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L_2 SPACE AND g-CALCULUS WITH APPLICATIONS

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ABSTRACT. J. Desatníková [L_p *spaces and g-calculus*, student scientific work, FPV UMB, Banská Bystrica, 2004] studied L_1, L_2 spaces with respect to the Pap's g-calculus (see [E. Pap: *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava, 1995]). She defined $(L_1^{\oplus}, d_1^{\oplus})$, $(L_2^{\oplus}, d_2^{\oplus})$ spaces and proved that they are complete $\oplus$-pseudometric spaces, where $d_1^{\oplus}(f,h) = \int^{\oplus} |f \ominus h|^{\oplus}\, dP$ and $d_2^{\oplus}(f,h) = \left(\int^{\oplus} (f \ominus h)^{2\oplus}\, dP\right)^{\frac{1}{2}\oplus}$. The aim of this paper is to show a relation between L_2 space and $L_2^{\oplus}$ space. We define a mapping $(\cdot,\cdot)^{\oplus} : L_2^{\oplus} \times L_2^{\oplus} \to \mathbb{R}$ satisfying the following conditions:

(i) $(f,f)^{\oplus} \geqslant 0$ for each $f \in L_2^{\oplus}$,

(ii) $(f,h)^{\oplus} = (h,f)^{\oplus}$ for each $f,h \in L_2^{\oplus}$,

(iii) $(\lambda \odot f, h)^{\oplus} = \lambda \odot (f,h)^{\oplus}$ for each $f,h \in L_2^{\oplus}$ and $\lambda \in \mathbb{R}$,

(iv) $(f, h_1 \oplus h_2)^{\oplus} = (f,h_1)^{\oplus} \oplus (f,h_2)^{\oplus}$ for each $f, h_1, h_2 \in L_2^{\oplus}$.

We prove the Cauchy inequality for $L_2^{\oplus}$ space and the Chebyshev inequality for pseudo-probability P defined by L. M. Nedović and T. Grbić: [*The pseudo probability*, Journal of Electrical Engineering **53** (2002), 27–31]. We show the applications of $L_2^{\oplus}$ space like a pseudo-dispersion $\sigma^{2\oplus}$ and pseudo-mean value $E^{\oplus}$ in a pseudo-probability.

1. Introduction

One of the ways of using the L_2 space is the probability theory. It can be seen in works of B. Riečan [7]–[9], D. Pollard [6] and P. Billingsley [1]. Hence it is interesting to study the L_p spaces with respect to the Pap g-calculus (see [5]). The $(L_1^{\oplus}, d_1^{\oplus})$, $(L_2^{\oplus}, d_2^{\oplus})$ spaces, where $d_1^{\oplus}(f,h) = \int^{\oplus} |f \ominus h|^{\oplus}\, dP$ and $d_2^{\oplus}(f,h) = \left(\int^{\oplus} (f \ominus h)^{2\oplus}\, dP\right)^{\frac{1}{2}\oplus}$, were studied by J. Desatníková in the paper [2]. She proved that these spaces are complete $\oplus$-pseudometric spaces.

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In this paper we show the relation between L_2 space and $L_2^{\oplus}$ space. We define the mapping $(\cdot,\cdot)^{\oplus}\colon L_2^{\oplus}\times L_2^{\oplus}\to\mathbb{R}$ satisfying the following four conditions:

(i) $(f,f)^{\oplus}\geqslant 0$ for each $f\in L_2^{\oplus}$,

(ii) $(f,h)^{\oplus}=(h,f)^{\oplus}$ for each $f,h\in L_2^{\oplus}$,

(iii) $(\lambda\odot f,h)^{\oplus}=\lambda\odot(f,h)^{\oplus}$ for each $f,h\in L_2^{\oplus}$ and $\lambda\in\mathbb{R}$,

(iv) $(f,h_1\oplus h_2)^{\oplus}=(f,h_1)^{\oplus}\oplus(f,h_2)^{\oplus}$ for each $f,h_1,h_2\in L_2^{\oplus}$.

We prove the Cauchy inequality for $L_2^{\oplus}$ space and the Chebyshev inequality for pseudo-probability P defined by L. M. Nedović and T. Grbić in [4]. We show the applications of $L_2^{\oplus}$ space like pseudo-dispersion $\sigma^{2\oplus}$ and pseudo-mean value $E^{\oplus}$ in pseudo-probability, too.

2. L_2 space and g-calculus

In many directions there have been used the pseudo-arithmetical operations based on generator g (see [5])

pseudo-addition: $x\oplus y:=g^{-1}\big(g(x)+g(y)\big)$,

pseudo-subtraction: $x\ominus y:=g^{-1}\big(g(x)-g(y)\big)$,

pseudo-multiplication: $x\odot y:=g^{-1}\big(g(x)\cdot g(y)\big)$,

pseudo-division: $x\oslash y:=g^{-1}\left(\frac{g(x)}{g(y)}\right)$,

for $x,y\in\mathbb{R}$, where $g\colon\mathbb{R}\to\mathbb{R}$ is an increasing bijection with $g(0)=0$.

Similarly we can define

$$|x|^{\oplus}:=g^{-1}\big(|g(x)|\big),\ \text{for } x\in\mathbb{R},$$
$$x^{2\oplus}:=g^{-1}\big((g(x))^2\big),\ \text{for } x\in\mathbb{R},$$
$$\sqrt{x}^{\,\oplus}:=g^{-1}\big(\sqrt{g(x)}\,\big),\ \text{for } x\in[0,\infty).$$

In the paper [3] the pseudo-Lebesgue integral is defined by formula

$$\int^{\oplus} f\,dP=g^{-1}\left(\int g\circ f\ dp\right),$$

for each measurable function f.

We can find the following two definitions in the paper [2].

Definition 2.1. Let $(\Omega,\mathcal{S},p)$ be a measure space. We say that $(\Omega,\mathcal{S},P)$ is *g-measure space* if P is defined by formula $P=g^{-1}\circ p$, where p is measure. P is called *g-measure*.

Definition 2.2. Let $(\Omega, \mathcal{S}, P)$ be a g-measure space. We denote $L_2^{\oplus} = L_2^{\oplus}(\Omega, \mathcal{S}, P)$ the class of all measurable real functions f for which $f \odot f$ is integrable. We write

$$L_2^{\oplus} = \left\{ f \colon \Omega \to \mathbb{R} \colon \int^{\oplus} f \odot f \, dP < \infty \right\}.$$

The following theorem says about the relation between L_2 space and $L_2^{\oplus}$ space.

Theorem 2.3. *Let $(\Omega, \mathcal{S}, p)$ be a measure space and $(\Omega, \mathcal{S}, P)$ be the corresponding g-measure space. Let $L_2 = L_2(\Omega, \mathcal{S}, p)$ be the class of all measurable real functions f for which $f \cdot f$ is integrable and $L_2^{\oplus} = L_2^{\oplus}(\Omega, \mathcal{S}, P)$ be the class of all measurable real functions f for which $f \odot f$ is integrable. Then*

$$g \circ f \in L_2 \iff f \in L_2^{\oplus}.$$

Proof. "$\Rightarrow$" Let $g \circ f \in L_2$. Then there exists $\int (g \circ f)^2 \, dp < \infty$. Since

$$\begin{aligned} \int (g \circ f)^2 \, dp &= \int g \circ g^{-1}\big((g \circ f)^2\big) \, dp = \int g(f \odot f) \, dp \\ &= g \circ g^{-1}\Big(\int g(f \odot f) \, dp \Big) = g\Big(\int^{\oplus} f \odot f \, dP \Big), \end{aligned}$$

then there exists $\int^{\oplus} f \odot f \, dP < \infty$. Hence $f \in L_2^{\oplus}$.

"$\Leftarrow$" Let $f \in L_2^{\oplus}$. Then there exists $\int^{\oplus} f \odot f \, dP < \infty$. Since

$$\begin{aligned} \int^{\oplus} f \odot f \, dP &= g^{-1}\Big(\int g(f \odot f) \, dp \Big) = g^{-1}\Big(\int g \circ g^{-1}\big((g \circ f)^2\big) \, dp \Big) \\ &= g^{-1}\Big(\int (g \circ f)^2 \, dp \Big), \end{aligned}$$

then there exists $\int (g \circ f)^2 \, dp < \infty$. Hence $g \circ f \in L_2$. □

Definition 2.4. We say that M is $\oplus$*-linear space* if the following two conditions are satisfied:

(i) $\forall f, h \in M \colon f \oplus h \in M$,

(ii) $\forall f \in M, \lambda \in \mathbb{R} \colon \lambda \odot f \in M$.

Proposition 2.1. *$L_2^{\oplus}$ is a $\oplus$-linear space and for each $f, h \in L_2^{\oplus}$ there exists $\int^{\oplus} f \odot h \, dP < \infty$. If we denote*

$$(f, h)^{\oplus} = \int^{\oplus} f \odot h \, dP = g^{-1}\big((g \circ f, g \circ h)\big),$$

then the following four conditions are satisfied:

(i) $(f,f)^{\oplus} \geqslant 0$ *for each* $f \in L_2^{\oplus}$;

(ii) $(f,h)^{\oplus} = (h,f)^{\oplus}$ *for each* $f,h \in L_2^{\oplus}$;

(iii) $(\lambda \odot f, h)^{\oplus} = \lambda \odot (f,h)^{\oplus}$ *for each* $f,h \in L_2^{\oplus}$ *and* $\lambda \in \mathbb{R}$;

(iv) $(f, h_1 \oplus h_2)^{\oplus} = (f,h_1)^{\oplus} \oplus (f,h_2)^{\oplus}$ *for each* $f, h_1, h_2 \in L_2^{\oplus}$.

Proof. 1. We show that $L_2^{\oplus}$ is a $\oplus$-linear space.

Let $f,h \in L_2^{\oplus}$. By Theorem 2.3 we obtain that $g \circ f, g \circ h \in L_2$. Since L_2 is a linear space, then $g \circ f + g \circ h \in L_2$. But $g \circ f + g \circ h = g \circ g^{-1}(g \circ f + g \circ h) = g(f \oplus h)$. Hence $g(f \oplus h) \in L_2$. Therefore by Theorem 2.3 we have that $f \oplus h \in L_2^{\oplus}$.

Let $f \in L_2^{\oplus}$, $\lambda \in \mathbb{R}$. Then $g \circ f \in L_2$ and $g \circ \lambda \in \mathbb{R}$. Since L_2 is a linear space, then $g \circ \lambda \cdot g \circ f \in L_2$. But $g \circ \lambda \cdot g \circ f = g \circ g^{-1}(g \circ \lambda \cdot g \circ f) = g(\lambda \odot f)$. Hence $g(\lambda \odot f) \in L_2$. By Theorem 2.3 we obtain that $\lambda \odot f \in L_2^{\oplus}$.

2. Now we prove four conditions of the mapping $(\cdot,\cdot)^{\oplus}$.

(i) Let $f \in L_2^{\oplus}$. Then $g \circ f \in L_2$. Hence $(g \circ f, g \circ f) \geqslant 0$. Since g is an increasing bijection with $g(0) = 0$, then we have

$$g^{-1}\big((g \circ f, g \circ f)\big) \geqslant g^{-1}(0),$$

$$(f,f)^{\oplus} \geqslant 0.$$

(ii) Let $f,h \in L_2^{\oplus}$. By Theorem 2.3 we have that $g \circ f, g \circ h \in L_2$. Then

$$(f,h)^{\oplus} = g^{-1}\big((g \circ f, g \circ h)\big) = g^{-1}\big((g \circ h, g \circ f)\big) = (h,f)^{\oplus}.$$

(iii) Let $f,h \in L_2^{\oplus}$, $\lambda \in \mathbb{R}$. By Theorem 2.3 we obtain that $g \circ f, g \circ h \in L_2$.

Then

$$\begin{aligned}(\lambda \odot f, h)^{\oplus} &= g^{-1}\big((g(\lambda \odot f), g \circ h)\big) = g^{-1}\big((g \circ g^{-1}(g(\lambda) \cdot g \circ f), g \circ h)\big)\\ &= g^{-1}\big((g(\lambda) \cdot g \circ f, g \circ h)\big) = g^{-1}\big(g(\lambda) \cdot (g \circ f, g \circ h)\big)\\ &= g^{-1}\big(g(\lambda) \cdot g \circ g^{-1}\big((g \circ f, g \circ h)\big)\big) = g^{-1}\big(g(\lambda) \cdot g\big((f,h)^{\oplus}\big)\big)\\ &= \lambda \odot (f,h)^{\oplus}.\end{aligned}$$

(iv) Let $f, h_1, h_2 \in L_2^{\oplus}$. By Theorem 2.3 we have that $g \circ f, g \circ h_1, g \circ h_2 \in L_2$.

$$\begin{aligned}(f, h_1 \oplus h_2)^{\oplus} &= g^{-1}\big((g \circ f, g(h_1 \oplus h_2))\big) = g^{-1}\big((g \circ f, g \circ g^{-1}(g \circ h_1 + g \circ h_2))\big)\\ &= g^{-1}\big((g \circ f, g \circ h_1 + g \circ h_2)\big) = g^{-1}\big((g \circ f, g \circ h_1) + (g \circ f, g \circ h_2)\big)\\ &= g^{-1}\big(g \circ g^{-1}\big((g \circ f, g \circ h_1)\big) + g \circ g^{-1}\big((g \circ f, g \circ h_2)\big)\big)\\ &= g^{-1}\big(g\big((f,h_1)^{\oplus}\big) + g\big((f,h_2)^{\oplus}\big)\big) = (f,h_1)^{\oplus} \oplus (f,h_2)^{\oplus}.\end{aligned}$$

□

Proposition 2.2. *Let $L_2^{\oplus}$ be a $\oplus$-linear space with mapping $(.,.)^{\oplus}\colon L_2^{\oplus}\times L_2^{\oplus}\to\mathbb{R}$ which satisfies the conditions* (i)–(iv) *from Proposition 2.1. Then for each $f,h\in L_2^{\oplus}$ we have*

$$\left|(f,h)^{\oplus}\right|^{\oplus}\leqslant\sqrt{(f,f)^{\oplus}}^{\oplus}\odot\sqrt{(h,h)^{\oplus}}^{\oplus}.$$

Proof. Let $f,h\in L_2^{\oplus}$. By *Theorem 2.3* we have that $g\circ f,\, g\circ h\in L_2$. Hence

$$\left|(g\circ f,g\circ h)\right|\leqslant\sqrt{(g\circ f,g\circ f)}\cdot\sqrt{(g\circ h,g\circ h)}$$

$$\left|g\circ g^{-1}\big((g\circ f,g\circ h)\big)\right|\leqslant\sqrt{g\circ g^{-1}\big((g\circ f,g\circ f)\big)}\cdot\sqrt{g\circ g^{-1}\big((g\circ h,g\circ h)\big)}$$

$$\left|g\big((f,h)^{\oplus}\big)\right|\leqslant\sqrt{g\big((f,f)^{\oplus}\big)}\cdot\sqrt{g\big((h,h)^{\oplus}\big)}$$

$$g\circ g^{-1}\Big(\left|g\big((f,h)^{\oplus}\big)\right|\Big)\leqslant g\circ g^{-1}\Big(\sqrt{g\big((f,f)^{\oplus}\big)}\Big)\cdot g\circ g^{-1}\Big(\sqrt{g\big((h,h)^{\oplus}\big)}\Big)$$

$$g\big(\left|(f,h)^{\oplus}\right|^{\oplus}\big)\leqslant g\big(\sqrt{(f,f)^{\oplus}}^{\oplus}\big)\cdot g\big(\sqrt{(h,h)^{\oplus}}^{\oplus}\big).$$

Since g is an increasing bijection, then we obtain that

$$g^{-1}\circ g\Big(\left|(f,h)^{\oplus}\right|^{\oplus}\Big)\leqslant g^{-1}\Big(g\Big(\sqrt{(f,f)^{\oplus}}^{\oplus}\Big)\cdot g\Big(\sqrt{(h,h)^{\oplus}}^{\oplus}\Big)\Big)$$

$$\left|(f,h)^{\oplus}\right|^{\oplus}\leqslant\sqrt{(f,f)^{\oplus}}^{\oplus}\odot\sqrt{(h,h)^{\oplus}}^{\oplus}.$$

□

Remark 2.5. The inequality $\left|(f,h)^{\oplus}\right|^{\oplus}\leqslant\sqrt{(f,f)^{\oplus}}^{\oplus}\odot\sqrt{(h,h)^{\oplus}}^{\oplus}$ is Cauchy inequality for $L_2^{\oplus}$ space.

Proposition 2.3. *Let $L_2^{\oplus}$ be a $\oplus$-linear space with a mapping $(\cdot,\cdot)^{\oplus}\colon L_2^{\oplus}\times L_2^{\oplus}\to\mathbb{R}$ satisfying the properties* (i)–(iv) *from Proposition 2.1. If we denote $\|f\|^{\oplus}=\sqrt{(f,f)^{\oplus}}^{\oplus}$, then $\|f\|^{\oplus}$ is $\oplus$-seminorm, that is*

(i) $\|f\|^{\oplus}\geqslant 0$ *for each* $f\in L_2^{\oplus}$,

(ii) $\|\lambda\odot f\|^{\oplus}=|\lambda|^{\oplus}\odot\|f\|^{\oplus}$ *for each* $f\in L_2^{\oplus}$ *and* $\lambda\in\mathbb{R}$,

(iii) $\|f\oplus h\|^{\oplus}\leqslant\|f\|^{\oplus}\oplus\|h\|^{\oplus}$ *for each* $f,h\in L_2^{\oplus}$.

Proof.

(i) By Proposition 2.1 we have that $(f,f)^{\oplus}\geqslant 0$ for each $f\in L_2^{\oplus}$. Because g is an increasing bijection with $g(0)=0$, we obtain $g\big((f,f)^{\oplus}\big)\geqslant g(0)=0$. Hence $\|f\|^{\oplus}=\sqrt{(f,f)^{\oplus}}^{\oplus}=g^{-1}\Big(\sqrt{g\big((f,f)^{\oplus}\big)}\Big)\geqslant 0$.

(ii)

$$
\begin{aligned}
\|\lambda \odot f\|^{\oplus} &= \sqrt{(\lambda \odot f, \lambda \odot f)^{\oplus}}^{\oplus} = \sqrt{g^{-1}\big((g(\lambda \odot f), g(\lambda \odot f))\big)}^{\oplus} \\
&= \sqrt{g^{-1}\big((g \circ g^{-1}(g(\lambda) \cdot g \circ f), g \circ g^{-1}(g(\lambda) \cdot g \circ f))\big)}^{\oplus} \\
&= \sqrt{g^{-1}\big((g(\lambda) \cdot g \circ f, g(\lambda) \cdot g \circ f)\big)}^{\oplus} \\
&= g^{-1}\Big(\sqrt{g \circ g^{-1}\big((g(\lambda) \cdot g \circ f, g(\lambda) \cdot g \circ f)\big)}\Big) \\
&= g^{-1}\big(\sqrt{(g(\lambda) \cdot g \circ f, g(\lambda) \cdot g \circ f)}\big) = g^{-1}\big(\|g(\lambda) \cdot g \circ f\|\big) \\
&= g^{-1}\big(\big|g(\lambda)\big| \cdot \|g \circ f\|\big) = g^{-1}\big(\big|g(\lambda)\big| \cdot \sqrt{(g \circ f, g \circ f)}\big) \\
&= g^{-1}\Big(g \circ g^{-1}\big(\big|g(\lambda)\big|\big) \cdot g \circ g^{-1}\big(\sqrt{(g \circ f, g \circ f)}\big)\Big) \\
&= g^{-1}\Big(g\big(|\lambda|^{\oplus}\big) \cdot g \circ g^{-1}\big(\sqrt{(g \circ f, g \circ f)}\big)\Big) = |\lambda|^{\oplus} \odot g^{-1}\big(\sqrt{(g \circ f, g \circ f)}\big) \\
&= |\lambda|^{\oplus} \odot g^{-1}\Big(\sqrt{g \circ g^{-1}((g \circ f, g \circ f))}\Big) = |\lambda|^{\oplus} \odot g^{-1}\Big(\sqrt{g\big((f,f)^{\oplus}\big)}\Big) \\
&= |\lambda|^{\oplus} \odot \sqrt{(f,f)^{\oplus}}^{\oplus}.
\end{aligned}
$$

(iii) Let $f, h \in L_2^{\oplus}$. By Theorem 2.3 we have that $g \circ f,\ g \circ h \in L_2$. Then

$$\|g \circ f + g \circ h\| \leqslant \|f\| + \|h\|,$$

$$\sqrt{(g \circ f + g \circ h, g \circ f + g \circ h)} \leqslant \sqrt{(g \circ f, g \circ f)} + \sqrt{(g \circ h, g \circ h)}$$

$$
\begin{aligned}
&\sqrt{(g \circ g^{-1}(g \circ f + g \circ h), g \circ g^{-1}(g \circ f + g \circ h))} \\
&\qquad\qquad \leqslant \sqrt{g \circ g^{-1}\big((g \circ f, g \circ f)\big)} + \sqrt{g \circ g^{-1}\big((g \circ h, g \circ h)\big)}.
\end{aligned}
$$

$$\sqrt{\big(g(f \oplus h), g(f \oplus h)\big)} \leqslant \sqrt{g\big((f,f)^{\oplus}\big)} + \sqrt{g\big((h,h)^{\oplus}\big)}$$

$$\sqrt{g \circ g^{-1}\Big(\big(g(f \oplus h), g(f \oplus h)\big)\Big)} \leqslant \sqrt{g\big((f,f)^{\oplus}\big)} + \sqrt{g\big((h,h)^{\oplus}\big)}$$

$$\sqrt{g\big((f \oplus h, f \oplus h)^{\oplus}\big)} \leqslant \sqrt{g\big((f,f)^{\oplus}\big)} + \sqrt{g\big((h,h)^{\oplus}\big)}$$

$$g \circ g^{-1}\Big(\sqrt{g\big((f \oplus h, f \oplus h)^{\oplus}\big)}\Big) \leqslant g \circ g^{-1}\Big(\sqrt{g\big((f,f)^{\oplus}\big)}\Big) + g \circ g^{-1}\Big(\sqrt{g\big((h,h)^{\oplus}\big)}\Big)$$

$$g\Big(\sqrt{(f \oplus h, f \oplus h)^{\oplus}}^{\oplus}\Big) \leqslant g\Big(\sqrt{(f,f)^{\oplus}}^{\oplus}\Big) + g\Big(\sqrt{(h,h)^{\oplus}}^{\oplus}\Big).$$

Since g is an increasing bijection, then we obtain

$$g^{-1} \circ g\Big(\sqrt{(f \oplus h, f \oplus h)^{\oplus}}^{\,\oplus}\Big) \leqslant g^{-1}\Big(g\Big(\sqrt{(f,f)^{\oplus}}^{\,\oplus}\Big) + g\Big(\sqrt{(h,h)^{\oplus}}^{\,\oplus}\Big)\Big)$$

$$\sqrt{(f \oplus h, f \oplus h)^{\oplus}}^{\,\oplus} \leqslant \sqrt{(f,f)^{\oplus}}^{\,\oplus} \oplus \sqrt{(h,h)^{\oplus}}^{\,\oplus}$$

$$\|f \oplus h\|^{\oplus} \leqslant \|f\|^{\oplus} \oplus \|h\|^{\oplus}.$$

The notion of a $\oplus$-pseudometric space and the notion of a complete space were introduced by J. Desatníková in [2]. □

Definition 2.6. Let X be a set and $d\colon X \times X \to \mathbb{R}$ be a mapping with the following properties:

(i) $d(x,y) \geqslant 0$ and $d(x,x) = 0$ for each $x, y \in X$,

(ii) $d(x,y) = d(y,x)$ for each $x, y \in X$,

(iii) $d(x,z) \leqslant d(x,y) \oplus d(y,z)$ for each $x, y, z \in X$.

Then we say that (X, d) is *$\oplus$-pseudometric space.*

Definition 2.7. A sequence $(x_n)_1^{\infty}$ of a $\oplus$-pseudometric space (X,d) is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists n_0 such that for each $n, m \geqslant n_0$ it is $d(x_n, x_m) < \varepsilon$. A $\oplus$-pseudometric space is a *complete space* if each Cauchy sequence is convergent.

Proposition 2.4. *If for each $f, h \in L_2^{\oplus}$ we denote $\rho_2^{\oplus}(f,h) = \|f \ominus h\|^{\oplus} = \big(\int^{\oplus} (f \ominus h)^{2\oplus}\, dP\big)^{\frac{1}{2}\oplus}$, then $(L_2^{\oplus}, \rho_2^{\oplus})$ is $\oplus$-pseudometric space.*

Proof. In [2] Theorem 3.2 we put $\rho_2^{\oplus} = d_2^{\oplus}$. □

Proposition 2.5. *$(L_2^{\oplus}, \rho_2^{\oplus})$ is a complete $\oplus$-pseudometric space.*

Proof. In [2] Theorem 3.3. □

In [2] we can find the definition of a $\oplus$-metric space, too.

Definition 2.8. Let X be a set and $d\colon X \times X \to \mathbb{R}$ be a mapping with the following properties:

(i) $d(x,y) \geqslant 0$ and $d(x,x) = 0 \iff x = y$ for each $x, y \in X$,

(ii) $d(x,y) = d(y,x)$ for each $x, y \in X$,

(iii) $d(x,z) \leqslant d(x,y) \oplus d(y,z)$ for each $x, y, z \in X$.

Then we say that (X, d) is *$\oplus$-metric space.*

Theorem 2.9. *For $f \in L_2^{\oplus}$ we denote $[f]^{\oplus} = \{h \in L_2^{\oplus} : P$-almost everywhere $f = h\}$. Then $\mathbf{L}_2^{\oplus} = \{[f]^{\oplus}, f \in L_2^{\oplus}\}$ is a $\oplus$-linear space with operations*

$$[f]^{\oplus} \oplus [h]^{\oplus} = [f \oplus h]^{\oplus}, \quad [\alpha \odot f]^{\oplus} = \alpha \odot [f]^{\oplus},$$

which is a $\oplus$-metric space, where $\rho^{\oplus}([f]^{\oplus}, [h]^{\oplus}) = \rho_2^{\oplus}(f, h) = \sqrt{(f \ominus h, f \ominus h)^{\oplus}}^{\,\oplus}$ $= \left(\int^{\oplus} (f \ominus h)^{2\oplus}\, dP\right)^{\frac{1}{2}\oplus}$.

P r o o f . All propositions are now evident. We prove that $\rho^{\oplus}$ is a $\oplus$-metric (not only $\oplus$-pseudometric). Let $[f]^{\oplus}, [h]^{\oplus} \in \mathbf{L}_2^{\oplus}$ and $\rho^{\oplus}([f]^{\oplus}, [h]^{\oplus}) = 0$. Then

$$\left(\int^{\oplus} (f \ominus h)^{2\oplus}\, dP\right)^{\frac{1}{2}\oplus} = 0\,,$$

$$\int^{\oplus} (f \ominus h)^{2\oplus}\, dP = 0\,.$$

Hence $(f \ominus h)^{2\oplus} = 0$ P-almost everywhere $\iff$ $f = h$ P-almost everywhere $\iff$ $[f]^{\oplus} = [h]^{\oplus}$. ☐

Proposition 2.6. *Let $(\Omega, \mathcal{S}, P)$ be a space with a finite g-measure P, $\mathcal{S}$ be a σ-algebra. Then $\mathbf{L}_2^{\oplus} \subset \mathbf{L}_1^{\oplus}$, where $\mathbf{L}_1^{\oplus} = \{[f]^{\oplus} : f \in L_1^{\oplus}\}$, $L_1^{\oplus} = \{f : \Omega \to \mathbb{R} : \exists \int^{\oplus} f\, dP < \infty\}$.*

P r o o f . 1. First proof.

Let $[f]^{\oplus} \in \mathbf{L}_2^{\oplus}$. Then $f \in L_2^{\oplus}$. By Theorem 2.3 we have that $g \circ f \in L_2$. Since $\mathbf{L}^2 \subset \mathbf{L}^1$, then $[g \circ f] \in \mathbf{L}^1$. Hence $[f]^{\oplus} \in \mathbf{L}_1^{\oplus}$.

2. Second proof.

If we put $h = 1$ in Cauchy inequality for $L_2^{\oplus}$ space in Proposition 2.2, then we obtain that

$$\left| \int^{\oplus} f \odot 1\, dP \right|^{\oplus} \leqslant \sqrt{\int^{\oplus} f \odot f\, dP}^{\,\oplus} \cdot \sqrt{\int^{\oplus} 1 \odot 1\, dP}^{\,\oplus}.$$

Since there exists $\int^{\oplus} f \odot f dP$, then there exists $\int^{\oplus} f\, dP$. Hence $[f]^{\oplus} \in \mathbf{L}_1^{\oplus}$. ☐

3. Pseudo-dispersion

In this section we introduce the notion of pseudo-mean value and the notion of pseudo-dispersion. We prove the Chebyshev inequality for pseudo-probability P.

In [4] L. M. N e d o v i ć and T. G r b i ć defined the pseudo-probability P as a function $P : \mathcal{S} \to I$ with the following properties:

(i) $P(\emptyset) = 0$ and $P(\Omega) = 1$,

(ii) $P\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \bigoplus\limits_{i=1}^{\infty} P(A_i)$ for pairwise disjoint sets A_i from $\mathcal{S}$, $i \in \mathbb{N}$,

where $\mathcal{S}$ is a σ-algebra of subsets of a set Ω and $(I, \oplus, \odot)$ is a semiring.

We can express the pseudo-probability P by formula $P = g^{-1} \circ p$, where p is the probability. The triple $(\Omega, \mathcal{S}, P)$ is called pseudo-probability space.

Definition 3.1. Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space. If a random variable $\xi\colon \Omega \to \mathbb{R}$ is integrable, then the number

$$E^{\oplus}(\xi) = \int_{\Omega}^{\oplus} \xi \, dP < \infty$$

is called the *pseudo-mean value* (*expectation*) of the random variable ξ.

Theorem 3.2. *Let $(\Omega, \mathcal{S}, p)$ be a probability space and $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space. Let $\xi\colon \Omega \to \mathbb{R}$ be a random variable. Then the random variable $g \circ \xi$ is integrable in probability space if and only if the random variable ξ is integrable in pseudo-probability space and the pseudo mean-value $E^{\oplus}$ of the random variable ξ can be expressed by formula*

$$E^{\oplus}(\xi) = g^{-1}\big(E(g \circ \xi)\big).$$

Proof. "$\Rightarrow$" Let ξ be integrable in $(\Omega, \mathcal{S}, P)$. Then there exists $\int_{\Omega}^{\oplus} \xi \, dP < \infty$. Since $\int_{\Omega}^{\oplus} \xi \, dP = g^{-1}\left(\int_{\Omega} g \circ \xi \, dp\right)$, then there exists $\int_{\Omega} g \circ \xi \, dp < \infty$. Hence $g \circ \xi$ is integrable in $(\Omega, \mathcal{S}, p)$.

"$\Leftarrow$" Proof is similar to proof "$\Rightarrow$".

$$E^{\oplus}(\xi) = \int_{\Omega}^{\oplus} \xi \, dP = g^{-1}\left(\int_{\Omega} g \circ \xi \, dp\right) = g^{-1}\big(E(g \circ \xi)\big).$$

$\square$

Definition 3.3. Let $\xi\colon \Omega \to \mathbb{R}$ be an integrable random variable in $(\Omega, \mathcal{S}, P)$. If the random variable $\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}$ is integrable, then we say that ξ has *pseudo-dispersion* $\sigma^{2\oplus}$ defined by formula

$$\sigma^{2\oplus}(\xi) = E^{\oplus}\Big(\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}\Big).$$

Theorem 3.4. *Let $(\Omega, \mathcal{S}, P)$ be a pseudo-probability space and $(\Omega, \mathcal{S}, p)$ be a probability space. Let $\xi\colon \Omega \to \mathbb{R}$ be a random variable. Then ξ has pseudo-dispersion $\sigma^{2\oplus}(\xi)$ if and only if $g \circ \xi$ has dispersion $\sigma^2(g \circ \xi)$ and then the pseudo-dispersion can be expressed by formula*

$$\sigma^{2\oplus}(\xi) = g^{-1}\big(\sigma^2(g \circ \xi)\big).$$

Proof. Let ξ have pseudo-dispersion $\sigma^{2\oplus}$. Then there exist $E^{\oplus}(\xi) < \infty$ and $E^{\oplus}\Big(\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}\Big) < \infty \iff$ there exist $g^{-1}\big(E(g \circ \xi)\big) < \infty$ and $g^{-1}\big(E\big((g\circ\xi - E(g\circ\xi)\big)^2\big)\big) < \infty \iff$ there exist $E(g\circ\xi) < \infty$ and $E\big((g \circ \xi - E(g \circ \xi)\big)^2\big) < \infty \Leftrightarrow g \circ \xi$ has dispersion σ^2.

By Definition 3.3 we have

$$\begin{aligned}
\sigma^{2\oplus}(\xi) &= E^{\oplus}\Big(\big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}\Big) = g^{-1}\Big(E\big(g \circ \big(\xi \ominus E^{\oplus}(\xi)\big)^{2\oplus}\big)\Big) \\
&= g^{-1}\Big(E\Big(g \circ g^{-1}\big(\big(g \circ \big(\xi \ominus E^{\oplus}(\xi)\big)\big)^2\big)\Big)\Big) \\
&= g^{-1}\Big(E\Big(\big(g \circ g^{-1}\big(g \circ \xi - g \circ E^{\oplus}(\xi)\big)\big)^2\Big)\Big) \\
&= g^{-1}\Big(E\big(\big(g \circ \xi - E(g \circ \xi)\big)^2\big)\Big) = g^{-1}\big(\sigma^2(g \circ \xi)\big).
\end{aligned}$$

□

Proposition 3.1. *The random variable ξ has pseudo-dispersion $\sigma^{2\oplus}$ if and only if $\xi \in L_2^{\oplus}(\Omega, \mathcal{S}, P)$ and then*

$$\sigma^{2\oplus}(\xi) = E^{\oplus}\big(\xi^{2\oplus}\big) \ominus \big(E^{\oplus}(\xi)\big)^{2\oplus}.$$

Proof. By Theorem 3.4 we obtain

$$\begin{aligned}
\sigma^{2\oplus}(\xi) &= g^{-1}\big(\sigma^2(g \circ \xi)\big) = g^{-1}\Big(E\big((g \circ \xi)^2\big) - \big(E(g \circ \xi)\big)^2\Big) \\
&= g^{-1}\Big(E\big(g \circ g^{-1}\big((g \circ \xi)^2\big)\big) - \big(g \circ g^{-1}\big(E(g \circ \xi)\big)\big)^2\Big) \\
&= g^{-1}\Big(E\big(g \circ \xi^{2\oplus}\big) - \big(g \circ E^{\oplus}(\xi)\big)^2\Big) \\
&= g^{-1}\Big(g \circ g^{-1}\big(E\big(g \circ \xi^{2\oplus}\big)\big) - g \circ g^{-1}\big(\big(g \circ E^{\oplus}(\xi)\big)^2\big)\Big) \\
&= g^{-1}\Big(g \circ E^{\oplus}\big(\xi^{2\oplus}\big) - g \circ \big(E^{\oplus}(\xi)\big)^{2\oplus}\Big) = E^{\oplus}\big(\xi^{2\oplus}\big) \ominus \big(E^{\oplus}(\xi)\big)^{2\oplus}.
\end{aligned}$$

□

THEOREM 3.5. (Chebyshev inequality for pseudo-probability) *If a random variable $\xi\colon \Omega \to \mathbb{R}$ has pseudo-dispersion $\sigma^{2\oplus}$, then for each $\varepsilon > 0$ it is satisfied inequality*

$$P\Big(\big\{\omega\colon \big|\xi(\omega) \ominus E^{\oplus}(\xi)\big|^{\oplus} \geqslant \varepsilon\big\}\Big) \leqslant \sigma^{2\oplus}(\xi) \oslash \varepsilon^{2\oplus}.$$

P r o o f. If ξ has the pseudo-dispersion, then $g \circ \xi$ has the dispersion. Then

$$\begin{aligned}
P\Big((\big\{\omega\colon \big|\xi(\omega) \ominus E^{\oplus}(\xi)\big|^{\oplus} \geqslant \varepsilon\big\}\Big) &= P\Big(\big\{\omega\colon g^{-1}\big(\big|g \circ \big(\xi(\omega) \ominus E^{\oplus}(\xi)\big)\big|\big) \geqslant \varepsilon\big\}\Big) \\
&= P\Big(\big\{\omega\colon g^{-1}\big(\big|g \circ g^{-1}\big(g \circ \xi(\omega) - g \circ E^{\oplus}(\xi)\big)\big|\big) \geqslant \varepsilon\big\}\Big) \\
&= P\Big(\big\{\omega\colon g^{-1}\big(\big|g \circ \xi(\omega) - g \circ E^{\oplus}(\xi)\big|\big) \geqslant \varepsilon\big\}\Big) \\
&= P\Big(\big\{\omega\colon g^{-1}\big(\big|g \circ \xi(\omega) - g \circ g^{-1}\big(E(g \circ \xi)\big)\big|\big) \geqslant \varepsilon\big\}\Big) \\
&= P\Big(\big\{\omega\colon g^{-1}\big(\big|g \circ \xi(\omega) - E(g \circ \xi)\big|\big) \geqslant \varepsilon\big\}\Big) \\
&= P\Big(\big\{\omega\colon \big|g \circ \xi(\omega) - E(g \circ \xi)\big| \geqslant g(\varepsilon)\big\}\Big) \\
&= g^{-1} \circ p\Big(\big\{\omega\colon \big|g \circ \xi(\omega) - E(g \circ \xi)\big| \geqslant g(\varepsilon)\big\}\Big).
\end{aligned}$$

Since

$$p\big(\big\{\omega\colon \big|g \circ \xi(\omega) - E(g \circ \xi)\big| \geqslant g(\varepsilon)\big\}\big) \leqslant \frac{\sigma^2(g \circ \xi)}{\big(g(\varepsilon)\big)^2}$$

$$= \frac{g \circ g^{-1}\big(\sigma^2(g \circ \xi)\big)}{g \circ g^{-1}\big(\big(g(\varepsilon)\big)^2\big)} = \frac{g\big(\sigma^{2\oplus}(\xi)\big)}{g\big(\varepsilon^{2\oplus}\big)} = g \circ g^{-1}\left(\frac{g\big(\sigma^{2\oplus}(\xi)\big)}{g\big(\varepsilon^{2\oplus}\big)}\right) = g\big(\sigma^{2\oplus}(\xi) \oslash \varepsilon^{2\oplus}\big),$$

then

$$g^{-1} \circ p\Big(\big\{\omega\colon \big|g \circ \xi(\omega) - E(g \circ \xi)\big| \geqslant g(\varepsilon)\big\}\Big) \leqslant g^{-1} \circ g\big(\sigma^{2\oplus}(\xi) \oslash \varepsilon^{2\oplus}\big),$$

$$P\big(\big\{\omega\colon \big|\xi(\omega) \ominus E^{\oplus}(\xi)\big|^{\oplus} \geqslant \varepsilon\big\}\big) \leqslant \sigma^{2\oplus}(\xi) \oslash \varepsilon^{2\oplus}.$$

□

REFERENCES

[1] BILLINGSLEY, P.: *Probability and Measure*, Wiley Interscience Publication, New York, 1995.

[2] DESATNÍKOVÁ, J.: *L_p spaces and g-calculus*, student scientific work, FPV UMB, Banská Bystrica, 2004.

[3] LENDELOVÁ, K.: *On the pseudo-Lebesgue-Stieltjes integral* (*accepted in Novi Sad Journal of Mathematics*) (to appear).

[4] NEDOVIĆ, L. M.—GRBIĆ, T.: *The pseudo probability*, J. Electr. Eng. **53** (2002), 27–31.
[5] PAP, E.: *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava, 1995.
[6] POLLARD, D.: *A User's Guide to Measure Theoretic Probability*, Cambridge University Press, New York, 2002.
[7] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*, Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava, 1997.
[8] RIEČAN, B.—NEUBRUNN, T.: *Measure Theory*, Veda, Bratislava, 1992.
[9] RIEČAN, B.—NEUBRUNN, T.: *Measure and Integral*, Veda, Bratislava, 1981.

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