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CARATHÉODORY MEASURABILITY REVISITED

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ABSTRACT. Carathéodory measurability is studied starting with outer measures with values in a Dedekind σ-complete Abelian lattice ordered groups: the measurable elements create a σ-algebra and the restriction of the outer measure on them is a measure. On the base the group-valued measure extension theorem is proved, of course, for a special class of groups.

1. Introduction

We shall work with a Dedekind σ-complete Abelian lattice ordered group, i.e., with a structure $(G, +, \leqslant)$ such that $(G, +)$ is an Abelian group, $(G, \leqslant)$ is a Dedekind σ-complete lattice (i.e., any countable upper bounded subset of G has the supremum), and $a \leqslant b$ implies $a + c \leqslant b + c$ for any $c \in G$.

It is almost surprised that the whole theory of measurability also works in such a general structure (see Section 2). Of course, the main aim of the Carathéodory theory is the measure extension theorem. The outer measure μ^* induced by a measure μ can be defined for the case of any Dedekind complete group (σ-completeness is not sufficient). And again, in the general case μ^* is an extension of μ, its restriction to σ-algebra $\mathcal{M}$ of measurable elements is additive and σ-superadditive.

The only exception but a crucial one, $\mu^*|_{\mathcal{M}}$, is σ-additive, if G is weakly σ-distributive and of countable type. Recall that by the Fremlin theorem, weak σ-distributivity is a necessary and sufficient condition for a Riesz space G for every G-valued measure can be extended from a ring of sets to the generated σ-ring of sets.

Moreover, we prove the G-valued Choquet lemma stating that μ^* is continuous from bellow not only on measurable sets, what is evident, but moreover on the family of all subsets of the supporting space.

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The paper is dedicated to the memory of three author's students. For more information see the Appendix.

2. Measurable sets

Let G be a σ-complete Abelian l-group, $0 \in G$ a neutral element (i.e., $a + 0 = a$ for any $a \in G$), $G^+ = \{a \in G : a \geqslant 0\}$. By ∞ denote an ideal element,

$$G^* = G^+ \cup \{\infty\},$$

where $a + \infty = \infty + a = \infty + \infty = \infty$ for any $a \in G$, and $a \leqslant \infty$, $\infty \leqslant \infty$ for any $a \in G$.

Definition 2.1. Let X be a non-empty set, $\mathcal{P}(X)$ be the family of all subsets of X. Under an *outer measure* we mean a mapping $\mu^* : \mathcal{P}(X) \to G^*$ satisfying the following conditions:

(i) $\mu^*(\emptyset) = 0$;

(ii) $A \subset B$ implies $\mu^*(A) \leqslant \mu^*(B)$;

(iii) $\mu^*(A \cup B) \leqslant \mu^*(A) + \mu^*(B)$ for any $A, B \in \mathcal{P}(X)$.

An outer measure μ^* is called *σ-subadditive* if

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i)$$

for any $A_n \in \mathcal{P}(X)$ $(n = 1, 2, \ldots)$.

Definition 2.2. Let $\mu^* : \mathcal{P}(X) \to G^*$ be an outer measure. A set $A \subset X$ is called *measurable* if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A')$$

for any $E \subset X$. By $\mathcal{M}$ denote the family of all measurable subsets of X.

Theorem 2.3. *The set $\mathcal{M}$ of all measurable sets is an algebra and $\mu^*|_{\mathcal{M}}$ is additive. If μ^* is σ-subadditive, then $\mathcal{M}$ is a σ-algebra, and $\mu^*|_{\mathcal{M}}$ is a measure.*

Proof. If $A, B \in \mathcal{M}$, then for any $E \subset X$

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A') \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B') + \mu^*(E \cap A' \cap B) + \mu^*(E \cap A' \cap B') \\ &\geqslant \mu^*\big((E \cap A \cap B) \cup (E \cap A \cap B') \cup (E \cap A' \cap B)\big) + \mu^*(E \cap A' \cap B') \\ &= \mu^*\big(A \cap (A \cup B)\big) + \mu^*\big(E \cap (A \cup B)'\big) \geqslant \mu^*(E), \end{aligned}$$

hence $A \cup B \in \mathcal{M}$. Evidently also $A' \in \mathcal{M}$, hence $\mathcal{M}$ is an algebra.

If $A \cap B = \emptyset$ then

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*\big((A \cup B) \cap A\big) + \mu^*\big((A \cup B) \cap A'\big)\\ &= \mu^*(A) + \mu^*(B)\,.\end{aligned}$$

Moreover, for any $E \subset X$

$$\mu^*\big(E \cap (A \cup B)\big) = \mu^*(E \cap A) + \mu^*(E \cap B)\,,$$

and generalizing by induction

$$\mu^*\left(E \cap \bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$$

whenever $A_i \in \mathcal{M}$ $(i = 1, 2, \ldots)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$.

Let $A_i \in \mathcal{M}$ $(i = 1, 2, \ldots)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and μ^* be σ-subadditive now. Then

$$\begin{aligned}\mu^*(E) &= \mu^*\left(E \cap \bigcup_{i=1}^{n} A_i\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)'\right)\\ &\geqslant \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right)\end{aligned}$$

for any $n \in \mathbb{N}$, hence

$$\begin{aligned}\mu^*(E) &\geqslant \sum_{n=1}^{\infty} \mu^*(E \cap A_n) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right)\\ &\geqslant \mu^*\left(\bigcup_{n=1}^{\infty} (E \cap A_n)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right)\\ &= \mu^*\left(E \cap \left(\bigcup_{n=1}^{\infty} A_i\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right) \geqslant \mu^*(E)\,.\end{aligned}$$

We have obtained that $\bigcup\limits_{n=1}^{\infty} A_i \in \mathcal{M}$, whenever A_n are disjoint. Generally, if $B_n \in \mathcal{M}$ $(n = 1, 2, \ldots)$, put $A_1 = B_1$, $A_n = B_n \Big\backslash \left(\bigcup\limits_{i=1}^{n-1} B_i\right)$ $(i = 2, 3, \ldots)$. Then $A_n \in \mathcal{M}$, A_n are disjoint, hence $\bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{M}$.

If $A_n \in \mathcal{M}$, A_n are disjoint, we have just proved

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap A_n) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right),$$

hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} \mu^*\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap A_n\right) + \mu^*\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap \left(\bigcup_{i=1}^{\infty} A_i\right)'\right)$$
$$= \sum_{n=1}^{\infty} \mu^*(A_n) + \mu^*(\emptyset) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

3. Induced outer measure

In the next we shall assume that G is Dedekind complete, i.e., any upper bounded subset of G has the supremum. Recall that a G-valued measure on an algebra $\mathcal{R}$ of subsets of X is a function $\mu\colon \mathcal{R} \to G^+$ such that $\mu(A) = \sum_{i=1}^{\infty} \mu(A_n)$, whenever A_n are disjoint sets from $\mathcal{R}$, and $A = \bigcup_{n=1}^{\infty} A_n$.

Definition 3.1. Let μ be a G-valued measure defined on an algebra $\mathcal{R}$ of subsets of X. Then for any $A \subset X$ we define

$$\mu^*(A) = \bigwedge\left\{ \sum_{n=1}^{\infty} \mu(A_n)\colon A_n \in \mathcal{R},\ A \subset \bigcup_{n=1}^{\infty} A_n \right\}$$

Theorem 3.2. *The function μ^* is an outer measure. Any set of $\mathcal{R}$ is measurable and μ^* is an extension of μ. Moreover, μ^* is superadditive on the family $\mathcal{M}$ of all measurable elements, i.e., $\mu^*(A) \geqslant \sum_{n=1}^{\infty} \mu^*(A_n)$, whenever A_n are disjoint and $A \supset \bigcup_{n=1}^{\infty} A_n$.*

P r o o f. Evidently $0 \leqslant \mu^*(\emptyset) \leqslant \mu(\emptyset) + \mu(\emptyset) + \dots = 0$, hence $\mu^*(\emptyset) = 0$. If $A \subset B$, then

$$\left\{ \sum_{i=1}^{\infty} \mu(A_i)\colon \bigcup_{i=1}^{\infty} A_i \supset B,\ A_i \in \mathcal{R} \right\} \subset \left\{ \sum_{i=1}^{\infty} \mu(A_i)\colon \bigcup_{i=1}^{\infty} A_i \supset A,\ A_i \in \mathcal{R} \right\}.$$

We see that $\mu^*(A)$ is an infimum of a larger set, and therefore $\mu^*(A) \leqslant \mu^*(B)$. Finally, let A, B be arbitrary, $A_i, B_i \in \mathcal{R}$ $(i = 1, 2, \dots)$, and $\bigcup_{i=1}^{\infty} A_i \supset A$, $\bigcup_{i=1}^{\infty} B_i \supset B$. Therefore $\bigcup_{i=1}^{\infty} A_i \cup \bigcup_{i=1}^{\infty} B_i \supset A \cup B$, and

$$\mu^*(A \cup B) \leqslant \sum_{i=1}^{\infty} \mu(A_i) + \sum_{i=1}^{\infty} \mu(B_i).$$

Fix $(B_i)_{i=1}^{\infty}$ for a moment. Since the preceding inequality holds for any $(A_i)_{i=1}^{\infty}$, we obtain

$$\mu^*(A \cup B) \leqslant \mu^*(A) + \sum_{i=1}^{\infty} \mu(B_i)\,,$$

and by a similar argument also

$$\mu^*(A \cup B) \leqslant \mu^*(A) + \mu^*(B)\,.$$

Let now $A_n \in \mathcal{M}$ be disjoint. Then

$$\mu^*(A) \geqslant \mu^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i)\,.$$

Since the preceding inequality holds for any $n \in \mathbb{N}$, we have

$$\mu^*(A) \geqslant \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

Lemma 3.3. *For any $A \subset X$ we have*

$$\mu^*(A) = \bigwedge \left\{ \bigvee_{n=1}^{\infty} \mu(B_n)\colon B_n \in \mathcal{R},\, B_n \subset B_{n+1},\, \bigcup_{n=1}^{\infty} B_n \supset A \right\}.$$

Proof. Consider any sequence $B_n \in \mathcal{R}$, $B_n \subset B_{n+1}$ $(n = 1, 2, \ldots)$, $\bigcup\limits_{n=1}^{\infty} B_n \supset A$. Put $A_1 = B_1$, $A_2 = B_2 \setminus B_1$, $A_3 = B_3 \setminus B_2, \ldots$ Then $\sum\limits_{i=1}^{n} \mu(A_i) = \mu(B_n)$, hence

$$\mu^*(A) \leqslant \sum_{i=1}^{\infty} \mu(A_i) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(A_i) = \bigvee_{i=1}^{\infty} \mu(B_n)\,.$$

On the other hand consider any sequence $(A_n)_{n=1}^{\infty}$, $A_n \in \mathcal{R}$, $A \subset \bigcup\limits_{n=1}^{\infty} A_n$. Put $B_n = \bigcup\limits_{i=1}^{n} A_i$. Then $B_n \in \mathcal{R}$, $B_n \subset B_{n+1}$, $\bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} A_n \supset A$, hence

$$\bigvee_{n=1}^{\infty} \mu(B_n) = \bigvee_{n=1}^{\infty} \mu\left(\bigvee_{i=1}^{n} A_i\right) \leqslant \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

and therefore

$$\bigwedge \left\{ \bigvee_{n=1}^{\infty} \mu(B_n)\colon B_n \subset B_{n+1},\, B_n \in \mathcal{R},\, \bigcup_{n=1}^{\infty} B_n \supset A \right\} \leqslant \sum_{i=1}^{\infty} \mu(A_i)\,.$$

Since the last inequality holds for any sequence $(A_i)_{i=1}^\infty$, we also have

$$\bigwedge\left\{\bigvee_{n=1}^\infty \mu(B_n)\colon B_n \subset B_{n+1},\ B_n \in \mathcal{R},\ \bigcup_{n=1}^\infty B_n \supset A\right\} \leqslant \mu^*(A).$$

$\square$

Remark 3.4. By $\mathcal{R}^+$ denote the set of all unions $B = \bigcup\limits_{n=1}^\infty B_n$ such that $B_n \in \mathcal{R}$, $B_n \subset B_{n+1}$ $(n = 1,2,\dots)$. It is not difficult to prove that the value $\bigvee\limits_{n=1}^\infty \mu(B_n)$ does not depends on the sequence $(B_n)_{n=1}^\infty$ but on the union B only. Therefore, one can define $\mu^+\colon \mathcal{R}^+ \to G^*$ by the formula

$$\mu^+(B) = \bigvee_{n=1}^\infty \mu(B_n),$$

where $B = \bigcup\limits_{n=1}^\infty B_n$, $B_n \in \mathcal{R}$, $B_n \subset B_{n+1}$ $(n = 1,2,\dots)$. Using this notation we can reformulate Lemma 3.3 by the following way

$$\mu^*(A) = \bigwedge\{\mu^+(B)\colon B \in \mathcal{R}^+,\ B \supset A\}.$$

4. Choquet lemma

The Choquet lemma states that $A_n \nearrow A$ implies $\mu^*(A_n) \nearrow \mu^*(A)$. In the group-valued case we shall assume the two following conditions:

1. Dedekind complete l-group G is called to be of *countable type*, if to any bounded set $A \subset G$ there exists such a countable subset $B \subset A$ that

$$\bigwedge A = \bigwedge B.$$

2. Dedekind σ-complete l-group G is called to be *weakly σ-distributive* if for any bounded double sequence (a_{ij}) such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1,2,\dots)$ it is

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^\infty a_{i\varphi(i)} = 0.$$

The second condition is connected with the fact that the usual ε-technique cannot be used in general l-groups. Namely, if $G = \mathbb{R}$ then to any $\emptyset \neq C \subset \mathbb{R}$ and any $\varepsilon > 0$ there exists $c \in C$ such that

$$\bigwedge C - \varepsilon < c.$$

Instead of this technique one can use the double sequence technique. Namely, if $a_{ij} \in \mathbb{R}$, $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) then to any $\varepsilon > 0$ and any i there exists $\varphi(i)$ such that

$$a_{ij} < \varepsilon$$

for any $j \geqslant \varphi(i)$. Since $a_{i\varphi(i)} < \varepsilon$ for any $i = 1, 2, \dots$ then $\bigvee_{i=1}^{\infty} a_{i\varphi(i)} \leqslant \varepsilon$ for any $\varepsilon > 0$, hence $\bigwedge_{\varphi} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0$.

Definition 4.1. An l-group G is called *weakly σ-distributive*, if it is Dedekind σ-complete and if for any bounded double sequence (a_{ij}) such that $a_{ij} \searrow 0$ if $j \to \infty$ for any i there is

$$\bigwedge_{\varphi} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0 .$$

Lemma 4.2. *If G is a Dedekind complete l-group of countable type then to any $A \subset X$ there exists a bounded double sequence $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$) such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $B \in \mathbb{R}^{+}$, $B \supset A$, such that*

$$\mu^*(A) + \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \geqslant \mu^+(B) .$$

Proof. Since G is of countable type, by Lemma 3.3 there exists a sequence $(C_n)_{n=1}^{\infty}$ of elements of $\mathcal{R}^+$ such that $C_n \supset A$ and

$$\mu^*(A) = \bigwedge_{n=1}^{\infty} \mu^+(C_n) .$$

Put $B_n = \bigcap_{i=1}^{n} C_i$. Then evidently $B_n \supset A$, $B_n \in \mathcal{R}^+$, $B_n \supset B_{n+1}$ and

$$\mu^*(A) = \bigwedge_{n=1}^{\infty} \mu^+(B_n) .$$

Define $a_{ij} = \mu^+(B_j) - \mu^*(A)$, $j = 1, 2, \dots$, $i = 1, 2, \dots$ Then $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \dots$). Let $\varphi\colon \mathbb{N} \to \mathbb{N}$. Then

$$\bigvee_{i=1}^{\infty} a_{i\varphi(i)} \geqslant a_{i\varphi(i)} = \mu^+(B_{\varphi(i)}) - \mu^*(A) .$$

Therefore we may put $B = B_{\varphi(i)}$ to obtain the inequality

$$\mu^*(A) + \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \geqslant \mu^+(B) .$$

$\square$

Theorem 4.3. *Let G be a Dedekind complete weakly σ-distributive l-group of countable type. Let $A_n \nearrow A$. Then $\mu^*(A_n) \nearrow \mu^*(A)$.*

P r o o f. We shall use two properties of μ^+ which are consequences of analogous properties of μ:

(1) $\mu^+(B_1) + \mu^+(B_2) = \mu^+(B_1 \cup B_2) + \mu^+(B_1 \cap B_2)$ for any $B_1, B_2 \in \mathcal{R}^+$;

(2) if $C_n \nearrow C$, $C_n \in \mathcal{R}^+$, then $C \in \mathcal{R}^+$, too, and $\mu^+(C) = \bigvee\limits_{n=1}^{\infty} \mu^+(C_n)$.

Evidently $\mu^*(A_n) \leqslant \mu^*(A)$ and the equality holds if $\bigvee\limits_{n=1}^{\infty} \mu^*(A_n) = \infty$. Therefore we may assume that $\bigvee\limits_{n=1}^{\infty} \mu^*(A_n) \in G$. By Lemma 4.2 for any n there exists a bounded sequence $(a_{nij})_{ij}$ such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $B_n \in \mathcal{R}^+$, $B_n \supset A_n$ such that

$$\mu^*(A_n) + \bigvee_{i=1}^{\infty} a_{ni\varphi(n+i)} \geqslant \mu^+(B_n).$$

By the Fremlin theorem (see [19], Theorem 3.2.3) there exists a bounded double sequence $(a_{ij})_{i,j} \searrow 0$ $(j \to \infty,\ i = 1, 2, \dots)$ and such that

$$\sum_{n=1}^{k} \bigvee_{i=1}^{\infty} a_{ni\varphi(n+i)} \leqslant \bigvee_{i=1}^{\infty} a_{i\varphi(i)}$$

for every $k \in \mathbb{N}$. Put $C_n = \bigcup\limits_{i=1}^{n} B_i$. Then $C_n \in \mathcal{R}^+$, $C_n \subset C_{n+1}$ $(n = 1, 2, \dots)$, $\bigcup\limits_{n=1}^{\infty} C_n \supset \bigcup\limits_{n=1}^{\infty} A_n = A$. We see that

$$\begin{aligned}
\mu^*(A_2) + \bigwedge_{i=1}^{\infty} a_{2i\varphi(2+i)} &\geqslant \mu^+(B_2) \\
&= \mu^+(B_1 \cup B_2) + \mu^+(B_1 \cap B_2) - \mu^+(B_1) \\
&\geqslant \mu^+(C_2) + \mu^*(A_1 \cap A_2) - \mu^*(A_1) - \bigvee_{i=1}^{\infty} a_{1i\varphi(1+i)} \\
&= \mu^+(C_2) - \bigvee_{i=1}^{\infty} a_{1i\varphi(1+i)},
\end{aligned}$$

hence

$$\mu^*(A_2) \geqslant \mu^+(C_2) - \bigvee_{i=1}^{\infty} a_{1i\varphi(1+i)} - \bigvee_{i=1}^{\infty} a_{2i\varphi(2+i)}.$$

Similarly,

$$\mu^*(A_n) \geqslant \mu^+(C_n) - \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{ki\varphi(k+i)} \geqslant \mu^+(C_n) - \bigvee_{i=1}^{\infty} a_{i\varphi(i)} .$$

Therefore

$$\bigvee_{n=1}^{\infty} \mu^*(A_n) + \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \geqslant \bigvee_{n=1}^{\infty} \mu^+(C_n) = \mu^+\left(\bigcup_{n=1}^{\infty} C_n \right) \geqslant \mu^*(A) .$$

Since

$$\bigvee_{n=1}^{\infty} a_{i\varphi(i)} \geqslant \mu^*(A) - \bigvee_{n=1}^{\infty} \mu^*(A_n)$$

holds for any $\varphi\colon \mathbb{N} \to \mathbb{N}$, and G is weakly σ-distributive, we obtain

$$0 = \bigwedge_{\varphi} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \geqslant \mu^*(A) - \bigvee_{n=1}^{\infty} \mu^*(A_n) .$$

□

THEOREM 4.4. *If G is a Dedekind complete weakly σ-distributive l-group of countable type then μ^* is σ-subadditive.*

P r o o f. Let $A = \bigcup_{i=1}^{\infty} A_i$. Put $B_n = \bigcup_{i=1}^{n} A_i$. Then $B_n \nearrow A$, hence by Theorem 4.3 and Theorem 3.2

$$\mu^*(A) = \bigvee_{n=1}^{\infty} \mu^*(B_n) \leqslant \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i) .$$

□

5. Measure extension theorem

We have started with a measure $\mu\colon \mathcal{R} \to G^+$ defined on an algebra $\mathcal{R}$. Then we have defined a function $\mu^*\colon \mathcal{R} \to G^*$ by Definition 3.1. We have seen that μ^* is an outer measure that is an extension of μ (Theorem 3.2). It follows particularly that $\mu^*(X) = \mu(X) \in G$, hence the function $\mu^*\colon \mathcal{P}(X) \to G$ is finite. Moreover, $\mathcal{R} \subset \mathcal{M}$ where $\mathcal{M}$ is the family of measurable sets.

If G is a Dedekind complete weakly σ-distributive l-group of countable type then μ^* is a σ-subadditive outer measure, hence we may use Theorem 2.3: $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a measure.

THEOREM 5.1. *Let G be a Dedekind complete weakly σ-distributive l-group of countable type, μ be a measure defined on an algebra $\mathcal{R}$ of subsets of a set X. Then there exists exactly one extension $\bar{\mu}$ of μ to the σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$.*

Proof. 1. Existence.

Since $\mathcal{R} \subset \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra, then $\sigma(\mathcal{R}) \subset \mathcal{M}$. Put $\bar{\mu} = \mu^*|_{\sigma(\mathcal{R})}$. Then $\bar{\mu}$ is a measure on $\sigma(\mathcal{R})$ extending μ.

2. Uniqueness.

Let $\nu\colon \sigma(\mathcal{R}) \to G$ be a measure extending μ. Put

$$\mathcal{K} = \bigl\{A \in \sigma(\mathcal{R}) \colon \nu(A) = \bar{\mu}(A)\bigr\}.$$

Then $\mathcal{K} \supset \mathcal{R}$ and $\mathcal{K}$ is a q-σ-algebra, hence $\mathcal{K} \supset q$-$\sigma(\mathcal{R}) = \sigma(\mathcal{R})$. Therefore, any set $A \in \sigma(\mathcal{R})$ belongs to $\mathcal{K}$; $\nu(A) = \bar{\mu}(A)$ for any $A \in \sigma(\mathcal{R})$. □

6. Concluding remarks

Of course, the G-valued measure extension theorem is Theorem 3.4.2 in [19]. It was published in [17] for the first time. The theorem has now some important applications in the probability theory on MV-algebras ([18]). As we have already mentioned the weak σ-distributivity is a necessary and sufficient condition for every G-valued measure to be extendable from a ring to a generated σ-ring in the case that G is a Riesz space. In [18] it has been proved that any probability MV-algebra with a positive probability p (i.e., such that $p(a) = 0 \iff a = 0$) is weakly σ-distributive automatically. Moreover, an MV-algebra is weakly σ-distributive if and only if the generated l-group G ([16]) is weakly σ-distributive ([18, Proposition 9.4.5]).

On the other hand our approach has a larger area of applications. E.g. it includes not only the Borel measurability but also the Lebesgue measurability.

7. Appendix

The paper is dedicated to the memory of three author's former students, Eugen Futáš, Jozef Kalas and Ervín Hrachovina.

Eugen Futáš was born on in Trenčín 8. 8. 1943. In 1965 he obtained his master degree in mathematics, specialization mathematical analysis, from the Faculty of Sciences, Comenius University in Bratislava. In 1971 he obtained the RNDr. and in 1980 his PhD degrees from the same university. In 1987 he obtained a scientific degree equivalent to Associate Professor from the Slovak Academy of

Sciences. He started his research in Measure Theory on lattices. Later he continued his studies in applied mathematics (civil engineering, operational research, data processing). He died on 19. 1. 2003 in Košice.

Jozef K a l a s was born in Bratislava on 8. 9. 1949. In 1972 he obtained his master degree in mathematics, specialization probability and statistics, from the Faculty of Sciences, Comenius University in Bratislava. After earning his RNDr. and PhD. (1978) degrees from the same university, in 1989 he was promoted to the position of Associate Professor at the Faculty of Mathematics and Physics, Comenius University. He started his scientific research in the theory of subadditive measures. Then he became a specialist in the theory of stochastic processes and discrete probability distributions. He died in Bratislava 3. 11. 2000.

Ervín H r a c h o v i n a was born in Bratislava on 1. 8. 1958. In 1982 he obtained his master degree in mathematics, specialization probability and mathematical statistics, from the Faculty of Mathematics and Physics, Comenius University. In 1982 he obtained the RNDr. degree and 1986 his PhD degree. He started to work in the theory of Riesz spaces valued measures. He planned to work in statistical information theory. He died in Bratislava 20. 2. 1990.

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