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ON CONCIRCULAR φ-RECURRENT SASAKIAN MANIFOLD

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ABSTRACT. In this paper we study concircular φ-recurrent Sasakian manifold and show that a concircular φ-recurrent Sasakian manifold is an Einstein manifold and a concircular φ-recurrent Sasakian manifold having a non-zero constant sectional curvature is locally concircular φ-symmetric.

1. Introduction

In 1977, T. Takahashi [4] introduced the notion of locally φ-symmetric Sasakian manifold and obtained some interesting properties. The papers like [2] and [7], have extended this notion to 3-dimensional Kenmotsu and trans--Sasakian manifolds, respectively. Also φ-recurrent Sasakian and Kenmotsu manifold were studied in the papers [3] and [6], respectively. In this paper we study concircular φ-recurrent Sasakian manifold and obtain some interesting results. Here we show that a concircular φ-recurrent Sasakian manifold is an Einstein manifold and also locally concircular φ-symmetric if it has a non-zero constant sectional curvature.

2. Preliminaries

Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a Sasakian manifold with the structure (φ,ξ,η,g). Then the following relations hold ([1], [3]).

$$\varphi^2 = -I + \eta\otimes\xi\,, \tag{2.1}$$

$$\eta(\xi) = 1\,, \quad \varphi\xi = 0\,, \quad \eta\circ\varphi = 0\,, \quad g(X,\xi) = \eta(X)\,, \tag{2.2}$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)\,, \tag{2.3}$$

$$R(\xi,X)Y = (\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X\,, \tag{2.4}$$

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$$\text{(a) } \nabla_X \xi = -\varphi X\,, \quad \text{(b) } (\nabla_X \eta)Y = g(X, \varphi Y)\,, \tag{2.5}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi\,, \tag{2.7}$$

$$\eta\big(R(X,Y)Z\big) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\,, \tag{2.8}$$

$$S(X,\xi) = 2n\,\eta(X)\,, \tag{2.9}$$

$$S(\varphi X, \varphi Y) = S(X,Y) - 2n\eta(X)\eta(Y)\,, \tag{2.10}$$

for all vector fields, X, Y, Z, where ∇ denotes the operator of covariant differentiation with respect to g, φ is a (1,1) tensor field, S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

Definition 2.1. ([4]) A Sasakian manifold is said to be *locally φ-symmetric* if

$$\varphi^2\big((\nabla_W R)(X,Y)Z\big) = 0\,, \tag{2.11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. A Sasakian manifold is said to be *locally concircular φ-symmetric* if

$$\varphi^2\big((\nabla_W \widetilde{C})(X,Y)Z\big) = 0\,, \tag{2.12}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.3. A Sasakian manifold is said to be *concircular φ-recurrent manifold* if there exists a non-zero 1-form A such that

$$\varphi^2\big((\nabla_W \widetilde{C})(X,Y)Z\big) = A(W)\widetilde{C}(X,Y)Z\,, \tag{2.13}$$

for arbitrary vector fields X, Y, Z, W, where $\widetilde{C}$ is a concircular curvature tensor given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\big[g(Y,Z)X - g(X,Z)Y\big]\,, \tag{2.14}$$

where R is the curvature tensor, and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to a locally concircular φ-symmetric manifold.

3. Concircular φ-recurrent Sasakian manifold

Let us consider a concircular φ-recurrent Sasakian manifold. Then by virtue of (2.1) and (2.13) we have,

$$-(\nabla_W \widetilde{C})(X,Y)Z + \eta\big((\nabla_W \widetilde{C})(X,Y)Z\big)\xi = A(W)\widetilde{C}(X,Y)Z\,, \tag{3.1}$$

from which it follows that

$$-g\big((\nabla_W\widetilde{C})(X,Y)Z,U\big)+\eta\big((\nabla_W\widetilde{C})(X,Y)Z\big)\eta(U)=A(W)g\big(\widetilde{C}(X,Y)Z,U\big)\,. \tag{3.2}$$

Let $\{e_i\}$, $i=1,2,\dots,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (3.2) and taking summation over i, $1\leqslant i\leqslant 2n+1$, we get

$$(\nabla_W S)(Y,Z)=\frac{dr(W)}{2n+1}g(Y,Z)\,. \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.2), we have

$$(\nabla_W S)(Y,\xi)=\frac{dr(W)}{2n+1}\eta(Y)\,. \tag{3.4}$$

Now we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi)\,.$$

Using (2.5a) and (2.9) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi)=S(Y,\varphi W)-2ng(Y,\varphi W)\,. \tag{3.5}$$

In view of (3.4) and (3.5) we have

$$S(Y,\varphi W)=\frac{dr(W)}{2n+1}\eta(Y)+2ng(Y,\varphi W) \tag{3.6}$$

Replacing Y by φY in (3.6) and using (2.3), (2.10), we obtain

$$S(Y,W)=2ng(Y,W)\,. \tag{3.7}$$

This leads to the following theorem:

Theorem 3.1. *Concircular φ-recurrent Sasakian manifold (M^{2n+1},g) is an Einstein manifold.*

Now from (3.1) we have

$$(\nabla_W\widetilde{C})(X,Y)Z=\eta\big((\nabla_W\widetilde{C})(X,Y)Z\big)\xi-A(W)\widetilde{C}(X,Y)Z\,. \tag{3.8}$$

This implies

$$\begin{aligned}(\nabla_W R)(X,Y)Z&=\eta\big((\nabla_W R)(X,Y)Z\big)\xi-A(W)R(X,Y)Z\\&\quad+\frac{dr(W)}{2n(2n+1)}\big[g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi\big]\\&\qquad+\frac{r}{2n(2n+1)}A(W)\big[g(Y,Z)X-g(X,Z)Y\big]\,.\end{aligned} \tag{3.9}$$

From (3.9) and the Bianchi identity we get

$$\begin{aligned} A(W)\eta\big(R(X,Y)Z\big) + A(X)\eta\big(R(Y,W)Z\big) + A(Y)\eta\big(R(W,X)Z\big) \\ = \frac{r}{2n(2n+1)} A(W)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big] \\ + \frac{r}{2n(2n+1)} A(X)\big[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)\big] \\ + \frac{r}{2n(2n+1)} A(Y)\big[g(X,Z)\eta(W) - g(W,Z)\eta(X)\big] \end{aligned} \tag{3.10}$$

By virtue of (2.8) we obtain from (3.10) that

$$\begin{aligned} A(W)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big] + A(X)\big[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)\big] \\ + A(Y)\big[g(X,Z)\eta(W) - g(W,Z)\eta(X)\big] \\ = \frac{r}{2n(2n+1)} A(W)\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big] \\ + \frac{r}{2n(2n+1)} A(X)\big[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)\big] \\ + \frac{r}{2n(2n+1)} A(Y)\big[g(X,Z)\eta(W) - g(W,Z)\eta(X)\big] \end{aligned} \tag{3.11}$$

Putting $Y = Z = e_i$ in (3.11) and taking summation over i, $1 \leqslant i \leqslant 2n+1$, we get

$$A(W)\eta(X) = A(X)\eta(W)\,, \tag{3.12}$$

for all vector fields X, W.

Replacing X by ξ in (3.12), we get

$$A(W) = \eta(W)\eta(\rho)\,, \tag{3.13}$$

for all vector fields W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A, i.e.,

$$g(X,\rho) = A(X)\,.$$

From (3.12) and (3.13) we can state the following.

Theorem 3.2. *In a concircular φ-recurrent Sasakian manifold (M^{2n+1}, g), $n \geqslant 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by* (3.13).

From (2.14) it follows that

$$(\nabla_W \widetilde{C})(X,Y)\xi = (\nabla_W R)(X,Y)\xi - \frac{dr(W)}{2n(2n+1)}\big[\eta(Y)X - \eta(X)Y\big]\,. \tag{3.14}$$

In view of (2.5)(a) and (2.6) it can be easily seen that in a Sasakian manifold the following relation holds:

$$(\nabla_W R)(X,Y)\xi = g(X,\varphi W)Y - g(Y,\varphi W)X + R(X,Y)\varphi W. \tag{3.15}$$

Substituting (3.15) in (3.14), we obtain

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)\xi = g(X,\varphi W)Y - g(Y,\varphi W)X + R(X,Y)\varphi W \\ - \frac{dr(W)}{2n(2n+1)}\big[\eta(Y)X - \eta(X)Y\big].\end{aligned} \tag{3.16}$$

By virtue of (2.8) it follows from (3.16) that

$$\eta\big((\nabla_W \widetilde{C})(X,Y)\xi\big) = 0. \tag{3.17}$$

Again from T a n n o [5] we have

$$\begin{aligned}R(X,Y)\varphi W = g(\varphi X,W)Y - g(Y,W)\varphi X \\ + g(X,W)\varphi Y - g(\varphi Y,W)X + \varphi R(X,Y)W.\end{aligned} \tag{3.18}$$

Using (3.18) in (3.16) we get

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)\xi = g(X,W)\varphi Y - g(Y,W)\varphi X + \varphi R(X,Y)W \\ - \frac{dr(W)}{2n(2n+1)}\big[\eta(Y)X - \eta(X)Y\big].\end{aligned} \tag{3.19}$$

We now suppose that a Sasakian manifold (M^{2n+1},g), $n>1$, is concircular φ-recurrent. Then from (3.8) and (3.19), it follows that

$$\begin{aligned}(\nabla_W \widetilde{C})(X,Y)Z = \big[g(Y,W)g(\varphi X,Z) - g(X,W)g(\varphi Y,Z) + g\big(\varphi R(X,Y)W,Z\big) \\ + \frac{dr(W)}{2n(2n+1)}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\big]\xi - A(W)\widetilde{C}(X,Y)Z.\end{aligned} \tag{3.20}$$

Next, we suppose that in a concircular φ-recurrent Sasakian manifold, the sectional curvature of a plane $\pi \subset T_pM$ defined by

$$K_p(\pi) = g\big(R(X,Y)Y,X\big), \tag{3.21}$$

is a non-zero constant k, where $\{X,Y\}$ is any orthonormal basis of π. Then we have

$$g\big((\nabla_Z R)(X,Y)Y,X\big) = 0. \tag{3.22}$$

Again from (2.14) we have

$$(\nabla_Z \widetilde{C})(X,Y)Y = (\nabla_Z R)(X,Y)Y - \frac{dr(Z)}{2n(2n+1)}\big[g(Y,Y)X - g(X,Y)Y\big]. \tag{3.23}$$

In view of (3.22) it follows from (3.23) that

$$g\big((\nabla_Z \widetilde{C})(X,Y)Y,X\big) = 0\,. \tag{3.24}$$

By virtue of (3.24) and (3.1) we obtain

$$g\big((\nabla_Z \widetilde{C})(X,Y)Y,\xi\big)\,\eta(X) = A(Z)g\left(\widetilde{C}(X,Y)Y,X\right). \tag{3.25}$$

Since in a concircular φ-recurrent, the relation (3.20) holds, using (3.20) in (3.25) we get

$$\begin{aligned}\eta(X)\Big[g(Y,Z)g(\varphi X,Y) - g(X,Z)g(\varphi Y,Y) - g\left(\varphi R(X,Y)Z,Y\right) \\ + \frac{dr(Z)}{2n(2n+1)}\big[g(X,Y)\eta(Y) - g(Y,Y)\eta(X)\big]\Big] \\ - A(Z)\eta(X)\Big[\big[g(Y,Y)\eta(X) - g(X,Y)\eta(Y)\big] \\ - \frac{r}{2n(2n+1)}\big[g(Y,Y)\eta(X) - g(X,Y)\eta(Y)\big]\Big] = kA(Z)\,.\end{aligned} \tag{3.26}$$

Putting $Y = Z = \xi$ in (3.26) and simplifying we get

$$\eta(\rho) = 0\,.$$

Hence by (3.13), we obtain from (2.13) that

$$\varphi^2\big((\nabla_W \widetilde{C})(X,Y)Z\big) = 0\,.$$

This leads to the following theorem:

THEOREM 3.3. *If a concircular φ-recurrent Sasakian manifold (M^{2n+1}, g), $n > 1$, has a non-zero constant sectional curvature, then it reduces to a locally concircular φ-symmetric manifold.*

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