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THE ULAM PROBLEM FOR 3-DIMENSIONAL QUADRATIC MAPPINGS

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ABSTRACT. In 1940 and in 1964 S. M. Ulam proposed the general problem: When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In this paper we investigate the 3-dimensional quadratic mappings $Q\colon X \to Y$, satisfying the functional equation

$$Q(x_1+x_2+x_3)+Q(x_1-x_2+x_3)+Q(x_1+x_2-x_3)+Q(x_1-x_2-x_3) = 4[Q(x_1)+Q(x_2)+Q(x_3)]$$

and then solve the corresponding Ulam stability problem.

1. Two-dimensional mappings

S. M. Ulam ([10]) proposed the general problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" In this paper we investigate the Ulam stability problem for 3-dimensional mappings. In this section we consider the Ulam stability problem for 2-dimensional mappings.

Theorem 1.1. ([4]–[6]) *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f\colon X \to Y$ is a mapping for which there exists a constant $c \geqslant 0$ (independent of x_1, x_2) such that the quadratic functional inequality*

$$\left\| f(x_1+x_2)+f(x_1-x_2)-2\left[f(x_1)+f(x_2)\right] \right\| \leqslant c$$

holds for all $(x_1,x_2) \in X^2$. Then the limit $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ exists for all $x \in X$ and all $n \in \mathbb{N} = \{1,2,\dots\}$ and $Q\colon X \to Y$ is the unique quadratic mapping satisfying the functional equation

$$Q(x_1+x_2)+Q(x_1-x_2) = 2\left[Q(x_1)+Q(x_2)\right]$$

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for all $(x_1, x_2) \in X^2$, *such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{2}c$$

holds for all $x \in X$.

2. Three-dimensional mappings

In this section we establish the Ulam stability for 3-dimensional quadratic mappings.

Definition 2.1. Let X be a normed linear space and let Y be a real complete normed linear space. Then a mapping $Q\colon X \to Y$, is called *3-dimensional quadratic* if the functional equation

$$Q(x_1 + x_2 + x_3) + Q(x_1 - x_2 + x_3) + Q(x_1 + x_2 - x_3) + Q(x_1 - x_2 - x_3) = 4\big[Q(x_1) + Q(x_2) + Q(x_3)\big] \quad (2.1)$$

holds for all $(x_1, x_2, x_3) \in X^3$. Note that mapping Q is called *quadratic*, because the functional equation

$$Q(2^n x) = (2^n)^2 Q(x) \quad (2.2)$$

holds for all $x \in X$, and all $n \in \mathbb{N}$ ([4]–[6], [8], [9]).

In fact, substitution of $x_1 = x_2 = x_3 = 0$ in equation (2.1) yields that $Q(0) = 0$.

Substituting $x_1 = x_2 = x$, $x_3 = 0$ one gets that the functional equation

$$2Q(2x) + 2Q(0) = 4\big[2Q(x) + Q(0)\big] \quad \text{or} \quad Q(2x) = (2)^2 Q(x)$$

holds for all $x \in X$.

Then induction on $n \in \mathbb{N}$ with $x \to 2^{n-1}x$ yields equation (2.2).

Theorem 2.1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f\colon X \to Y$ is a 3-dimensional mapping for which there exists a constant c (independent of x_1, x_2, x_3) $\geqslant 0$ such that the quadratic functional inequality*

$$\big\|f(x_1 + x_2 + x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 - x_2 - x_3) - 4\big[f(x_1) + f(x_2) + f(x_3)\big]\big\| \leqslant c \quad (2.3)$$

holds for all $(x_1, x_2, x_3) \in X^3$. *Then the limit*

$$Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x) \quad (2.4)$$

exists for all $x \in X$ *and all* $n \in \mathbb{N}$ *and* $Q\colon X \to Y$ *is the unique 3-dimensional quadratic mapping satisfying the functional equation* (2.1), *such that*

$$\|f(x) - Q(x)\| \leqslant \frac{5}{24}c \tag{2.5}$$

holds for all $x \in X$. *Moreover, functional identity*

$$Q(x) = 2^{-2n}Q(2^n x)$$

holds for all $x \in X$, *and all* $n \in \mathbb{N}$.

P r o o f. Substitution of $x_1 = x_2 = x_3 = 0$ in inequality (2.3) yields that

$$\|f(0)\| \leqslant \frac{c}{8}. \tag{2.6}$$

Moreover, substituting $x_1 = x_2 = x$, $x_3 = 0$ in inequality (2.3) and employing (2.6) and the triangle inequality one concludes $\|2f(2x) + 2f(0) - 4[2f(x) + f(0)]\| \leqslant c$, or $\|f(2x) - 4f(x) - f(0)\| \leqslant \frac{c}{2}$, or $\|f(2x) - 4f(x)\| \leqslant \|f(2x) - 4f(x) - f(0)\| + \|f(0)\|$, or $\|f(2x) - 4f(x)\| \leqslant \frac{c}{2} + \frac{c}{8} = \frac{5}{8}c$, or thus one gets that *the basic functional inequality*

$$\|f(x) - 2^{-2}f(2x)\| \leqslant \frac{5}{32}c = c_1(1 - 2^{-2}) \tag{2.7}$$

holds for all $x \in X$, where $c_1 = (5/24)c$. Replacing now x with $2x$ in (2.7) one concludes that $\|f(2x) - 2^{-2}f(2^2x)\| \leqslant c_1(1 - 2^{-2})$ or

$$\|2^{-2}f(2x) - 2^{-4}f(2^2x)\| \leqslant c_1(2^{-2} - 2^{-4}) \tag{2.7a}$$

holds for all $x \in X$. Functional inequalities (2.7)–(2.7a) and the triangle inequality yield

$$\begin{aligned}\|f(x) - 2^{-4}f(2^2x)\| &\leqslant \|f(x) - 2^{-2}f(2x)\| + \|2^{-2}f(2x) - 2^{-4}f(2^2x)\| \\ &\leqslant c_1[(1 - 2^{-2}) + (2^{-2} - 2^{-4})]\end{aligned}$$

or that the functional inequality

$$\|f(x) - 2^{-4}f(2^2x)\| \leqslant c_1(1 - 2^{-4})$$

holds for all $x \in X$.

Similarly by induction on $n \in \mathbb{N}$ with $x \to 2^{n-1}x$ in the basic inequality (2.7) claim that the general functional inequality

$$\|f(x) - 2^{-2n}f(2^nx)\| \leqslant c_1(1 - 2^{-2n}) \tag{2.8}$$

holds for all $x \in X$ and all $n \in \mathbb{N}$. In fact, the basic inequality (2.7) with $x \to 2^{n-1}x$ yield the functional inequality $\|f(2^{n-1}x) - 2^{-2}f(2^nx)\| \leqslant c_1(1 - 2^{-2})$ or the inequality

$$\|2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^nx)\| \leqslant c_1(2^{-2(n-1)} - 2^{-2n}) \tag{2.8a}$$

holds for all $x \in X$. Moreover, by induction hypothesis with $n \to n-1$ in the general inequality (2.8) one gets that

$$\left\| f(x) - 2^{-2(n-1)} f(2^{n-1}x) \right\| \leqslant c_1\left(1 - 2^{-2(n-1)}\right) \tag{2.8b}$$

holds for all $x \in X$. Thus functional inequalities (2.8a)–(2.8b) and the triangle inequality imply

$$\begin{aligned}\left\| f(x) - 2^{-2n} f(2^n x) \right\| \leqslant \left\| f(x) - 2^{-2(n-1)} f(2^{n-1}x) \right\| \\ + \left\| 2^{-2(n-1)} f(2^{n-1}x) - 2^{-2n} f(2^n x) \right\|\end{aligned}$$

or

$$\left\| f(x) - 2^{-2n} f(2^n x) \right\| \leqslant c_1\left[\left(1 - 2^{-2(n-1)}\right) + \left(2^{-2(n-1)} - 2^{-2n}\right)\right] = c_1\left(1 - 2^{-2n}\right),$$

completing the proof of the required general functional inequality (2.8).

Claim now that the sequence $\left\{2^{-2n} f(2^n x)\right\}$ converges. Note that from the general inequality (2.8) and the completeness of Y, one proves that the above sequence is a *Cauchy sequence*. In fact, if $i > j > 0$, then

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 2^{-2j} \left\| 2^{-2(i-j)} f(2^i x) - f(2^j x) \right\|$$

holds for all $x \in X$, and all $i, j \in \mathbb{N}$. Setting $h = 2^j x$ in the above relation and employing the general inequality (2.8) one concludes that

$$\begin{aligned}\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 2^{-2j} \left\| 2^{-2(i-j)} f\left(2^{i-j} h\right) - f(h) \right\| \\ \leqslant 2^{-2j} c_1\left(1 - 2^{-2(i-j)}\right)\end{aligned}$$

or

$$\left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| \leqslant c_1\left(2^{-2j} - 2^{-2i}\right) < c_1 2^{-2j}$$

or

$$\lim_{j\to\infty} \left\| 2^{-2i} f(2^i x) - 2^{-2j} f(2^j x) \right\| = 0,$$

completing the proof that the sequence $\left\{2^{-2n} f(2^n x)\right\}$ converges. Hence $Q = Q(x)$ is a *well-defined mapping* via the formula (2.4). This means that the limit (2.4) exists for all $x \in X$. In addition claim that mapping Q *satisfies the functional equation* (2.1) for all $(x_1, x_2, x_3) \in X^3$. In fact, it is clear from the functional inequality (2.3) and the limit (2.4) that the following inequality

$$\begin{aligned}2^{-2n} \big\| f(2^n x_1 + 2^n x_2 + 2^n x_3) + f(2^n x_1 - 2^n x_2 + 2^n x_3) + f(2^n x_1 + 2^n x_2 - 2^n x_3) \\ + f(2^n x_1 - 2^n x_2 - 2^n x_3) - 4\left[f(2^n x_1) + f(2^n x_2) + f(2^n x_3)\right] \big\| \leqslant 2^{-2n} c\end{aligned}$$

holds for all $(x_1, x_2, x_3) \in X^3$, and all $n \in \mathbb{N}$. Therefore one gets

$$\begin{aligned}\Big\| \lim_{n\to\infty} 2^{-2n} f\left[2^n(x_1 + x_2 + x_3)\right] + \lim_{n\to\infty} 2^{-2n} f\left[2^n(x_1 - x_2 + x_3)\right] \\ + \lim_{n\to\infty} 2^{-2n} f\left[2^n(x_1 + x_2 - x_3)\right] + \lim_{n\to\infty} 2^{-2n} f\left[2^n(x_1 - x_2 - x_3)\right] \\ - 4\Big[\lim_{n\to\infty} 2^{-2n} f(2^n x_1) + \lim_{n\to\infty} 2^{-2n} f(2^n x_2) + \lim_{n\to\infty} 2^{-2n} f(2^n x_3)\Big] \Big\| \\ \leqslant \lim_{n\to\infty} (2^{-2n}) c = 0\end{aligned}$$

or mapping Q satisfies the equation (2.1) for all $(x_1, x_2, x_3) \in X^3$.

Thus Q is a *3-dimensional quadratic mapping*. It is clear now from the general functional inequality (2.8), $n \to \infty$, and the formula (2.4) that inequality (2.5) holds in X, completing *the existence proof* of this Theorem 2.1.

The proof of *uniqueness* is omitted as obvious (see: [1]–[9]) and thus the *stability* of this Theorem 2.1 is complete. $\square$

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