Tatra Mt. Math. Publ. **34** (2006), 385–391

TATRA MOUNTAINS
Mathematical Publications

SEMIGROUPS OF OPERATORS OF FACTOR SEQUENCE TYPE

Miloslav Duchoň

ABSTRACT. There are studied semigroups of factor sequence type in the case of homogeneous Banach spaces on the circle group.

The object of this paper is a study of semigroups of operators of factor sequence type in case of homogeneous Banach spaces. We consider strongly continuous semigroups of operators of factor sequence type on homogeneous Banach spaces X of integrable functions on the circle group $\mathbb{T}$ $\big(X \subset L^1(\mathbb{T})\big)$ as well as their dual semigroups on the corresponding dual spaces.

1. Through this paper $\mathbb{T}$ or the circle mean the group of real numbers defined modulo 2π, x is a point on the circle, and all integrals are taken over the circle $\mathbb{T}$. The homogeneous Banach spaces over $\mathbb{T}$ are, by definition, the linear subspaces $B(\mathbb{T})$ of $L^1(\mathbb{T})$ which are endowed with a norm $\|\cdot\|_B$ under which we have a Banach space such that

(i) $\|f\|_1 \leqslant \|f\|_B$ for all $f \in B(\mathbb{T})$;

(ii) if $f \in B(\mathbb{T})$ and $x \in \mathbb{T}$, then $f_x \in B(\mathbb{T})$ and $\|f_x\|_B = \|f\|_B$, $\big[f_x(u) = f(u-x)\big]$;

(iii) $\lim\limits_{u\to 0} \|f_u - f\|_B = 0$ for all $f \in B(\mathbb{T})$.

Let X and Y be two homogeneous Banach spaces over $\mathbb{T}$. Recall that a sequence of complex numbers $(c_k)_{k=-\infty}^{\infty}$ is said to be a factor sequence of type (X,Y) if for each $f \in X$ the formal series

$$\sum_{k=-\infty}^{\infty} c_k \hat{f}(k) e^{ikx}, \qquad \hat{f}(k) = \int e^{-ikx} f(x)\,dx\,,\ \ k = 0, \pm 1, \dots$$

2000 Mathematics Subject Classification: 47D05, 42A38.

Keywords: semigroup of operators, factor sequence, homogeneous Banach space, Fourier series.

Supported by grant VEGA 4137, by project SK–CZ–07906, and APVV 51 006904

is the Fourier series of some function in Y. It is easy to show that each factor sequence of type (X, Y) defines a bounded linear mapping L from X into Y. Similar definitions can be given for duals of X and Y, respectively.

Examples of homogeneous Banach spaces.

1. All spaces $L^p(\mathbb{T})$, $1 \leqslant p < \infty$.
2. $C(\mathbb{T})$—the space of all continuous 2π-periodic functions with the max norm.
3. $C^n(\mathbb{T})$—the subspace of $C(\mathbb{T})$ of all n-times continuously differentiable functions with the common norm.
4. $L^{(1)}(\mathbb{T})$—the Banach space of all functions f on $\mathbb{T}$ which are absolutely continuous with the common norm.
5. $\mathrm{lip}_q(\mathbb{T})$, $0 < q \leqslant 1$,—("small") Lipschitz spaces with the usual norm.

Note that $L^\infty(\mathbb{T})$—the space of all essentially bounded functions in $L^1(\mathbb{T})$ with the ess sup norm is not a homogeneous Banach space. The maximal homogeneous Banach space in it is the space $C(\mathbb{T})$.

Recall that a trigonometric polynomial on $\mathbb{T}$ is a function $a = a(t)$ defined on $\mathbb{T}$ by $a(t) = \sum\limits_{-n}^{n} a_j e^{ijt}$. Denote by $p(\mathbb{T})$ the set of all trigonometric polynomials on $\mathbb{T}$.

Remark. If $B(\mathbb{T})$ is a homogeneous Banach space over $\mathbb{T}$, then for every $f \in B(\mathbb{T})$ and $n \in \mathbb{Z}$ the function $x \to \hat{f}(n)e^{inx}$ belongs to $B(\mathbb{T})$ [5K, p. 6, Lemma 1.9 and p. 17, Ex. 13]. It follows that if the Fourier-Lebesgue coefficient $\hat{f}(n)$ of f is not zero, then the exponential $x \to e^{inx}$ belongs to the space $B(\mathbb{T})$ for this $n \in \mathbb{Z}$. So we may say that the exponential $x \to e^{inx}$ belongs to $B(\mathbb{T})$ if there exists a function $f \in B(\mathbb{T})$ such that $\hat{f}(n)$ is not zero. With regard to this fact we will suppose tacitly for the sake of simplicity that $B(\mathbb{T})$ is a homogeneous Banach space containing all exponentials $x \to e^{inx}$, $n \in \mathbb{Z}$.

Now we may claim that trigonometric polynomials are dense in a homogeneous Banach space $B(\mathbb{T})$, namely we shall need the following result [5K, Th. 2.12].

Theorem. *Let $B(\mathbb{T})$ be a homogeneous Banach space. For every $f \in B(\mathbb{T})$ we have $\sigma_n(f) \to f$ for $n \to \infty$, in the $B(\mathbb{T})$-norm.*

Recall that the Fejér sums of the Fourier series of a function f are the trigonometric polynomials

$$\sigma_k(f,t) = \sigma_k(t) = \sum_{-k}^{k} \left(1 - \frac{|n|}{k+1}\right) \hat{f}(n) e^{int}, \qquad k = 1, 2, \dots$$

Similarly as for the spaces $C(\mathbb{T})$ and $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, in [1B] in solving Fourier problem of the ring for the homogeneous Banach space X the solution $w(s; f_0)$ is given by a semigroup of operators $W(s)$, $0 \leqslant s < \infty$, acting on the initial value $f_0 \in X$. In terms of the finite Fourier transform the semigroup operators themselves are characterized by the sequence of real numbers

$$w_k(s) = e^{-|k|^2 s}, \qquad k = 0, \pm 1, \pm 2, \dots,\ 0 \leqslant s < \infty,$$

and thus corresponding semigroup $W(s)$, $0 \leqslant s < \infty$, is represented by a semigroup of factor sequence of type (X, X).

2. Let $B(\mathbb{T})$ be a homogeneous Banach space over $\mathbb{T}$ and let $T(s)$, $0 \leqslant s < \infty$, be a family of factor sequence operators of type $\big(B(\mathbb{T}), B(\mathbb{T})\big)$ of the form

$$\big[T(s)f\big](x) \sim \sum_{k=-\infty}^{\infty} e^{\lambda_k t} \hat{f}(k) e^{ikx}, \tag{1}$$

where $\hat{f}(k)$ are Fourier coefficients of $f \in B(\mathbb{T})$ and λ_k, $k = 0, \pm 1, \pm 2, \dots$, are complex numbers.

The following theorem is an extension for the homogeneous Banach spaces of the result in [1B, Th. 1.5.4] and the proof is an adaptation of the proof given there.

THEOREM 1. *Let $B(\mathbb{T})$ be a homogeneous Banach space. A necessary and sufficient condition such that the family of operators* (1) *forms a strongly continuous semigroup in $L\big(B(\mathbb{T})\big)$ is that there exists a constant $M \geqslant 1$ such that $\|T\| \leqslant M$ for $0 \leqslant t \leqslant 1$. Furthermore, the infinitesimal generator A of $T(t)_{0 \leqslant t}$ is defined by*

$$[Af]\hat{}\,(k) = \lambda_k \hat{f}(k), \qquad k = 0, \pm 1, \pm 2, \dots,\ f \in D(A) \tag{2}$$

and

$$D(A) = \left\{ f \in B(\mathbb{T}) \colon g \sim \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k) e^{ikx},\ g \in B(\mathbb{T}) \right\}. \tag{3}$$

P r o o f. It is easily seen that the family of operators $T(t)$, $t \geqslant 0$, forms a semigroup in $L\big(B(\mathbb{T})\big)$. If $T(t)$, $t \geqslant 0$, is a strongly continuous semigroup in $L\big(B(\mathbb{T})\big)$, then by [1B, Prop. 1.1.2] there exists a constant $M \geqslant 1$ such that $\|T(t)\| \leqslant M$ for $0 \leqslant t \leqslant 1$. Conversely, setting $e_k(x) = e^{ikx}$ we have that $s\text{-}\lim_{t \downarrow 0} T(t) e_k = s\text{-}\lim_{t \downarrow 0} e^{\lambda_k t} e_k = e_k$ for all integers. Thus the semigroup satisfies the condition of strong continuity for every trigonometric polynomial in $B(\mathbb{T})$. Since these are dense in $B(\mathbb{T})$ by Theorem and the norms $\|T(t)\|$ are uniformly bounded by M in $0 \leqslant t \leqslant 1$ we obtain by the Banach–Steinhaus theorem that $s\text{-}\lim_{t \downarrow 0} T(t) f = f$ for all $f \in B(\mathbb{T})$, proving the first part of this theorem. Now suppose that $f \in D(A)$. Then by definition of Fourier coefficients

$$\begin{aligned} [Af]\hat{}(k) &= \lim_{\tau \downarrow 0} \frac{1}{2\pi} \int \tau^{-1}\big\{\big[T(\tau)f\big](x) - f(x)\big\} e^{-ikx}\,dx \\ &= \lim_{\tau \downarrow 0} \tau^{-1}\big\{e^{\lambda_k \tau} - 1\big\}\hat{f}(k) = \lambda_k \hat{f}(k)\,, \qquad (k = 0, \pm 1, \pm 2, \dots)\,, \end{aligned}$$

proving relation (2) and

$$D(A) \subset \left\{ f \in B(\mathbb{T}) \colon g \sim \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k) e^{ikx},\ g \in B(\mathbb{T}) \right\}.$$

On the other hand, if f belongs to the set on the right-hand of the latter relation, we obtain for the nth Fejér sum of the element $g(x) \sim \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k) e^{ikx}$ that $A\sigma_n(f) = \sigma_n(g)$, $n = 1, 2, \dots$ Since A is a closed operator and $\sigma_n(f)$ and $A\sigma_n(f)$ converge in $B(\mathbb{T})$-norm to f and g, respectively, as $n \to \infty$, this proves that $f \in D(A)$ and $Af = g$, which completes the proof. □

Remark. The operators defined by (1) commute with the semigroup of translations. This follows directly by definition.

The following example is an extension of example in [1B, Ex.1.5.5].

Example. Let $B(\mathbb{T})$ be a homogeneous Banach space. The sequence of constants $\lambda_k = ik$, $k = 0, \pm 1, \pm 2, \dots$, generates the semigroup of left translations

$$\big[T(t)f\big](x) = f(x+t) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik(x+t)}$$

and

$$\begin{aligned} D(A) &= \big\{ f \in B(\mathbb{T}) \colon f \in AC(\mathbb{T}) \text{ and } f' \in B(\mathbb{T}) \big\} \\ &= \left\{ f \in B(\mathbb{T}) \colon g \sim \sum_{k=-\infty}^{\infty} (ik)\hat{f}(k) e^{ikx},\ g \in B(\mathbb{T}) \right\}, \end{aligned}$$

where

$$[Af]\hat{}(k) = (ik)\hat{f}(k)\,, \qquad k = 0, \pm 1, \pm 2, \dots$$

In the following theorem we discuss the dual of a semigroup of factor sequence operators of type (1), cf. [1B,Th. 1.5.7.]

Theorem 2. *Let $B(\mathbb{T})$ be a homogeneous Banach space. The dual semigroup $T^*(t)$, $0 \leqslant t < \infty$, on $B^*(\mathbb{T})$ is again a family of factor sequence operators of type $\big(B(\mathbb{T})^*(\mathbb{T}), B^*(\mathbb{T})\big)$, and for $f^* \in B^*(\mathbb{T})$*

$$\big[T^*(t)f^*\big] \sim \sum_{k=-\infty}^{\infty} e^{\overline{\lambda}_k t} [f^*]\hat{}(k) e^{ikx}, \qquad (t > 0)\,. \tag{4}$$

If $T(t)$, $0 \leqslant t < \infty$, is strongly continuous in $L\big(B(\mathbb{T})\big)$, then $T^*(t)$, $0 \leqslant t < \infty$, is weakly star continuous in $L\big(B^*(\mathbb{T})\big)$

$$D(A^*) = \left\{ f^* \in B^*(\mathbb{T})\colon g^* \sim \sum_{k=-\infty}^{\infty} \overline{\lambda}_k [f^*]\hat{\ }(k) e^{ikx},\ g^* \in B^*(\mathbb{T}) \right\} \tag{5}$$

and

$$[A^* f^*]\hat{\ }(k) = \overline{\lambda}_k [f^*]\hat{\ }(k), \qquad k = 0, \pm 1, \pm 2, \dots \tag{6}$$

Proof. By definition, $\big(T^*(t) f^*, f\big) = \big(f^*, T(t) f\big)$ for all $f \in B(\mathbb{T})$ and all $f^* \in B^*(\mathbb{T})$. Thus for $f = e_k$, $\big(e_k(x) = e^{ikx}\big)$, we have

$$\begin{aligned} \overline{[T^*(t) f^*]\hat{\ }}(k) &= \big(T^*(t) f^*, e_k\big) = \big(f^*, T(t) e_k\big) \\ &= (f^*, e^{\lambda_k t} e_k) = e^{\lambda_k t} (f^*, e_k) = e^{\lambda_k t} \overline{[f^*]\hat{\ }}(k), \end{aligned}$$

i.e., for all integers k we have

$$\big[T^*(t) f^*\big]\hat{\ }(k) = e^{\overline{\lambda}_k t} [f^*]\hat{\ }(k),$$

proving (4).

By the same method one obtains for each $f^* \in D(A^*)$ that the relation (6) is valid for the Fourier coefficients of $A^* f^*$. Thus $D(A^*)$ is a subset of the set of elements

$$\left\{ f^* \in B^*(\mathbb{T})\colon g^* \sim \sum_{k=-\infty}^{\infty} \overline{\lambda}_k [f^*]\hat{\ }(k) e^{ikx},\ g^* \in B^*(\mathbb{T}) \right\}.$$

On the other hand, if f^* belongs to the latter subset then we have for the nth Fejér mean of $g^* \sim \sum_{k=-\infty}^{\infty} \overline{\lambda}_k [f^*]\hat{\ }(k) e^{ikx}$ that $A^* \sigma_n(f^*) = \sigma_n(g^*)$, and since $\sigma_n(f^*) \to f^*$ and $A^* \sigma_n(f^*) \to g^*$ in the weak star topology as $n \to \infty$, [5K] and since A^* is weakly star closed, we have that $f^* \in D(A^*)$ with $A^* f^* = g^*$. □

Example 2. For the semigroup of left translations on the homogeneous Banach space $B(\mathbb{T})$ the dual semigroup $T^*(t)$, $0 \leqslant t < \infty$, on $B^*(\mathbb{T})$ will be the semigroup of right translations, i.e., for $f^* \in B^*(\mathbb{T})$ we have

$$T^*(t) f^* \sim \sum_{k=-\infty}^{\infty} [f^*]\hat{\ }(k) e^{ik(x-t)}, \qquad t > 0,$$

$$D(A^*) = \left\{ f^* \in B^*(\mathbb{T})\colon g^* \sim \sum_{k=-\infty}^{\infty} (-ik) [f^*]\hat{\ }(k) e^{ikx},\ g^* \in B^*(\mathbb{T}) \right\}$$

and

$$[A^* f^*]\hat{\ }(k) = (-ik) [f^*]\hat{\ }(k), \qquad k = 0, \pm 1, \pm 2, \dots$$

The following fact is required in the sequel.

Lemma. *Let $B(\mathbb{T})$ be a homogeneous Banach space over $\mathbb{T}$ such that the elements of B^* are (Borel) measures. Then the following characterizations of an element $\mu \in B^*$ are equivalent:*

(i) *there exists a $\nu \in B^*$ such that* $\sum\limits_{k=-\infty}^{\infty} (ik)\check{\mu}(k)e^{ikx} = \sum\limits_{k=-\infty}^{\infty} \check{\nu}(k)e^{ikx}$,

(ii) $\mu \in AC(T)$ *and* $\mu' \in B^*$.

P r o o f. We shall prove this lemma for the case that $B^* = M(\mathbb{T})$ is the Banach space of complex (Borel) measures over $\mathbb{T}$. Let for $\mu \in B^*$ there exist a $\nu \in B^*$ such that

$$d\nu(x) \sim \sum_{k=-\infty}^{\infty} (ik)\check{\mu}(k)e^{ikx} = \sum_{k=-\infty}^{\infty} \check{\nu}(k)e^{ikx},$$

i.e., the Fourier–Stieltjes coefficients of ν are $\check{\nu}(0) = 0$, $\check{\nu}(k) = (ik)\check{\mu}(k)$, $k \neq 0$, hence $\sum\limits_{k=-\infty}^{\infty} \check{\mu}(k)e^{ikx}$ is the Fourier-Stieltjes series of some function F of bounded variation over $\mathbb{T}$. For every $f \in C(\mathbb{T})$ we have

$$\int f(t)\,\sigma_n(d\mu)(t) \to \int f(t)\,d\mu(t)\,, \qquad n \to \infty\,,$$

$$\int f(t)\,\sigma_n(d\mu)(t) = \int f(t)\,\sigma_n(F,t)\,dt \to \int f(t)F(t)\,dt\,, \qquad n \to \infty\,,$$

since $\sigma_n(d\mu) \to \mu$ in the weak-star topology and $\sigma_n(F,t) \to F(t)$ almost everywhere and $\sigma_n(F,t) \leqslant K$ (K being some constant). Hence $d\mu(t) = F(t)dt$, that is μ is absolutely continuous with respect to Lebesgue measure, hence $\mu' = F$ a.e. and $d\mu' = dF \in B^*$ and $dF = d\mu' = d\nu$.

Conversely if for $\mu \in B^*$ we have $\mu \in AC(\mathbb{T})$ and $d\mu' = dF \in B^*$, that is there exists F of bounded variation such that $\mu(E) = \int\limits_E F(t)dt$, let $d\nu = dF$, then $\nu \in B^*$ and

$$\begin{aligned}\check{\mu}(k) &= \int e^{-ikx}\,d\mu(x) = \int e^{-ikx}F(x)\,dx\\ &= \frac{1}{ik}\int e^{-ikx}\,dF(x) = \frac{1}{ik}\int e^{-ikx}\,d\nu(x) = \frac{1}{ik}\,\check{\nu}(k)\,, \qquad k \neq 0\,.\end{aligned}$$

That is

$$\sum_{k=-\infty}^{\infty} (ik)\check{\mu}(k)e^{ikx}$$

is the Fourier-Stieltjes series of a $\nu \in B^*$. □

In Example 2 we studied the semi-group of left translations on the homogeneous Banach space B. By Theorem 2 the dual semi-group $\{T^*(t)\colon 0 \leqslant t < \infty\}$, on B^* will be the semi-group of right translations, i.e., for $\mu \in B^*$

$$d\big[T^*(t)\mu\big](x) = d\mu(x-t) \sim \sum_{k=-\infty}^{\infty} \check{\mu}(k) e^{ik(x-t)} .$$

Now if B is such a homogeneous Banach space that B^* is a Banach space of Borel measures then by Lemma

$$\begin{aligned} D(A^*) &= \left\{\mu \in B^*\colon \sim \sum_{k=-\infty}^{\infty} (-ik)\check{\mu}(k) e^{ikx} \in B^*\right\} \\ &= \big\{\mu \in B^* : \mu \in AC(\mathbb{T}),\, \mu' \in B^*\big\} \end{aligned}$$

and $\check{[A^*\mu]}(k) = (-ik)\check{\mu}(k)$ for all $k = 0, \pm 1, \pm 2, \dots$

By Proposition the semi-group $\{T^*(t)\colon 0 \leqslant t < \infty\}$ is continuous in the weak-star operator topology on B^*. It is strongly continuous on the closed subspace $[B^*]_0$ equal to the strong closure of $D(A^*)$. For $B = C(\mathbb{T})$ this subspace $\big(C(\mathbb{T})\big)_0^*$ is isometrically isomorphic to $AC(\mathbb{T})$ or equivalently to $L^1(\mathbb{T})$. It is thus shown that the set of all functions $\mu \in M(\mathbb{T})$ with continuous right translations is precisely the set of all absolutely continuous functions, i.e., the subset $AC(\mathbb{T})$ of $M(\mathbb{T})$.

REFERENCES

[1B] BUTZER, P. L.—BERENS, H.: *Semi-Groups of Operators and Approximations*, Grundlehrer Math. Wiss., Springer-Verlag, New York, 1967.

[2E] EDWARDS, R. E.: *Fourier Series I*, 2nd ed., Springer-Verlag, Berlin, 1979.

[3E] EDWARDS, R. E.: *Fourier Series II.*, 2nd ed., Springer-Verlag, Berlin, 1982.

[4K] KAHANE, J. P.: *Some Random Series of Functions*, DC Heath and Company, Lexington, Mass., 1968.

[5K] KATZNELSON, Y.: *An Introduction to Harmonic Analysis*, 2nd ed., Dower Publications, Inc., New York, 1976.

[6S] SHILOV, G. E.: *Homogeneous rings of functions*, Uspekhi Math. Nauk **41** (1951), 91–137.

[7S] SHILOV, G. E.: *Homogeneous rings of functions*, Amer. Math. Soc. **920** (1953), 392–453.

[8S] SHILOV, G. E.: *Mathematical Analysis*, Moscow, 1961.

Received July 1, 2005

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA
E-mail: duchon@mat.savba.sk