexity/concavity of g, the mapping

$$x \mapsto g(x+y+z) - g(x+y) \qquad \big(x \in \,]0,\, \alpha - y - z[\big) \tag{7}$$

is strictly monotone for all fixed $y, z > 0$ with $y + z < \alpha$. Thus, the right hand side and also the left hand side of (4) is a strictly monotone function of variable x on the interval $]0, \alpha - y - z[$. Being the variable z arbitrarily small, it follows that the function

$$x \mapsto f(x+y) - f(x) \qquad \big(x \in \,]0, \alpha - y[\big) \tag{8}$$

is also a strictly monotone function for all $y \in I$. Therefore, f (exactly as g) is strictly convex or concave.

In view of the well-known regularity properties of convex/concave functions, it follows that f and g are differentiable from the left and from the right at each point of I and, except at countably many points, they are differentiable everywhere. Their inverses are also strictly convex or concave, therefore, they enjoy similar differentiability properties. The continuity of the function g implies that J and H are intervals. Since g is strictly monotone, hence j is an open interval. The strict convexity/concavity of g yields that H is also open.

In the next part of the proof, applying Lebesgue's theorem on the almost everywhere differentiability of monotone functions we show that ϕ is differentiable on I. The repeated application of Lebesgue's theorem will then give us also the differentiability of f, g and ψ.

Let $u_0 \in J$ be arbitrary. Then there exist $x_0, y_0, z_0 \in I$, such that $x_0 + y_0 + z_0 < \alpha$ and $u_0 = g(y_0 + z_0) - g(y_0)$. Then we have that $z_0 = g^{-1}\big(u_0 + g(y_0)\big) - y_0$, therefore the map $(u, y) \mapsto g^{-1}\big(u + g(y)\big) - y$ is defined in a neighborhood of the point (u_0, y_0). Substituting $z = g^{-1}\big(u + g(y)\big) - y$ into (4), we get that the point (u_0, x_0, y_0) has a neighborhood $U \times V \times W$ such that

$$\phi(u) = f(x) - f(x+y) + \psi\Big(g\big(x + g^{-1}(u + g(y))\big) - u - g(x+y)\Big) \qquad \big((u,x,y) \in U \times V \times W\big) \quad (9)$$

holds. We show that x and y can be chosen so that the right hand side of (9) is o be differentiable with respect to u at the point u_0, which yields the differentiability of ϕ at u_0.

Let A, B, and C be the sets of those points where g, g^{-1}, and ψ are non-differentiable. Then A and B are at most countable, and C is of Lebesgue measure zero (by Lebesgue's theorem). Obviously, the set of those points $y \in W$ such that $u_0 + g(y) \in B$ holds is at most countable. Therefore, there exists a point $y^* \in W$ such that $g^{-1}\big(u + g(y^*)\big)$ is differentiable with respect to u at u_0. The set

$$S := \big\{x \in V \colon x + g^{-1}\big(u_0 + g(y^*)\big) \in A\big\}$$

is also at most countable, therefore, for every x but at the points of the countable set S, the composite map $u \mapsto g\big(x + g^{-1}\big(u + g(y^*)\big)\big)$ is differentiable at u_0. Finally, we show that $x \in V \setminus S$ can be chosen so that

$$\gamma(x) := g\Big(x + g^{-1}\big(u_0 + g(y^*)\big)\Big) - u_0 - g(x + y^*) \notin C\,.$$

By the strict convexity/concavity of g and by $u_0 \neq 0$, γ is strictly monotone on V, therefore $\gamma(V \setminus S) = \gamma(V) \setminus \gamma(S)$, and $\gamma(V)$ is open. Thus $\gamma(V \setminus S)$ cannot be of measure zero, that implies contains also the elements of a complement of C. If $x = x^* \in V \setminus S$ is chosen in such a way that $\gamma(x^*) \notin C$, then the map

$$u \mapsto \psi\Big(g\big(x^* + g^{-1}\big(u + g(y^*)\big)\big) - u - g(x^* + y^*)\Big) \qquad (u \in U)$$

is differentiable at u_0. Hence the right-hand side of (9) is differentiable with respect to u at the point u_0. Thus the differentiability of ϕ is verified.

In the next step of the proof we show that f is also differentiable everywhere. Therefore, we rewrite (4) as

$$f(x) = f(x+y) + \phi\big(g(y+z) - g(y)\big) - \psi\big(g(x+y+z) - g(y+z) - g(x+y) + g(y)\big) \qquad (x, y, z > 0,\ x+y+z < \alpha) \quad (10)$$

and we show that for a fixed $x_0 \in I$, y, z can be chosen so that the right-hand side of (10) is to be differentiable with respect to x at the point x_0.

Let $y_0, z_0 > 0$ be fixed such that $x_0 + y_0 + z_0 < \alpha$. Then (x_0, y_0, z_0) has a neighborhood $V \times W \times U$ such that (10) is valid for all $(x, y, z) \in V \times W \times U$. As functions f and g are differentiable everywhere but at countably many points of $x_0 + W$, there exists $y^* \in W$, such that f and g are differentiable at $x_0 + y^*$.

Thus, the maps $x \mapsto f(x+y^*)$ and $x \mapsto g(x+y^*)$ are differentiable at x_0. The choice of z should satisfy two conditions: (a) Let g be differentiable at x_0+y^*+z, and (b) let ψ be differentiable at the point

$$\delta(z) := g(x_0+y^*+z) - g(y^*+z) - g(x_0+y^*) + g(y^*).$$

Condition (a) is satisfied by all but countably many points of U. Being the function g strictly convex/concave, δ is a strictly monotone function. Thus $\delta(U)$ is open, whence omitting countably many points from U that do not satisfy (a), the map of a residual set by δ cannot be of measure zero. Thus, by Lebesgue's theorem, $z=z^*\in U$ can be chosen so that (a) and (b) hold simultaneously. Substituting $y=y^*$ and $z=z^*$ into (10), we can see that the right-hand side of this equation is differentiable with respect to x at x_0. This shows the differentiability of f. Taking the convexity/concavity of f also into consideration, it follows that f is continuously differentiable as well.

To show the differentiability of g, using the substitution $u=x+y$ we rewrite (4) as

$$g(u) = g(u+z) - g(y+z) + g(y) - \psi^{-1}\Big(f(u) - f(u-y) + \phi\big(g(y+z) - g(y)\big)\Big)$$
$$(0<y<u,\ 0<z<\alpha-u). \quad (11)$$

Let $u_0\in I$ be fixed and let $0<y_0<u_0$, $0<z_0<\alpha-u_0$. Then choose $z\in\,]0,\alpha-u_0[$ such that (a') g is differentiable at u_0+z and (b') ψ^{-1} is differentiable at the point

$$\eta(z) := f(u_0) - f(u_0-y_0) + \phi\big(g(y_0+z) - g(y)\big).$$

Since η is strictly monotone and g is differentiable everywhere but at countably many points, therefore by repeating a former argument and using the Lebesgue's theorem again, we can see that $z=z^*$ can be chosen from any neighborhood of z_0 such that (a') and (b') are satisfied. Substituting $y=y_0$ and $z=z^*$ and applying the differentiability of f and ϕ, the right-hand side of (11) is differentiable with respect to u at u_0. This proves that g is differentiable everywhere. The concavity/convexity of g yields that the derivative of g is also continuous.

In the last step of the proof we show that ψ is also differentiable at each point of the interval H. Let $t_0\in H$ be arbitrary. Then there exist $x_0, y_0, z_0>0$ such that $x_0+y_0+z_0<\alpha$ and

$$t_0 = g(x_0+y_0+z_0) - g(x_0+y_0) - g(y_0+z_0) + g(y_0).$$

Let us introduce the notation:

$$h(u) := g(x_0+u) - g(u) \qquad (0<u<\alpha-x_0).$$

Due to strict convexity/concavity of g, the function h is strictly monotone and continuous. Thus it is also invertible. Using this notation and substitutions $x = x_0$, $u = y + z$ (4) can be rewritten as

$$f(x_0 + y) - f(x_0) + \phi\big(g(u) - g(y)\big) = \psi\big(h(u) - h(y)\big) \qquad (0 < y < u < \alpha - x_0). \tag{12}$$

Introduce now a new variable $t = h(u) - h(y)$. Then y has to be replaced by $h^{-1}\big(h(u) - t\big)$ in (12). Since $t_0 = h(y_0 + z_0) - h(y_0)$, therefore there exist neighborhoods V of t_0 and U of $u_0 = y_0 + z_0$ such that, after having made the above substitutions, we get

$$\psi(t) = f\big(x_0 + h^{-1}(h(u) - t)\big) - f(x_0) + \phi\Big(g(u) - g\big(h^{-1}(h(u) - t)\big)\Big) \qquad (u \in U,\ t \in V). \tag{13}$$

An inverse function h^{-1} is strictly monotone, therefore, by Lebesgue's theorem, $u = u^* \in U$ can be chosen so that h^{-1} is differentiable at $h(u^*) - t_0$. Then, using the choice $u = u^*$ and the differentiability properties concerning the f, g, ϕ, it follows that the right-hand side of (13) is a differentiable function of t at the point t_0. This proves the differentiability of ψ and completes the proof of the theorem. □

In order to describe the solutions of (4) in a unified structure, let us introduce the following notation:

$$L_\gamma(x) := \begin{cases} \ln x & \text{if } \gamma = 0, \\ \ln \dfrac{e^{\gamma x} - 1}{\gamma} & \text{if } \gamma \in \mathbb{R} \setminus \{0\}, \end{cases} \qquad (x > 0). \tag{14}$$

Theorem 2. *Assume that $f, g\colon I \to \mathbb{R}$, $\phi\colon J \to \mathbb{R}$ and $\psi\colon H \to \mathbb{R}$ are strictly monotone functions satisfying the functional equation (4). Then there exist constants $a, b, c, d, \gamma \in \mathbb{R}$ with $ac \neq 0$ such that*

$$f(x) = aL_\gamma(x) + b, \qquad g(x) = cL_\gamma(x) + d, \qquad (x \in I), \tag{15}$$

furthermore, $J = c\,]0, \infty[$, $H = c\,]-\infty, 0[$., and there exists a constant $\delta \in \mathbb{R}$ such that

$$\phi(x) = -a \ln\big(1 - e^{-x/c}\big) + \delta \qquad (x \in J), \tag{16}$$

$$\psi(x) = -a \ln\big(1 - e^{x/c}\big) + \delta \qquad (x \in H). \tag{17}$$

P r o o f. By the previous theorem, f, g, ϕ, ψ are differentiable, f', g' are continuous and non-vanishing on I. Differentiating (4) with respect to the variables x, y, z and using the notations

$$\xi := 1,$$
$$\eta := \phi'\big(g(y + z) - g(y)\big)$$

and

$$\zeta := -\psi'\big(g(x+y+z) - g(y+z) - g(x+y) + g(y)\big)$$

we get that (ξ, η, ζ) is a solution of the following system of homogeneous linear equations:

$$\begin{aligned}
\big[f'(x+y) - f'(x)\big]\cdot\xi + {}& 0\cdot\eta + \big[g'(x+y+z) - g'(x+y)\big]\cdot\zeta = 0\,,\\
f'(x+y)\cdot\xi + {}& \big[g'(y+z) - g'(y)\big]\cdot\eta + \big[g'(x+y+z) - g'(y+z)\\
&\qquad\qquad -g'(x+y) + g'(y)\big]\cdot\zeta = 0\,,\\
0\cdot\xi + {}& g'(y+z)\cdot\eta + \big[g'(x+y+z) - g'(y+z)\big]\cdot\zeta = 0\,.
\end{aligned}$$

Since (ξ, η, ζ) is a nontrivial solution, hence the determinant of the coefficient matrix vanishes, i.e.,

$$\begin{vmatrix}
f'(x+y) - f'(x) & 0 & g'(x+y+z) - g'(x+y)\\
f'(x+y) & g'(y+z) - g'(y) & g'(x+y+z) - g'(y+z) - g'(x+y) + g'(y)\\
0 & g'(y+z) & g'(x+y+z) - g'(y+z)
\end{vmatrix} = 0\,.$$

Subtracting the last row from the second one, the second from the first then, and finally adding the second columns to the third one, we get

$$\begin{vmatrix}
-f'(x) & g'(y) & g'(x+y+z)\\
f'(x+y) & -g'(y) & -g'(x+y)\\
0 & g'(y+z) & g'(x+y+z)
\end{vmatrix} = 0 \qquad (x, y, z > 0,\ x+y+z < \alpha)\,,$$

which yields the equation

$$\begin{aligned}
f'(x)\big[g'(y)g'(x+y+z) - g'(x+y)g'(y+z)\big]&\\
= f'(x+y)g'\big(x+y+z\big[g'(y) - g'(y+z)\big]\big)&\\
(x, y, z > 0,\ x+y+z < \alpha)\,.&\quad (18)
\end{aligned}$$

Dividing (18) by $f'(x)g'(x+y+z) \neq 0$, we get

$$\frac{g'(x+y)g'(y+z)}{g'(x+y+z)} - g'(y) = \frac{f'(x+y)}{f'(x)}\big[g'(y+z) - g'(y)\big]$$
$$(x, y, z > 0,\ x+y+z < \alpha)\,. \quad (19)$$

The left-hand side is symmetric in x, z, hence

$$\frac{f'(z)}{f'(z+y)}\big[g'(y+z) - g'(y)\big] = \frac{f'(x)}{f'(x+y)}\big[g'(y+x) - g'(y)\big]\ (x, y, z > 0,\ x+y+z < \alpha)\,.$$

Therefore, using fixed $y \in I$, we see that the function

$$x \mapsto \frac{f'(x)}{f'(x+y)}\big[g'(x+y) - g'(y)\big] \qquad (x \in\,]0, \alpha - y[)$$

is constant. Thus there exists a function $h\colon I \to \mathbb{R}$ such that

$$\frac{f'(x)}{f'(x+y)}\big[g'(x+y) - g'(y)\big] = h(y) \qquad (x,y>0,\ x+y<\alpha)\,. \tag{20}$$

In the second step of the proof, we utilize the symmetry in x, z of the right-hand side of (4). Applying (4) twice, we get, for $x, y, z > 0$, with $x+y+z<\alpha$,

$$f(x+y) - f(x) + \phi\big(g(y+z) - g(y)\big) = f(y+z) - f(z) + \phi\big(g(x+y) - g(y)\big)\,. \tag{21}$$

Differentiating with respect to x, y, z, we can see that

$$\xi := 1, \qquad \eta := \phi'\big(g(y+z) - g(y)\big)\,, \qquad \zeta := -\phi'\big(g(x+y) - g(y)\big)\,,$$

are the solutions of the following system of homogeneous linear equations:

$$\begin{aligned}
\big[f'(x+y) - f'(x)\big]\cdot\xi + 0\cdot\eta \qquad\qquad\qquad\quad &+ g'(x+y)\cdot\zeta &&= 0\,,\\
\big[f'(x+y) - f'(y+z)\big]\cdot\xi + \big[g'(y+z) - g'(y)\big]\cdot\eta\ &+ \big[g'(x+y) - g'(y)\big]\cdot\zeta &&= 0\,,\\
\big[f'(z) - f'(y+z)\big]\cdot\xi + g'(y+z)\cdot\eta \qquad\qquad &+ 0\cdot\zeta &&= 0\,.
\end{aligned}$$

Subtract the first and the last equations form the second one. Then (ξ, η, ζ) is a nontrivial solution of the equations obtained in such a way. Hence, the following determinant is zero

$$\begin{vmatrix} f'(x+y) - f'(x) & 0 & g'(x+y) \\ f'(x) - f'(z) & -g'(y) & -g'(y) \\ f'(z) - f'(y+z) & g'(y+z) & 0 \end{vmatrix} = 0 \qquad (x,y,z>0,\ x+y+z<\alpha)\,.$$

After expanding and rearranging, it follows that

$$\big[f'(x+y) - f'(x)\big]\frac{g'(y)}{g'(x+y)} + f'(x) = \big[f'(z+y) - f'(z)\big]\frac{g'(y)}{g'(z+y)} + f'(z)$$

for $x, y, z > 0$ with $x+y+z<\alpha$. This equation shows that, for a fixed $y \in I$, the function

$$x \mapsto \big[f'(x+y) - f'(x)\big]\frac{g'(y)}{g'(x+y)} + f'(x) \qquad \big(x \in \left]0, \alpha - y\right[\big)$$

is constant. Therefore, there exists a function $k\colon I \to \mathbb{R}$ such that

$$\big[f'(x+y) - f'(x)\big]\frac{g'(y)}{g'(x+y)} + f'(x) = k(y) \qquad (x,y>0,\ x+y<\alpha)\,. \tag{22}$$

Multiply (22) by $g'(x+y)$ and subtract (20) multiplied by $f'(x+y)$. Then, after having $x+y=u$ substituted, we get

$$f'(u)\big[g'(y) + h(y)\big] = g'(u)k(y) \qquad \big(0<y<u<\alpha\big)\,. \tag{23}$$

From this equation it follows that if $y = y_0$ is such that $k(y_0) \neq 0$, then $g'(y_0) + h(y_0)$ is not zero either. Therefore, the function $u \mapsto f'(u)/g'(u)$ is constant on interval $]y_0, \alpha[$. If $k \equiv 0$, then let $\beta := \alpha$, otherwise define β by

$$\beta := \inf\{y \in]0, \alpha[\, : \, k(y) \neq 0\}.$$

Then f'/g' is constant on $]\beta, \alpha[$, i.e., there exists $q \neq 0$ such that $f'(u) = qg'(u)$ for $\beta < u < \alpha$. On the other hand, $k(y) = 0$ for $0 < y < \beta$, therefore, by equation (23), we have $h(y) = -g'(y)$.

In fact, we show that f'/g' is constant on the entire interval I. If $\beta = 0$, then there is nothing to prove. Assume that $0 < \beta$. Then, by (20),

$$\frac{f'(x)}{f'(x+y)}\big[g'(x+y) - g'(y)\big] = -g'(y) \qquad \big(x, y > 0,\, 0 < x+y < \beta\big). \tag{24}$$

Expressing the fraction $f'(x+y)/f'(x)$ from here and substituting it into (19), after some calculations, we get

$$\frac{1}{g'(x+y+z)} + \frac{1}{g'(y)} = \frac{1}{g'(y+z)} + \frac{1}{g'(x+y)} \qquad \big(x, y, z > 0,\, x+y+z < \beta\big).$$

If now $y < u < \beta$ and $x := z := (u-y)/2$, then the above equation is so-called Jensen equation with respect to the function $1/g'$. Thus, there exist constants a, b such that $1/g'(x) = ax + b$ for $0 < x < \beta$. Substituting this form of g' into (24), we obtain that

$$\frac{f'(x+y)}{f'(x)} = \frac{ax}{a(x+y)+b} \qquad \big(x, y > 0,\, x+y < \beta\big).$$

Taking the limit $y \to 0$, we can see that $b = 0$ and that $x \mapsto xf'(x)$ is constant on $]0, \beta[$. Since $b = 0$, hence $1/g'(x) = ax$, therefore $f'(x)/g'(x) = axf'(x)$. Thus, we have shown that f'/g' is constant on $]0, \beta[$. If $\beta = \alpha$, then it means the entire interval I. If $\beta < \alpha$, then f'/g' is also constant on $]\beta, \alpha[$. By the continuity of f'/g' this is possible only if f'/g' takes the same constant value q on both subintervals.

Substitute now g' by f'/q in (22). Then we get

$$f'(x) + f'(y) - \frac{f'(x)f'(y)}{f'(x+y)} = k(y) \qquad \big(x, y > 0,\, x+y < \alpha\big). \tag{25}$$

Due to the symmetry of the left-hand side, we get that $k \equiv p$, constant. With the notation $F = 1/f'$, the equation (25) reduces to the equation

$$F(x) + F(y) = F(x+y) + pF(x)F(y) \qquad \big(x, y > 0,\, x+y < \alpha\big). \tag{26}$$

If $p = 0$, then this is a conditional Cauchy equation, whose regular solution is of the form $F(x) = x/a$ $(x \in I)$, where $0 \neq a \in \mathbb{R}$ is a constant (cf. Aczél [Acz66]). Thus, for the function f, we obtain that

$$f(x) = a \ln x + b = aL_0(x) + b \qquad (x \in I), \tag{27}$$

where b is a real constant.

If $p \neq 0$, then (25) can be rewritten in the form

$$1 - pF(x+y) = \big(1 - pF(x)\big)\big(1 - pF(y)\big) \qquad \big(x, y > 0,\ x + y < \alpha\big),$$

which is a conditional multiplicative Cauchy equation for a continuous function M defined by $M(x) = 1 - pF(x)$. Hence, there are two possibilities: either $M \equiv 0$, i.e.,

$$F(x) = \frac{1}{p}, \qquad f(x) = px + b \qquad (x \in I), \tag{28}$$

or there exists $\gamma \neq 0$ such that $M(x) = e^{-\gamma x}$, whence

$$F(x) = \frac{1 - e^{-\gamma x}}{p}, \qquad f(x) = a \ln \frac{e^{\gamma x} - 1}{\gamma} + b = aL_\gamma(x) + b \qquad (x \in I). \tag{29}$$

Here, $a = p/\gamma \neq 0$. Solution (28) leads to a contradiction, because f is not strictly convex/concave then. Thus we have proved that there exist the constants $a, b, \gamma \in \mathbb{R}$, with $a \neq 0$ such that

$$f(x) = aL_\gamma(x) + b \qquad (x \in I).$$

Since $f'/g' = q$, therefore $g' = f'/q = (a/q)L'_\gamma$. Hence there exist constants $c, d \in \mathbb{R}$ with $c \neq 0$ such that

$$g(x) = cL_\gamma(x) + d \qquad (x \in I).$$

Thus (4) has been proved.

Finally we determine the external unknown functions ϕ, ψ, and their domain J, H.

To determine the function ϕ, differentiate (21) with respect to the variable x. Then

$$f'(x+y) - f'(x) = \phi'\big(g(x+y) - g(y)\big) \cdot g'(x+y) \qquad \big(x, y > 0,\ x + y < \alpha\big).$$

(Since in (21), the variable z can be arbitrarily small.) Thus, in view of formulae (15), we have

$$\frac{a}{c} \cdot \frac{L'_\gamma(x+y) - L'_\gamma(x)}{L'_\gamma(x+y)} = \phi'\left(c \frac{L_\gamma(x+y)}{L_\gamma(y)}\right) \qquad \big(x, y > 0,\ x + y < \alpha\big). \tag{30}$$

In what follows, we make computations in details only in the case when $\gamma \neq 0$. The case $\gamma = 0$ can be treated in a similar and simpler way. Assuming $\gamma \neq 0$, and using (14) for L_γ, we can rewrite (30) as

$$\phi'\left(c \ln \frac{e^{\gamma(x+y)} - 1}{e^{\gamma(y)} - 1}\right) = \frac{a}{c} \cdot \frac{1 - e^{\gamma y}}{e^{\gamma(x+y)} - e^{\gamma y}} = \frac{a}{c} \cdot \frac{1}{1 - \frac{e^{\gamma(x+y)} - 1}{e^{\gamma y} - 1}} \ \big(x, y > 0,\ x+y < \alpha\big).$$

A substitution $t = t(x,y) = \frac{e^{\gamma(x+y)}-1}{e^{\gamma(y)}-1}$ turns this equation to the following equation

$$\phi'(c\ln t) = \frac{a}{c} \cdot \frac{1}{1-t} \qquad \big(t \in\,]1,\infty[\big) \tag{31}$$

Observe that $1 < t(x,y)$ for all $x,y>0$, with $x+y<\alpha$, furthermore, $t(x,y) \to 1$ as $x \to 0$, and $t(x,y) \to \infty$ as $y \to 0$. Thus, the domain of ϕ is $J = c\,]0,\infty[$. Substituting $t = e^{x/c}$ $(x \in J)$ into (31), we obtain

$$\phi'(x) = -\frac{a}{c} \cdot \frac{1}{e^{x/c}-1} \qquad (x \in J)\,.$$

Integrating this equation, it follows that ϕ is of the form (16). In the case $\gamma = 0$, a similar computation yields that ϕ is also given by (16).

We determine ψ and H also in the case $\gamma \neq 0$. Then we have that

$$\phi\big(g(y+z)-g(y)\big) = a\ln\frac{e^{\gamma(y+z)}-1}{e^{\gamma(y+z)}-e^{\gamma y}} + \delta \qquad \big(y,z>0,\ y+z<\alpha\big)\,.$$

Thus, (4) can be rewritten as

$$a\ln\frac{e^{\gamma(x+y)}-1}{e^{\gamma x}-1}\cdot\frac{e^{\gamma(y+z)}-1}{e^{\gamma(y+z)}-e^{\gamma y}} + \delta = \psi\left(c\ln\frac{e^{\gamma(x+y+z)}-1}{e^{\gamma(x+y)}-1}\cdot\frac{e^{\gamma y}-1}{e^{\gamma(y+z)}-1}\right), \tag{32}$$

for $x,y,z>0$ with $x+y+z<\alpha$. Introduce the notation

$$t(x,y,z) := \frac{e^{\gamma(x+y+z)}-1}{e^{\gamma(x+y)}-1}\cdot\frac{e^{\gamma y}-1}{e^{\gamma(y+z)}-1} \qquad \big(x,y,z>0,\ x+y+z<\alpha\big)$$

An elementary computation yields that $0 < t(x,y,z) < 1$

$$\lim_{x\to 0} t(x,y,z) = 1\,, \qquad \lim_{y\to 0} t(x,y,z) = 0\,.$$

Therefore, the range of the function t is an interval $]0,1[$. The substitution $t = t(x,y,z)$ simplifies equation (32) to

$$\psi(c\ln t) = a\ln\frac{1}{1-t} + \delta \qquad \big(t \in\,]0,1[\big)\,.$$

Thus, the domain of ψ is $H = c\,]{-\infty},0[$ and after the substitution $t = e^{x/c}$ $(x \in H)$ it follows that ψ is of a desired form (17). In the case $\gamma = 0$, a similar and a simpler argument shows that (17) is still valid. The determination of the function ψ also shows that the functions given by formulae (15), (16) and (17) satisfy the functional equation (4), indeed. $\square$

3. Concluding remarks

The applications of methods treated in this paper can be found in Aczél–Maksa–Páles [AMP01], Aczél–Maksa–Ng–Páles [AMNP01] and Maksa–Marley–Páles [MMP00].

The following functional equation is a possible generalization of (4)

$$\begin{aligned} f(x+y)-f(x)+\sum_{i=1}^{n}\phi_i\big[g_i(y+z_i)-g_i(y)\big] \\ =\sum_{i=1}^{n}\psi_i\big[g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y)\big] \\ \big(x,y,z_i>0,\ x+y+z_i<\alpha;\ i=1,\dots,n\big)\,. \end{aligned}$$

Extending the regularity improving methods, Gilányi–Páles [GP01] showed the differentiability of unknown functions (assuming monotonicity properties). The solutions of this equation are still unknown.

A number of problems related to quasi-arithmetic means lead to the following composite equation (called Matkowski--Sutô equation):

$$\phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \qquad (x,y\in I)\,, \tag{33}$$

where $\phi,\psi\colon I\to\mathbb{R}$ are continuous and strictly monotone functions. The C^∞-solutions were found by Sutô [Sut14a], [Sut14b], The solutions with C^2 assumptions were determined by Matkowski [Mat99a], with C^1 assumptions by Daróczy–Páles [DP01a], and with convexity assumptions by Daróczy–Maksa [DM99a]. In Daróczy–Maksa–Páles [DMP00] and Daróczy–Hajdu-Ng [DHN03] extension theorems were proved. Finally a complete solution was found by Daróczy–Páles in [DP02c] and [DP03c]. Further generalizations concerning the weighted quasi-arithmetic means and conjugate arithmetic means were found in [Dar00], [DN00], [DP02b], [DP03a], [DP03d], [Haj02], [Haj03].

There is no room here to describe all the methods applied in these papers. We only pick out one essential step: from (33) it easily follows that

$$t\to\phi^{-1}\left(\frac{s+t}{2}\right)-\phi^{-1}(t) \qquad \big(t\in\phi(I)\big) \tag{34}$$

is a strictly monotone function for all fixed $s\in\phi(I)$. This property does not imply that ϕ has Jensen-convexity or concavity properties. However, applying the monotonicity of function (34), a generalization of the Bernstein–Doetsch theorem yields that ϕ^{-1} and ϕ are locally Lipschitz functions. For more details concerning this approach, cf. [Pál03b]. After obtaining the Lipschitz properties of ϕ^{-1} and ϕ, and using Lebesgue's theorem again, one can show that they are differentiable in a subinterval of I. At this point, one has to apply Baire's and Piccard's theorem ([Pic42]) to get that these functions are continuously differentiable in a (possibly smaller) subinterval of I. The solution is then completed when applying the results concerning C^1 assumptions [DP01a] and extension theorem from [DMP00].

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Received November 12, 2004

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