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ON KUBOTA–TYPE INTEGRALS

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ABSTRACT. We consider four integrals that are common generalizations to the Burkill integral and the wide Denjoy integral. Our contribution is a complete chart of relations between these four generalizations.

1. Burkill integral vs wide Denjoy integral

John Burkill ([1]) introduced an approximate version of Perron integral.

Definition 1. We will say that an $f\colon [a,b] \to \mathbb{R}$ is *integrable in the sense of Burkill* if for each $\varepsilon > 0$ there exist two approximately continuous functions $M, m\colon [a,b] \to \mathbb{R}$ such that $M(a) = m(a) = 0$, $M(b) - m(b) < \varepsilon$ and

$$\underline{M}'_{\mathrm{ap}}(x) \geqslant f(x) \geqslant \overline{m}'_{\mathrm{ap}}(x)$$

for each $x \in [a,b]$; $\underline{M}'_{\mathrm{ap}}(x)$, $\overline{m}'_{\mathrm{ap}}(x)$ stand for the lower and upper approximate derivative of M and m at x, respectively. The integral of f is defined to be the common value of $\inf\limits_{M} M(b)$ and $\sup\limits_{m} m(b)$.

In 1939 Tolstov ([13]) gave an example of a function integrable in the wide Denjoy sense, but not integrable in the sense of Burkill. An easy counterexample in the opposite direction was also well-known by that time: an approximately differentiable discontinuous function which being a Burkill primitive is not a wide Denjoy primitive. So the problem of

$$\textit{defining an integral that would include both of these integrals} \tag{$\star$}$$

arose.

The following question remains unanswered.

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Question 2. *Is the approximate Kurzweil–Henstock integral ($\equiv$ the approximate Perron integral, defined alike the Burkill integral but without requiring M and m to be approximately continuous) a* ***strict*** *generalization of the Burkill integral?*

One meets claims in the negative in the literature, but they have never been completely justified. So, the problem we mentioned above may be replaced by the one that sounds:

$$\text{define an integral that would include both the wide Denjoy and the approximate Kurzweil–Henstock integrals.} \tag{$\star\star$}$$

2. $\mathcal{F}_i$-(Kubota–type) integrals

We are interested in three classical solutions to ($\star\star$). The first one is due to Ridder, [7] (β-integral, [8]) and Kubota, [4] (*AD*-integral). The second ($T_{\text{ap}}D$-integral) is due to Sarkhel and Kar, [11], [9]. The third one ($\mathrm{AK}_{\mathcal{N}}$-integral) is due to Gordon, [2].

Let $E \neq \emptyset$ and an $F\colon E \to \mathbb{R}$. We will say that the function F satisfies (Lusin's) condition $\mathcal{N}$ if $|F(N)| = 0$ for each $N \subset E$, $|N| = 0$. The function F is said to be AC if for every $\varepsilon > 0$ there exists $\eta > 0$, such that for any pairwise nonoverlapping intervals $[a_1, b_1], \dots, [a_n, b_n]$, with both endpoints in E,

$$\sum_{i=1}^{n}(b_i - a_i) < \eta \implies \sum_{i=1}^{n}\left|F(b_i) - F(a_i)\right| < \varepsilon\,.$$

The function F is said to be VB if there is a number $M > 0$, such that for any pairwise nonoverlapping intervals $[a_1, b_1], \dots, [a_n, b_n]$, with both endpoints in E,

$$\sum_{i=1}^{n}\left|F(b_i) - F(a_i)\right| < M\,.$$

Definition 3. We say that an $F\colon [a,b] \to \mathbb{R}$ is *[ACG]-, [VBG]-, ACG-*, and *VBG-function* if there is a set sequence $(E_n)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} E_n = [a,b]$, and for each n, respectively: $F \restriction E_n$ is AC and E_n is closed; $F \restriction E_n$ is VB and E_n is closed; $F \restriction E_n$ is AC; $F \restriction E_n$ is VB.

Consider the following four classes of functions defined on a fixed $[a,b]$: $\mathcal{L}_1$-[ACG]-functions, $\mathcal{L}_2$-[VBG]-functions satisfying $\mathcal{N}$, $\mathcal{L}_3$-ACG-functions, $\mathcal{L}_4$-VBG-functions satisfying $\mathcal{N}$.

Recall the definition of wide Denjoy (Denjoy–Khintchine) integral.

Definition 4. We say that an $f\colon [a,b] \to \mathbb{R}$ is *integrable in the wide Denjoy sense* if there exists a continuous function $F \in \mathcal{L}_3$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. The integral of f is defined to be $F(b) - F(a)$.

(Here $\mathcal{L}_3$ can be replaced by any $\mathcal{L}_i$, $i = 1,2,4$.)

Let $\mathcal{F}_i$ denote the class of approximately continuous functions from $\mathcal{L}_i$, $i = 1,2,3,4$.

Definition 5. We say that $f\colon [a,b] \to \mathbb{R}$ is $\mathcal{F}_i$*-integrable* if there exists $F \in \mathcal{F}_i$ such that $F'_{\mathrm{ap}}(x) = f(x)$ for almost all $x \in [a,b]$. The integral of f is defined to be $F(b) - F(a)$.

The $\mathcal{F}_1$-integral is the Ridder's β-integral (Kubota's AD-integral), the $\mathcal{F}_4$-integral is the Gordon's $\mathrm{AK}_{\mathcal{N}}$-integral, while the $\mathcal{F}_2$-integral is equivalent to the Sarkhel & Kar's $T_{\mathrm{ap}}D$-integral. The $\mathcal{F}_3$-integral was originally considered by Kubota in [3], but it was abandoned in favour of the $\mathcal{F}_1$-integral.

The fact that the Ridder's (Kubota's) integral solves problems ($\star$) and ($\star\star$) was shown only around 1997.

Theorem 6. (Lee [5], Sarkhel [10].) *The $\mathcal{F}_1$-integral includes (is more general than) the Burkill integral.*

Theorem 7. (Lu [6].) *The $\mathcal{F}_1$-integral includes the approximate Kurzweil–Henstock integral.*

Let us ask the following

Question 8. *Is the $\mathcal{F}_1$-integral,* ***the smallest*** *integral that includes both the approximate Kurzweil–Henstock integral and the wide Denjoy integral?*

We conjecture the answer is affirmative.

3. Relations between $\mathcal{F}_i$-integrals (our contribution)

Our result fills a hole in the theory of Kubota–type integrals. We have obtained a complete picture of relations between $\mathcal{F}_i$-integrals. Relations between classes of integrands are the same as relations between respective classes of primitives.

Obvious relations.

$$\mathcal{F}_1 \subset \mathcal{F}_2\,, \quad \mathcal{F}_3 \subset \mathcal{F}_4\,, \quad \mathcal{F}_1 \subset \mathcal{F}_3\,, \quad \mathcal{F}_2 \subset \mathcal{F}_4\,.$$

For construction of the counterexamples below it is essential to define a primitive on Cantor's ternary set $\mathbb{C}$, and next to extend it properly to an approximately continuous function on the whole interval. Details are given in [12].

Sarkhel & Kar's example [11].

$$\mathcal{F}_3 \not\subset \mathcal{F}_2 .$$

S k e t c h . Let $F \restriction \mathbb{C}$ be an increasing function with a dense (in $\mathbb{C}$) set of jumps. □

Example 1.

$$\mathcal{F}_2 \not\subset \mathcal{F}_3 .$$

S k e t c h . Let $F \restriction \mathbb{C}$ have 0 as the limit at each $x \in \mathbb{C}$ and have an unbounded variation in every portion of $\mathbb{C}$. □

Example 2.

$$\mathcal{F}_2 \cap \mathcal{F}_3 \not\subset \mathcal{F}_1 .$$

S k e t c h . Let $F \restriction \mathbb{C}$ have 0 as the limit at each $x \in \mathbb{C}$, be of bounded variation, and have a dense (in $\mathbb{C}$) set of discontinuity points. □

Example 3.

$$\mathcal{F}_4 \not\subset \mathcal{F}_2 + \mathcal{F}_3 .$$

S k e t c h . Let G be the Cantor's ternary function. We can pick a sequence $A_1, A_2, \dots$ of disjoint closed nowhere dense subsets of $\mathbb{C}$, such that

$$\sum_{k=1}^{\infty} \left|G(A_k)\right| = 1 = \left|G(\mathbb{C})\right| .$$

Now, $F \restriction \mathbb{C}$ is defined in the following way:

- F is equal to G on $\mathbb{C} \setminus \bigcup_{k=1}^{\infty} A_k$,
- each restriction $F \restriction A_k$ is piecewise linear and

$$\sup_{x \in A_k} \left|F(x) - G(x)\right| < \frac{1}{k} .$$

□

The following theorem shows that all the difference between primitives from classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, reveals (as the above examples suggest) in their behaviour on a nowhere dense set.

Theorem 9. *Every primitive for the $\mathcal{F}_i$-integral has a nowhere dense set of discontinuity points.*

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REFERENCES

[1] BURKILL, J.: *The approximately continuous Perron integral*, Math. Z. **34** (1931), 270–278.

[2] GORDON, R. A.: *Some comments on an approximately continuous Khintchine integral*, Real Anal. Exchange **20** (1994–1995), 831–841.

[3] KUBOTA, Y.: *On the approximately continuous Denjoy integral*, Tôhoku Math. J. **15** (1963), 253–264.

[4] KUBOTA, Y.: *An integral of the Denjoy type*, Proc. Japan Acad., Ser. A Math. Sci. **40** (1964), 713–717.

[5] LEE, C. M.: *Kubota's AD-integral is more general than Burkill's AP-integral*, Real Anal. Exchange **22** (1996–1997), 433–436.

[6] LU, S. P.: *Notes on the approximately continuous Henstock integral*, Real Anal. Exchange **22** (1996–1997), 377–381.

[7] RIDDER, J.: *Über approximativ stetige Denjoy-Integrale*, Fund. Math. **21** (1933), 1–10.

[8] RIDDER, J.: *Über die gegenseitigen Beziehungen verschiedener approximativ stetiger Denjoy-Perron Integrale*, Fund. Math. **22** (1934), 136–162.

[9] SARKHEL, D. N.: *A wide Perron integral*, Bull. Austral. Math. Soc. **34** (1986), 225–232.

[10] SARKHEL, D. N.: *On the approximately continuous integrals of Burkill and Kubota*, Real Anal. Exchange **23** (1997–1998), 735–742.

[11] SARKHEL, D. N.—KAR, A. B.: *[PVB] functions and integration*, J. Austral. Math. Soc. Ser. A **36** (1984), 335–353.

[12] SWOROWSKI, P.: *An answer to some questions of Ene*, Real Anal. Exchange **30** (2004–2005), 183–192.

[13] TOLSTOV, G. P.: *Sur l'intégrale de Perron*, Mat. Sb. **5** (1939), 647–659.

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