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THE EXTENSION PROBLEM FOR THE CARATHÉODORY TYPE FUNCTIONS

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ABSTRACT. In this paper we present the results of extension of the Carathéodory type function defined on a measurable subset which is not a rectangle.

1. Preliminaries

Let us start with the notation. Let $(T, \mathcal{M})$ be a measurable space and X, Y be topological spaces. We say that X is a Polish space if it is separable metric complete. We consider the Borel σ-algebra $\mathcal{B}(X)$ on X and the product σ-algebra $\mathcal{M} \otimes \mathcal{B}(X)$ on $T \times X$. If $D \subset T \times X$, then by D_t and D^x we denote vertical and horizontal sections of D, i.e., $D_t = \{x \in X : (t, x) \in D\}$ for all $t \in T$ and $D^x = \{t \in T : (t, x) \in D\}$ for all $x \in X$, respectively. The set D will be considered with the trace σ-algebra $D \cap \mathcal{M} \otimes \mathcal{B}(X) = \{D \cap A : A \in \mathcal{M} \otimes \mathcal{B}(X)\}$. Let p_T denote the projection from $T \times X$ onto T.

Let 2^Y denote the family of all nonempty subsets of Y. By a multifunction from X to Y, we mean any mapping from X that has the family 2^Y as a value space. We say that the multifunction $\Phi : T \to 2^Y$ is measurable if, for each open set $A \subset Y$, the set $\{t \in T : \Phi(t) \cap A \neq \emptyset\}$ is measurable. By the graph of multifunction, we mean the subset $\mathrm{Gr}(\Phi) = \{(t, y) \in T \times Y : y \in \Phi(t)\}$ (see also [2]). The function $\gamma : T \to Y$ is called a selection of Φ if $\gamma(t) \in \Phi(t)$ for every $t \in T$. We say that Φ has a Castaing representation if there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ of measurable selections of Φ such that $\Phi(t) = \mathrm{cl}_{\Phi(t)}\{\phi_n(t) : n \in \mathbb{N}\}$ for every $t \in T$ (see also [1], Theorem III.7).

Let $\{X, d_1\}$ and $\{Y, d_2\}$ be metric spaces. Let $K(x, r)$ denote a closed ball with the center x and radius r. We say that a pair X, Y fulfills the Kirszbraun intersection property if for all choices $x_i \in X$, $y_i \in Y$ and $r_i > 0$, $i \in J$, where J is an arbitrary index set such that the intersection of the closed balls

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$K(x_i, r_i)$ in X is nonempty and $d_2(y_i, y_j) \leqslant d_1(x_i, x_i)$ for all $i, j \in J$, then it follows that the intersection of the balls $K(y_i, r_i)$ in Y is also nonempty. Note that the Kirszbraun intersection property is satisfied in the case of Hilbert spaces (see [7, Theorem 3]).

We say that a function or multifunction $F\colon T \times X \to Y$ is Carathéodory if it is measurable in t and continuous in x. Instead of continuity of the function $F(t, \cdot)$, we consider the Lipschitz condition with the constant $L(t)$ for every t.

At this end of the section, we formulate some theorems that are commonly used in our calculations. The first one regards the product measurability of Carathéodory functions.

Theorem 1.1. (See [3, vol. I, Proposition 1.6, p. 142].) *Let X be a separable metric space and Y be a metric space. Let $F\colon T \times X \to Y$ be a Carathéodory function. Then F is measurable with respect to the product σ-algebra $\mathcal{M} \otimes \mathcal{B}(X)$.*

The second theorem considers the graph measurability.

Theorem 1.2. (See [3, vol. I, Proposition 1.7, p. 142].) *Let Y be a separable metric space and $\Phi\colon T \to 2^Y$ be a measurable multifunction with closed values. Then $\mathrm{Gr}(\Phi) \in \mathcal{M} \otimes \mathcal{B}(Y)$.*

The third theorem is a graph-conditioned measurable selection theorem.

Theorem 1.3. *Let $(T, \mathcal{M})$ be a complete measurable space, Y be a Polish space, and $\Phi\colon T \to 2^Y$ be a multifunction with a measurable graph. Then Φ has a Castaing representation.*

Theorems 1.2 and 1.3 are the particular cases of a more general theorem in [3] (see vol. I, Theorem 2.4, p. 156).

2. Main results

In this part of the paper, we present two results of extension of the Carathéodory type function $f\colon D \to Y$, defined on the measurable subset $D \subset T \times X$ that is not a rectangle. At the very beginning, it is noteworthy that from the product measurability of the Carathéodory type function it follows a necessary condition of the extension (see Theorem 1.1). Namely, under the assumption of the separability of X, the necessary condition of the extension is $D \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurability of F. This is the reason why we do not use the measurability of the function $F(\cdot, x)$ for every x in the presented theorems.

Now, we give two lemmas that are commonly used in proofs of the main theorems.

Lemma 2.1. *Let X be a separable metric space. Let us consider the subset $D \subset T \times X$ such that $D \in \mathcal{M} \otimes \mathcal{B}(X)$ and $p_T(D) \in \mathcal{M}$. Let $F\colon D \to Y$ be a function, which is $D \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurable. Let f be a measurable selector of the multifunction $t \to D_t$. Then a map $G\colon p_T(D) \to Y$ such that $G(t) = F\big(t, f(t)\big)$ is a measurable function.*

P r o o f. The function $G(t) = F\big(t, f(t)\big)$ is a superposition of F and $g(t) = \big(t, f(t)\big)$. Measurability of G follows from the measurability of component functions. $\square$

Lemma 2.2. *Let $(T, \mathcal{M}, \mu)$ be a measurable space and (X, d) be a separable metric space. Let $f_i\colon T \to X$ and $\rho_i\colon T \to [0, +\infty)$, where $i = 1, 2, \dots$, be measurable maps. Let us assume for every $t \in T$*

$$\Gamma(t) = \bigcap_{i \geqslant 1} K\big(f_i(t), \rho_i(t)\big) \neq \emptyset\,.$$

Then $\Gamma\colon T \to 2^X$ is a map with a measurable graph.

P r o o f. Let us consider a map: $\psi_i\colon t \to K\big(f_i(t), \rho_i(t)\big)$, for every i. The graph of ψ_i has the following description:

$$\mathrm{Gr}(\psi_i) = \big\{(t, x) \in T \times X : d\big(f_i(t), x\big) - \rho_i(t) \leqslant 0\big\}$$

and it is a preimage of the set $(-\infty, 0]$ with respect to measurable function. In order to finish the proof, it suffices to notice

$$\mathrm{Gr}\,\Gamma = \bigcap_{i \geqslant 1} \mathrm{Gr}\,\psi_i\,.$$

$\square$

The theorem given below is the first result of extension of the Carathéodory type function.

Theorem 2.3. *Let $(T, \mathcal{M})$ be a complete measurable space with respect to some σ-finite measure. Let X, Y be Banach spaces such that the pair $\{X, Y\}$ satisfies the Kirszbraun intersection property, and let X be a separable space. Let us consider the set $D \in \mathcal{M} \otimes \mathcal{B}(X)$ such that $pr_T(D) = T$. Let $F\colon D \to Y$ be $D \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurable map. Additionally let us assume that $F(t, \cdot)\colon D_t \to Y$ fulfills the Lipschitz condition with the constant $L(t)$ for every $t \in T$, where $L\colon T \to \mathbb{R}^+ \setminus \{0\}$ is a measurable map.*

Then there exists a map $\hat{F}\colon T \times X \to Y$ such that $\hat{F}|_D = F$ and

$\forall x \in X$ a map $\hat{F}(\cdot, x)$ is measurable,

$\forall t \in T$ a map $\hat{F}(t, \cdot)$ fulfills the Lipschitz condition with the constant $L(t)$.

P r o o f . Let x_0 be an arbitrary point of X such that $D^{x_0} \neq T$. At the beginning, we will present an extension $F_{x_0}\colon D \cup (T \times \{x_0\}) \to Y$ of F. A multifunction $\Phi_D\colon T \to X$ such that $\Phi_D(t) = D_t$ has a Castaing representation $\{f_n\}$, which follows from an assumption $D \in \mathcal{M} \otimes \mathcal{B}(X)$.

Let us define $\Gamma_{x_0}(t) = \bigcap_n K\Big(F\big(t, f_n(t)\big),\, L(t)\big\|f_n(t) - x_0\big\|\Big)$, for $(t, x_0) \notin D$, and $\Gamma_{x_0}(t) = \big\{F(t, x_0)\big\}$, for $(t, x_0) \in D$. Let us consider the sequences of closed balls with the centers in $\big\{L(t) \cdot f_n(t)\big\}_{n=1}^{+\infty} \subset X$, $\big\{F\big(t, f_n(t)\big)\big\}_{n=1}^{+\infty} \subset Y$ and radiuses $\big\{L(t)\big\|f_n(t) - x_0\big\|\big\}_{n=1}^{+\infty} \subset \mathbb{R}^+$. From the Lipschitz condition for $F(t, \cdot)$, it follows that the distances between the centers of the balls satisfy the following inequality:

$$\big\|F\big(t, f_n(t)\big) - F\big(t, f_k(t)\big)\big\| \leqslant \big\|L(t) \cdot f_n(t) - L(t) \cdot f_k(t)\big\| .$$

By means of the Kirszbraun intersection property for the pair X, Y, we obtain $\Gamma_{x_0}(t) \neq \emptyset$ for every t such that $(t, x_0) \notin D$. Additionally, the multifunction $\Gamma_{x_0}\colon T \to 2^Y$ for $t\colon (t, x_0) \notin D$ has a measurable graph, which follows from Lemma 2.2. Resuming, the multifunction $\Gamma_{x_0}\colon T \to 2^Y$ has closed, nonempty values and a measurable graph. There exists a measurable selection $\gamma_{x_0}\colon T \to Y$. Now, we can define the extension F_{x_0}, namely $F_{x_0}(t, x_0) = \gamma_{x_0}(t)$, for $(t, x_0) \notin D$ and $F_{x_0}(t, x) = F(t, x)$, for $(t, x) \in D$. It is clear that F_{x_0} fulfills the following conditions:

$F_{x_0}\big|_D = F$,

F_{x_0} is $\big(D \cup \big(T \times \{x_0\}\big)\big) \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurable.

Now, we will prove that $\forall t \in T$ a map $F_{x_0}(t, \cdot)$ fulfills the Lipschitz condition with the constant $L(t)$. It is sufficient only to check for the points such that $(t, x) \in D$ and $(t, x_0) \notin D$. In this case we have

$$\big\|F_{x_0}(t, x) - F_{x_0}(t, x_0)\big\| = \big\|F(t, x) - \gamma_{x_0}(t)\big\| .$$

For every fixed n, the following estimation is true:

$$\big\|F(t, x) - \gamma_{x_0}(t)\big\| \leqslant \big\|F(t, x) - F\big(t, f_n(t)\big)\big\| + \big\|F\big(t, f_n(t)\big) - \gamma_{x_0}(t)\big\| .$$

Let us notice that γ_{x_0} is a selection of Γ_{x_0}, then

$$\gamma_{x_0}(t) \in \bigcap_n K\Big(F\big(t, f_n(t)\big),\, L(t)\big\|f_n(t) - x_0\big\|\Big),$$

which means

$$\big\|F\big(t, f_n(t)\big) - \gamma_{x_0}(t)\big\| \leqslant L(t)\big\|f_n(t) - x_0\big\| .$$

The condition $x \in D_t = \operatorname{cl}_{D_t}\big\{f_n(t)\colon n \in \mathbb{N}\big\}$ implies that, for arbitrary $\varepsilon > 0$, there exists k such that $\big\|f_k(t) - x\big\| \leqslant \varepsilon$. We obtain

$$\big\|F_{x_0}(t, x) - F_{x_0}(t, x_0)\big\| \leqslant L(t)\varepsilon + L(t)\big\|f_k(t) - x_0\big\| \leqslant 2L(t)\varepsilon + L(t)\big\|x - x_0\big\| .$$

Since ε can be arbitrarily close to 0, the function $F_{x_0}(t,\cdot)$ fulfills the Lipschitz condition with the constant $L(t)$.

Let $\{x_k\}_{k=1}^{+\infty}$ be a countable and dense subset of X, and let $\{x_i\}_{i=1}^{+\infty}$ be a subsequence of $\{x_k\}_{k=1}^{+\infty}$ such that $D^{x_i} \neq T$ for every i. In order to extend the function F over the whole space $T \times X$, we define the sequences of subsets $\{D_i\}_{i=1}^{+\infty}$ and maps $\{F_i\}_{i=1}^{+\infty}$, using the subsequence $\{x_i\}_{i=1}^{+\infty}$. Let $D_1 = D \cup \big(T \times \{x_1\}\big)$, and for every $i = 2, 3, \ldots$ we define $D_i = D_{i-1} \cup \big(T \times \{x_i\}\big)$. Next we extend the functions over the appropriate sets. Let F_1 be defined over the set D_1 as an extension of F defined over D, which is the case of $i = 1$. We have $F_i \colon D_i \to Y$ as an extension of F_{i-1} from the set D_{i-1}, for any $i = 2, 3, \ldots$ In this way, we obtain the sequence of sets $\{D_i\}_{i=1}^{+\infty}$ and maps $\{F_i \colon D_i \to Y\}_{i=1}^{+\infty}$ such that for every $i = 1, 2, \ldots$:

F_i is defined over the set $D_{i-1} \subset D_i$ and $F_i\big|_{D_{i-1}} = F_{i-1}$,

F_i is $D_i \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurable,

$\forall t \in T$ the map $F_i(t,\cdot)$ fulfills the Lipschitz condition with the constant $L(t)$.

For every point $(t,x) \in (T \times X) \setminus D$ there exists a sequence $\{x_{n_i}\}_{i=1}^{+\infty} \subset \{x_i\}_{i=1}^{+\infty}$ such that $\lim_i x_{n_i} = x$. Next, let us consider the sequence

$$\big\{F_{n_i}(t, x_{n_i})\big\}_{i=1}^{+\infty}.$$

Let us look at two points x_{n_i} and x_{n_j} such that $n_i > n_j$. Then $F_{n_j}(t, x_{n_j}) = F_{n_i}(t, x_{n_j})$, and thanks to this we can use the Lipschitz condition for a map F_{n_i} with the constant $L(t)$:

$$\big\|F_{n_i}(t, x_{n_i}) - F_{n_i}(t, x_{n_j})\big\| \leqslant L(t)\big\|x_{n_i} - x_{n_j}\big\|.$$

We obtain that $\big\{F_{n_i}(t, x_{n_i})\big\}_{i=1}^{+\infty}$ is a Cauchy sequence by applying the convergence of $\{x_{n_i}\}_{i=1}^{+\infty}$. Now it suffices to prove that the limit is unique

$$\lim_{x_{n_i} \to x} F_{n_i}(t, x_{n_i}).$$

Let us assume that $\{x_{n_i}\}_{i=1}^{+\infty}$ and $\{x_{m_j}\}_{j=1}^{+\infty}$ are two sequences that converge to x. Let them be combined by choosing elements alternately one from the first sequence and another from the second sequence, to obtain the resulting sequence $\{x_{l_k}\}_{k=1}^{+\infty}$, which also converges to x. $\big\{F_{n_i}(t, x_{n_i})\big\}_{i=1}^{+\infty}$ and $\{F_{m_j}(t, x_{m_j})\}_{j=1}^{+\infty}$ are subsequences of $\big\{F_{l_k}(t, x_{l_k})\big\}_{k=1}^{+\infty}$ and every one of them is a Cauchy sequence. It proves that the limit of the sequence $\big\{F_{n_i}(t, x_{n_i})\big\}_{i=1}^{+\infty}$ is unique.

Now, we can define $\hat{F}(t,x)$, namely

$$\hat{F}(t,x) = \lim_{x_{n_i} \to x} F_{n_i}(t, x_{n_i}).$$

Let $x_1, x_2 \in X$ be two arbitrary points and x_{n_i} be elements of appropriate sequences, for $i = 1, 2$ such that $\|x_{n_i} - x_i\| \leqslant \varepsilon$ and $\|\hat{F}(t, x_i) - F_{n_i}(t, x_{n_i})\| \leqslant \varepsilon$. Then we have

$$\begin{aligned} \|\hat{F}(t, x_1) - \hat{F}(t, x_2)\| &\leqslant 2\varepsilon + \|F_{n_1}(t, x_{n_1}) - F_{n_2}(t, x_{n_2})\| \\ &\leqslant 2\varepsilon + L(t)\|x_{n_1} - x_{n_2}\| \leqslant 2(1 + L(t))\varepsilon + L(t)\|x_1 - x_2\|. \end{aligned}$$

The map $\hat{F}(t, \cdot)$ fulfills the Lipschitz condition for every t with the constant $L(t)$. The sequence $\{x_{n_i}\}_{i=1}^{+\infty}$ that converges to the given x does not depend on t, and thanks to this $\hat{F}(\cdot, x)$ is a limit of the sequence of measurable functions $\{F_{n_i}(\cdot, x_{n_i})\}_{i=1}^{+\infty}$. We have that $\hat{F}(\cdot, x)$ is a measurable function, which completes the proof. ☐

Remark 2.4. We can consider the Hölder condition with the constants $L(t)$ and α:

$$\|\hat{F}(t, x_1) - \hat{F}(t, x_2)\| \leqslant L(t)\|x_1 - x_2\|^{\alpha},$$

instead of the Lipshitz one for the function $F(t, \cdot)$ in Theorem 2.3. The proof is analogous to that presented above. Let us mention that in the case of the Hölder condition we use the Kirszbraun intersection property for balls with centers in $\{L(t)^{\frac{1}{\alpha}} \cdot f_n(t)\}_{n=1}^{+\infty} \subset X$, $\{F(t, f_n(t))\}_{n=1}^{+\infty} \subset Y$ and radiuses $\{L(t)\|f_n(t) - x_0\|^{\alpha}\}_{n=1}^{+\infty} \subset \mathbb{R}^+$.

Remark 2.5. We can find the result of extension of the Caratheodory type functions from the subset of $T \times X$, that is a rectangular in the paper by J. Myjak and W. Zygadlewicz [5]. The theorem presented here is a generalization to the arbitrary measurable sets.

Remark 2.6. There exists an analogous theorem in the paper by N. Shahzad [6], however, the proof is uncompleted.

Now, we give a theorem similar to Theorem 2.3, but obtained for the case when X is a Polish space and $Y = \mathbb{R}$. Theorem 2.3 implies the next result for the case when the pair $X, \mathbb{R}$ has the Kirszbraun intersection property and X and $\mathbb{R}$ are the Banach spaces. Nevertheless, there is another method of proving this.

Theorem 2.7. *Let $(T, \mathcal{M})$ be a measurable space, (X, d) be a Polish space, and $Y = \mathbb{R}$. Let us consider the set $D \in \mathcal{M} \otimes \mathcal{B}(X)$ such that $pr_T(D) = T$. Let $F\colon D \to Y$ be $D \cap \mathcal{M} \otimes \mathcal{B}(X)$-measurable map. Additionally, let us assume that $F(t, \cdot)\colon D_t \to Y$ fulfills the Lipschitz condition with the constant $L(t)$ for every $t \in T$, where $L\colon T \to \mathbb{R}^+ \setminus \{0\}$ is a measurable map.*

Then there exists a map $\hat{F}\colon T \times X \to Y$ such that $\hat{F}|_D = F$ and

$\forall x \in X$ a map $\hat{F}(\cdot, x)$ is measurable,
$\forall t \in T$ a map $\hat{F}(t, \cdot)$ fulfills the Lipschitz condition with the constant $L(t)$.

P r o o f. The multifunction $\Phi_D \colon T \to X$ defined as $\Phi_D(t) = D_t$ has the Castaing representation $\{f_n\}$, which follows from the assumption $D \in \mathcal{M} \otimes \mathcal{B}(X)$. Let us define $\hat{F} \colon T \times X \to \mathbb{R}$ in the following way:

$$\hat{F}(t,x) = \sup_n \bigl\{ F\bigl(t, f_n(t)\bigr) - L(t)\, d(f_n(t), x) \bigr\} .$$

Lemma 2.1 implies that $F\bigl(\cdot, f_n(\cdot)\bigr)$ is measurable for every n. The measurability of $\hat{F}(\cdot, x)$, for every x, follows from the definition.

Let us consider $(t,x) \in D$. According to the Lipschitz condition, we have

$$F\bigl(t, f_n(t)\bigr) - L(t)\, d\bigl(f_n(t), x\bigr) \leqslant F(t,x) ,$$

for every n. Additionally, if we chose a subsequence $\bigl\{f_{n_i}(t)\bigr\}_{i=1}^{+\infty}$ that converges to x, for a fixed t such that $(t,x) \in D$, then $\lim_i \bigl\{ F\bigl(t, f_{n_i}(t)\bigr) - L(t)\, d\bigl(f_{n_i}(t), x\bigr) \bigr\} = F(t,x)$. We get that $\hat{F}(t,x) = F(t,x)$ is finite for every $(t,x) \in D$. Let (t,x) be an arbitrary point such that $(t,x) \notin D$ and $k \in \mathbb{N}$, then

$$\begin{aligned} \hat{F}(t,x) - \hat{F}\bigl(t, f_k(t)\bigr) &= \sup_n \bigl\{ F\bigl(t, f_n(t)\bigr) - F\bigl(t, f_k(t)\bigr) - L(t)\, d\bigl(f_n(t), x\bigr) \bigr\} \\ &\leqslant L(t) \sup_n \bigl\{ d\bigl(f_n(t), f_k(t)\bigr) - d\bigl(f_n(t), x\bigr) \bigr\} \leqslant L(t)\, d\bigl(f_k(t), x\bigr) . \end{aligned}$$

From the given calculation, we see that the value $\hat{F}(t,x)$ is finite for every $(t,x) \in T \times X$.

Now, we will prove that the map $\hat{F}(t, \cdot) \colon X \to \mathbb{R}$ fulfills the Lipschitz condition with the constant $L(t)$. Let $x_1, x_2 \in X$ be any arbitrary points such that $\hat{F}(t, x_1) \geqslant \hat{F}(t, x_2)$, then

$$\begin{aligned} 0 \leqslant \hat{F}(t, x_1) - \hat{F}(t, x_2) = {} & \sup_n \bigl\{ F\bigl(t, f_n(t)\bigr) - L(t)\, d\bigl(f_n(t), x_1\bigr) \bigr\} \\ & \qquad - \sup_n \bigl\{ F\bigl(t, f_n(t)\bigr) - L(t)\, d\bigl(f_n(t), x_2\bigr) \bigr\} \\ \leqslant {} & \sup_n \bigl\{ L(t)\, d\bigl(f_n(t), x_2\bigr) - L(t)\, d\bigl(f_n(t), x_1\bigr) \bigr\} \\ \leqslant {} & L(t)\, d(x_1, x_2) . \end{aligned}$$

The derivation given above proves that a map $\hat{F}(t, \cdot)$ for every t fulfills the Lipschitz condition with the constant $L(t)$. $\square$

Remark 2.8. If we define the function $\hat{F}$ in the proof of Theorem 2.7 by the formula

$$\hat{F}(t,x) = \sup_n \{F(t, f_n(t)) - L(t)\, d(f_n(t), x)^{\alpha}\},$$

then we obtain the result with the Hölder condition instead of the Lipshitz one.

Remark 2.9. The presented proof of Theorem 2.7 is a parametrization of the result given by E. J. McShane. In McShane's paper [4] we can find the theorem on the extension of one-variable function defined on the subset of a metric space with values in $\mathbb{R}$.

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